

LONG RANGE SCATTERING FOR THE KLEIN-GORDON EQUATION WITH THE CRITICAL NONLINEARITY

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1. INTRODUCTION

This note is a survey of the papers [22, 23] which is a joint work with Satoshi Masaki (Osaka university). In this note, we consider the final state problem for the Klein-Gordon equation with a critical nonlinearity in space dimensions $d = 1, 2, 3$:

$$(1.1) \quad \begin{cases} (\square + 1)u = \lambda|u|^{2/d}u & t \in \mathbb{R}, x \in \mathbb{R}^d, \\ u - u_{\text{ap}} \rightarrow 0 & \text{in } L^2 \text{ as } t \rightarrow +\infty, \end{cases}$$

where $\square = \partial_t^2 - \Delta$ is d'Alembertian, $u : \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{R}$ is an unknown function, $u_{\text{ap}} : \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{R}$ is a given function, and λ is a non-zero real constant. To explain why we consider this problem, we briefly review known results on the global existence and long time behavior of solutions to the nonlinear Klein-Gordon equation

$$(1.2) \quad (\square + 1)u = \lambda|u|^{p-1}u, \quad t \in \mathbb{R}, x \in \mathbb{R}^d,$$

where $p > 1$ and $\lambda \in \mathbb{R} \setminus \{0\}$. The point-wise decay of a solution to the linear Klein-Gordon equation is $O(t^{-d/2})$ as $t \rightarrow \infty$, so the linear scattering theory indicates that the power $p = 1 + 2/d$ will be a borderline between the short and the long range scattering theories. This formal observation was firstly justified by Glassey [6], Matsumura [24] and Georgiev and Yordanov [4] for $p \leq 1 + 2/d$. More precisely, they proved that if $1 < p \leq 1 + 2/d$, then the equation (1.2) has no non-trivial solution which scatter to a solution to the linear Klein-Gordon equation as $t \rightarrow \infty$. On the other hand, Hayashi and Naumkin [10] proved that if $p > 1 + 2/d$ and $d = 1, 2$, then a small solution to (1.2) scatters to a solution to the linear Klein-Gordon equation. See also Georgiev and Lecente [3] for earlier results. The readers are referred to [7, 15, 28, 29, 30, 32] for the small data scattering when $d \geq 3$ and p is large.

From the above results we see that for the case where $p \leq 1 + 2/d$, the long time behavior of solution to (1.2) is different from that of the linear Klein-Gordon equation. So, we are interested in the long time behavior of solution to (1.2) for $p \leq 1 + 2/d$. For the critical case $p = 1 + 2/d$ and $d = 1$, Georgiev and Yordanov [4] studied point-wise decay of a solution to the initial value problem. Delort [1] obtained an asymptotic behavior of a global solution to (1.2) under the assumption that the support of the initial data is compact. See also Lindblad and Soffer [17] for an alternative proof of [1]. The compact support assumption in [1] was later removed by Hayashi and Naumkin in [8] by using the vector field approach by Klainerman [15].

Recently, the authors [22, 23] considered (1.2) with $p = 1 + 2/d$ and $d = 2, 3$ and specified an asymptotic profile u_{ap} that allows a unique solution u which converges to u_{ap} as $t \rightarrow \infty$.

To state the main theorems in [22, 23] precisely, we introduce an asymptotic profile u_{ap} which we work with. To this end, we first recall that a solution to the linear Klein-Gordon equation

$$\begin{cases} (\square + 1)v = 0 & t \in \mathbb{R}, x \in \mathbb{R}^d, \\ v(0, x) = \phi_0(x), \quad \partial_t v(0, x) = \phi_1(x) & x \in \mathbb{R}^d \end{cases}$$

behaves like

$$v = t^{-\frac{d}{2}} \mathbf{1}_{\{|x| < t\}}(t, x) A_1(\mu) \cos(\alpha - \beta) + o(t^{-\frac{d}{2}}),$$

in L^∞ as $t \rightarrow \infty$, where $\mathbf{1}_\Omega(t, x)$ is the characteristic function supported on $\Omega \subset \mathbb{R}^{1+d}$, $\mu = x/\sqrt{t^2 - |x|^2}$,

$$\begin{aligned} A_1(\mu) &= \sqrt{P_1^2(\mu) + Q_1^2(\mu)}, \\ P_1(\mu) &= \langle \mu \rangle^{\frac{d+2}{2}} \left\{ \cos\left(\frac{d\pi}{4}\right) \left(\operatorname{Re} \hat{\phi}_0(\mu) - \langle \mu \rangle^{-1} \operatorname{Im} \hat{\phi}_1(\mu) \right) \right. \\ &\quad \left. - \sin\left(\frac{d\pi}{4}\right) \left(\operatorname{Im} \hat{\phi}_0(\mu) + \langle \mu \rangle^{-1} \operatorname{Re} \hat{\phi}_1(\mu) \right) \right\}, \\ Q_1(\mu) &= \langle \mu \rangle^{\frac{d+2}{2}} \left\{ \sin\left(\frac{d\pi}{4}\right) \left(\operatorname{Re} \hat{\phi}_0(\mu) - \langle \mu \rangle^{-1} \operatorname{Im} \hat{\phi}_1(\mu) \right) \right. \\ &\quad \left. + \cos\left(\frac{d\pi}{4}\right) \left(\operatorname{Im} \hat{\phi}_0(\mu) + \langle \mu \rangle^{-1} \operatorname{Re} \hat{\phi}_1(\mu) \right) \right\}, \end{aligned}$$

$\alpha = \langle \mu \rangle^{-1} t$ and $\beta \in (0, 2\pi]$ is given by

$$\cos \beta = \frac{P_1}{A_1}, \quad \sin \beta = \frac{Q_1}{A_1},$$

see Hörmander's book [11] for instance. For given final state (ϕ_0, ϕ_1) , we define an asymptotic profile u_{ap} by

$$(1.3) \quad u_{\text{ap}}(t, x) := t^{-\frac{d}{2}} \mathbf{1}_{\{|x| < t\}}(t, x) A_1(\mu) \cos(\alpha + \Psi(\mu) \log t - \beta),$$

where the phase correction term is given by

$$(1.4) \quad \Psi(\mu) = \begin{cases} -\frac{3}{8} \lambda \langle \mu \rangle^{-1} A_1^2(\mu) & \text{if } d = 1, \\ -\frac{4\lambda}{3\pi} \langle \mu \rangle^{-1} A_1(\mu) & \text{if } d = 2, \\ -\frac{\Gamma(\frac{11}{6})}{\sqrt{\pi} \Gamma(\frac{7}{3})} \lambda \langle \mu \rangle^{-1} A_1(\mu)^{\frac{2}{3}} & \text{if } d = 3. \end{cases}$$

The final state (ϕ_0, ϕ_1) is taken from the function space Y defined by

$$\begin{aligned} Y &:= \{(\phi_0, \phi_1) \in \mathcal{S}'(\mathbb{R}^d) \times \mathcal{S}'(\mathbb{R}^d); \|(\phi_0, \phi_1)\|_Y < \infty\}, \\ \|(\phi_0, \phi_1)\|_Y &:= \|\phi_0\|_{H_x^2} + \|x\phi_0\|_{H_x^3} + \|x^2\phi_0\|_{H_x^4} \\ &\quad + \|\phi_1\|_{H_x^1} + \|x\phi_1\|_{H_x^2} + \|x^2\phi_1\|_{H_x^3}. \end{aligned}$$

The main results in [22, 23] are summarized as follows.

Theorem 1.1. *Let $d = 1, 2, 3$. Then for $d/4 < \gamma < 1$, there exist a sufficiently large number $T \geq e$ and a sufficiently small number $\varepsilon > 0$ such that if $\|(\phi_0, \phi_1)\|_Y < \varepsilon$ then there exists a unique solution $u(t)$ for the equation (1.1) satisfying*

$$(1.5) \quad \begin{aligned} u &\in C([T, \infty); L_x^2), \\ \sup_{t \geq T} t^\gamma \|u - u_{\text{ap}}\|_{L^\infty((t, \infty); L_x^2)} &< \infty, \end{aligned}$$

where the asymptotic profile u_{ap} is defined by (1.3).

Note that for the one dimensional case, Theorem 1.1 is proved by Hayashi and Naumkin [9] under weaker assumption on the final data. We also note that Lindblad and Soffer [16] showed existence of a modified wave operators for (1.2) for large final data in the case where $\lambda < 0$.

Remark 1.2. For the two and three dimensional cases, the coefficients of the phase function Ψ come from the first Fourier-cosine coefficients of a 2π -periodic function $|\cos \theta|^{2/d} \cos \theta$. See Sections 4 and 5 for the detail.

Remark 1.3. The global existence and asymptotic behavior of a solution to the Klein-Gordon equation with the cubic quasi-linear nonlinearity is studied by Moriyama [26], Katayama [12], and Sunagawa [33] in one space dimension. Concerning the Klein-Gordon equation with the quadratic nonlinearity in two dimensions, Ozawa, Tsutaya, and Tsutsumi [27] proved a global existence result and characterized the asymptotic behavior of a small solution to (1.2) with a smooth, quadratic, semi-linear nonlinearity, i.e., nonlinear term depends on $u, \partial_t u, \nabla u$. Delort, Fang, and Xue [2] extended Ozawa-Tsutaya-Tsutsumi's result to the case where the nonlinear term is quasi-linear. See also Kawahara and Sunagawa [14] and Katayama, Ozawa and Sunagawa [13] for related works.

The proof of Theorem 1.1 consists of two parts. As a first step, we solve a Cauchy problem at infinite initial time for the equation (1.1) for a given asymptotic profile which decays like a solution to the linear Klein-Gordon equation and approximately solves (1.1) for large time. Next, we construct an asymptotic profile satisfying those properties which is a crucial part of our proof. In Section 2 we solve a Cauchy problem at infinite initial time for the equation (1.1) in an abstract framework (Proposition 2.1). Then in Sections 3, 4 and 5, we explain how to construct a function which satisfies the assumptions in Proposition 2.1 for the case $d = 1, 2$ and 3, respectively.

2. ABSTRACT CAUCHY PROBLEM

For $T > 0$, we define the function spaces X_T by

$$\begin{aligned} X_T &:= \{w \in C([T, \infty); L_x^2); \|w\|_{X_T} < \infty\}, \\ \|w\|_{X_T} &:= \sup_{t \geq T} t^\gamma (\|w\|_{L_t^\infty((t, \infty); H_x^{1/2})} + \|w\|_{L^q((t, \infty); L_x^2)}), \end{aligned}$$

where $d/4 < \gamma < 1$ and

$$(q, r) = \begin{cases} (4, \infty) & \text{if } d = 1, \\ (4, 4) & \text{if } d = 2, \\ (\frac{10}{3}, \frac{10}{3}) & \text{if } d = 3. \end{cases}$$

Proposition 2.1. *Let $d = 1, 2, 3$ and let $N(u) = \lambda|u|^{2/d}u$. Let γ be a constant such that $d/4 < \gamma < 1$. Then there exist a sufficiently large $T > 0$ and a sufficiently small $\eta > 0$ such that if $A(t, x)$ satisfies*

$$(2.1) \quad \|A(t)\|_{L_x^\infty} \leq \eta t^{-1},$$

$$(2.2) \quad \|(\square + 1)A(t) - N(A)(t)\|_{L_x^2} \leq \eta t^{-1-\gamma},$$

then there exists a unique solution u for the equation (1.1) satisfying

$$u \in C([T, \infty); L_x^2),$$

$$(2.3) \quad \sup_{t \geq T} t^\gamma (\|u - A\|_{L^\infty((t, \infty); H_x^{1/2})} + \|u - A\|_{L^q((t, \infty); L_x^r)}) < \infty.$$

By Proposition 2.1, once we find a function A satisfying (2.1) and (2.2), we can show the existence of a unique solution u to the equation (1.1) satisfying $u - A \in X_T$. In Sections 3, 4 and 5, we construct a function A satisfying the conditions (2.1) and (2.2) for a given final state $(\phi_0, \phi_1) \in Y$.

Let us give an outline of proof for Proposition 2.1. To prove this proposition, we use the following inhomogeneous Strichartz estimates associated with the Klein-Gordon equation. Let

$$(2.4) \quad \mathcal{G}[g](t) := \int_t^\infty \sin((t - \tau)\sqrt{1 - \Delta})(1 - \Delta)^{-1/2} g(\tau) d\tau.$$

Lemma 2.2. *Let $2 \leq r < (2d)/(d - 2)$ and $2/q + d/r = d/2$. Then we have*

$$\|\mathcal{G}[g]\|_{L_t^q([T, \infty), L_x^r)} \leq C \|(1 - \Delta)^{\frac{d}{4} - \frac{d+2}{2r}} g\|_{L_t^{q'}([T, \infty), L_x^{r'})},$$

$$\|\mathcal{G}[g]\|_{L_t^\infty([T, \infty), L_x^2)} \leq C \|(1 - \Delta)^{\frac{d-2}{8} - \frac{d+2}{4r}} g\|_{L_t^{q'}([T, \infty), L_x^{r'})},$$

$$\|\mathcal{G}[g]\|_{L_t^q([T, \infty), L_x^r)} \leq C \|(1 - \Delta)^{\frac{d-2}{8} - \frac{d+2}{4r}} g\|_{L_t^1([T, \infty), L_x^2)}.$$

Proof of Lemma 2.2. The above inequalities follow from the L^p - L^q estimate for the solution to the Klein-Gordon equation by [18] and the duality argument by [34]. Since the proof is now standard, we omit the detail. $\square$

Outline of the proof of Proposition 2.1. We put $v = u - A$ and $F = (\square + 1)A - N(A)$. Then the equation (1.1) is equivalent to

$$(2.5) \quad (\square + 1)v = N(v + A) - N(A) - F.$$

The associate integral equation to the equation (2.5) is

$$(2.6) \quad v = \mathcal{G}[\{N(v + A) - N(A)\} - F],$$

where $\mathcal{G}$ is given by (2.4). We show the existence of a unique solution v to the equation (2.6) in X_T for sufficiently large $T > 0$ and sufficiently small

$\eta > 0$ by the contraction argument. To this end, we define the nonlinear operator Φ by

$$\Phi v := \mathcal{G}[\{N(v + A) - N(A)\} - F]$$

for $v \in \tilde{X}_T(\rho)$ and the function space $\tilde{X}_T(\rho)$ by

$$\tilde{X}_T(\rho) = \{w \in C([T, \infty); L_x^2); \|w\|_{X_T} \leq \rho\},$$

where $\rho > 0$ and $T > 0$. Note that $\tilde{X}_T(\rho)$ is a complete metric space with the $\|\cdot\|_{X_T}$ -metric. By using Lemma 2.2, we are able to show that for any $\rho > 0$, Φ is a contraction map on $\tilde{X}_T(\rho)$ if $T > 0$ is sufficiently large and $\eta > 0$ is sufficiently small. Hence the Banach fixed point theorem yields Proposition 2.1. $\square$

3. OUTLINE OF THE PROOF OF THEOREM 1.1 CASE: $d = 1$

In this section, we give an outline of the proof of Theorem 1.1 for $d = 1$ by using the argument by Delort [1]. We now explain how to construct the function $A = A(t, x)$ satisfying the conditions (2.1) and (2.2). It will turn out that $A = u_{\text{ap}}$ does not work well, and so that we need further modification. The conclusion is that the choice $A := u_{\text{ap}} + v_{\text{ap}}$ works, where u_{ap} is the *first approximation* given by (1.3) and v_{ap} is the *second approximation* which is of the form

$$(3.1) \quad v_{\text{ap}} := t^{-\frac{3}{2}} \mathbf{1}_{\{|x| < t\}} A_3(\mu) \cos(3(\alpha + \Psi(\mu) \log t - \beta)).$$

Here the phase function Ψ is the same as (1.4), and choice of A_3 will be specified later. Remark that $v_{\text{ap}}(t) = O(t^{-1})$ in L_x^2 . Toward the conclusion, we will observe (i) why the second approximation v_{ap} is required, and (ii) what is the appropriate choice of A_3 . Hereafter, we consider the case $|x| < t$ only because u_{ap} and v_{ap} are identically zero in the region $|x| \geq t$.

We first focus on the nonlinear part $N(u_{\text{ap}}) = \lambda |u_{\text{ap}}|^2 u_{\text{ap}}$. Since $N(u) = \lambda |u|^2 u$ is polynomial in $(u, \bar{u})$, it is easy to pick up a *resonant part* from $N(u_{\text{ap}})$. Indeed, we have

$$\begin{aligned} (3.2) \quad N(u_{\text{ap}}) &= \lambda t^{-\frac{3}{2}} A_1(\mu)^3 \cos^3(\alpha + \Phi(\mu) \log t - \beta) \\ &= \frac{3}{4} \lambda t^{-\frac{3}{2}} A_1^3(\mu) \cos(\alpha + \Phi(\mu) \log t - \beta) \\ &\quad + \frac{1}{4} \lambda t^{-\frac{3}{2}} A_1^3(\mu) \cos(3(\alpha + \Phi(\mu) \log t - \beta)) \\ &=: N_{\text{r}}(u_{\text{ap}}) + N_{\text{nr}}(u_{\text{ap}}). \end{aligned}$$

Since both of the resonant and non-resonant parts are $O(t^{-1})$ in L_x^2 , we need to cancel out those terms by the linear part, otherwise (2.2) fails. Thanks to the phase correction Ψ , we have the desired cancellation of the resonant part. Namely, we have

$$(\square + 1)u_{\text{ap}} = N_{\text{r}}(u_{\text{ap}}) + O(t^{-2}(\log t)^2)$$

in L^2 as $t \rightarrow \infty$. We then add a *second approximation* v_{ap} of u , given in (3.1), in order to cancel the non-resonant term $N_{\text{nr}}(u_{\text{ap}})$ out. This is the reason why we need the second approximation v_{ap} .

To obtain the desired cancellation, we will choose suitable A_3 . More precisely, we choose A_3 so that the leading term of $(\square + 1)v_{\text{ap}}$ and $N_{\text{nr}}(u_{\text{ap}})$ coincide. By a computation, we have

$$(\square + 1)v_{\text{ap}} = -8t^{-\frac{3}{2}}A_3(\mu) \cos(3(\alpha + \Phi(\mu) \log t - \beta)) + O(t^{-2}(\log t)^2)$$

in L^2 as $t \rightarrow \infty$. Hence, we obtain the specific choice

$$(3.3) \quad A_n(\mu) = -\frac{\lambda}{32}A_1^3(\mu).$$

With this choice, the leading term of $(\square + 1)v_{\text{ap}}$ and $N_{\text{nr}}(u_{\text{ap}})$ successfully cancel out each other. Thus, we see that $A = u_{\text{ap}} + v_{\text{ap}}$ satisfies the conditions (2.1) and (2.2).

Notice that this kind of approximation was introduced in Hörmander [11] for the Klein-Gordon equation with *polynomial* nonlinearity in $(u, \bar{u})$. See also [25, 31] for the nonlinear Schrödinger equation with polynomial nonlinearity in $(u, \bar{u})$.

4. OUTLINE OF THE PROOF OF THEOREM 1.1 CASE: $d = 2$

In this section, we give an outline of the proof of Theorem 1.1 for $d = 2$ which is given by [22].

We now explain how to construct the function $A = A(t, x)$ satisfying the conditions (2.1) and (2.2). We choose $A := u_{\text{ap}} + v_{\text{ap}}$, where u_{ap} is the *first approximation* given by (1.3) and v_{ap} is the *second approximation* which is of the form

$$(4.1) \quad v_{\text{ap}} := t^{-2} \mathbf{1}_{\{|x| < t\}} \sum_{n=2}^{\infty} A_n(\mu) \cos(n(\alpha + \Psi(\mu) \log t - \beta)).$$

Here the phase function Ψ is given by (1.4), and choice of A_n will be specified later. Remark that $v_{\text{ap}}(t) = O(t^{-1})$ in L_x^2 . Hereafter, we consider the case $|x| < t$ only because u_{ap} and v_{ap} are identically zero in the region $|x| \geq t$.

We first focus on the nonlinear part $N(u_{\text{ap}}) = \lambda|u_{\text{ap}}|u_{\text{ap}}$. Unlike the one dimensional case, the nonlinear term $N(u) = \lambda|u|u$ is not polynomial in $(u, \bar{u})$, so it becomes difficult to pick up a *resonant part* from $N(u_{\text{ap}})$. Taking a hint from our previous paper [21], we use the Fourier series expansion of $N(u_{\text{ap}})$ to decompose $N(u_{\text{ap}})$ into the resonant part and the rest, the *non-resonant part*. This decomposition is done as follows.

$$\begin{aligned} (4.2) \quad N(u_{\text{ap}}) &= \lambda t^{-2} A_1(\mu)^2 |\cos(\alpha + \Phi(\mu) \log t - \beta)| \cos(\alpha + \Phi(\mu) \log t - \beta) \\ &= \lambda t^{-2} A_1(\mu)^2 \sum_{n \geq 1} c_n \cos(n(\alpha + \Phi(\mu) \log t - \beta)) \\ &= c_1 \lambda t^{-2} A_1(\mu)^2 \cos(\alpha + \Phi(\mu) \log t - \beta) \\ &\quad + \sum_{n \geq 2} \lambda c_n t^{-2} A_1(\mu)^2 \cos(n(\alpha + \Phi(\mu) \log t - \beta)) \\ &=: N_{\text{r}}(u_{\text{ap}}) + N_{\text{nr}}(u_{\text{ap}}), \end{aligned}$$

where c_n is the n -th Fourier coefficients for the function $|\cos \theta| \cos \theta$:

$$c_n = \frac{1}{\pi} \int_0^{2\pi} |\cos \theta| \cos \theta \cos n\theta d\theta = \begin{cases} -\frac{8}{\pi} \frac{\sin(\frac{n}{2}\pi)}{n(n^2-4)} & \text{if } n \text{ is odd,} \\ 0 & \text{if } n \text{ is even.} \end{cases}$$

This kind of technique was also used in Sunagawa [33] to pick up the resonant term from the cubic nonlinearity in one space dimension. As we explained in Section 2, for the one dimensional case, the Fourier series for $N(u_{\text{ap}})$ consists of only two terms. We would emphasize that, in our setting, the Fourier series consists of *infinitely many terms*, so we need to take care of the convergence of the Fourier series, which seems a new ingredient. Fortunately, it turns out that the nonlinearity $|u|u$ has enough smoothness to ensure the convergence of the Fourier series for $|u|u$. We mention similar but slightly different expansion of a nonlinearity into a infinite Fourier sires is used by the first author and Miyazaki [19] in the context of nonlinear Schrödinger equation.

Since both of the resonant and non-resonant parts are $O(t^{-1})$ in L_x^2 , we need to cancel out those terms by the linear part, otherwise (2.2) fails. Thanks to the phase correction Ψ , we have the desired cancellation of the resonant part. Namely, we have

$$(\square + 1)u_{\text{ap}} = N_{\text{r}}(u_{\text{ap}}) + O(t^{-2}(\log t)^2), \quad \text{in } L^2$$

as $t \rightarrow \infty$. We then add a *second approximation* v_{ap} of u , given in (4.1), in order to cancel the non-resonant term $N_{\text{nr}}(u_{\text{ap}})$ out.

To obtain the desired cancellation, we will choose suitable A_n . More precisely, we choose them so that the leading term of n -th term of $(\square + 1)v_{\text{ap}}$ and n -th term of the Fourier expansion of $N_{\text{nr}}(u_{\text{ap}})$ coincide. By a computation, we have

$$\begin{aligned} (\square + 1)v_{\text{ap}} &= t^{-2} \sum_{n=2}^{\infty} (1 - n^2) A_n(\mu) \cos(n(\alpha + \Phi(\mu) \log t - \beta)) \\ &\quad + O(t^{-2}(\log t)^2), \quad \text{in } L^2 \end{aligned}$$

as $t \rightarrow \infty$. Hence, we obtain the specific choice

$$(4.3) \quad A_n(\mu) = \begin{cases} \frac{8 \sin(\frac{n}{2}\pi)}{\pi n(n^2-1)(n^2-4)} \lambda A_1^2(\mu) & \text{if } n \text{ is odd,} \\ 0 & \text{if } n \text{ is even.} \end{cases}$$

With this choice, the leading term of the n -th term of $(\square + 1)v_{\text{ap}}$ and the n -th term of the Fourier expansion for $N_{\text{nr}}(u_{\text{ap}})$ successfully cancel out each other. Further, it turns out that the error term can be handled thanks to fast decay of A_n in n . Remark that the coefficients of A_n is order $O(|n|^{-5})$ as $|n| \rightarrow \infty$. The decay rate of the Fourier coefficients reflects the smoothness of the nonlinearity $\lambda|u|u$. Thus, we see that $A = u_{\text{ap}} + v_{\text{ap}}$ satisfies the conditions (2.1) and (2.2).

5. OUTLINE OF THE PROOF OF THEOREM 1.1 CASE: $d = 3$

In this section, we give an outline of the proof of Theorem 1.1 for $d = 3$ which is given by [23]. In this case, the power becomes a fractional number,

so the argument in the two dimensional case [22] is not directly applicable. To overcome this difficulty, we use the argument by Ginibre and Ozawa [5].

We now explain how to construct the function $A = A(t, x)$ satisfying the conditions (2.1) and (2.2). The conclusion is that the choice $A := \tilde{u}_{\text{ap}} + \tilde{v}_{\text{ap}}$ works, where $\tilde{u}_{\text{ap}}$ is the *first approximation* given by

$$\tilde{u}_{\text{ap}} := t^{-\frac{3}{2}} \mathbf{1}_{\{|x| < t\}} A_1(\mu) \cos(\alpha + \tilde{\Psi}(\mu) \log t - \beta),$$

where $\tilde{\Psi}$ is given by

$$\tilde{\Psi}(\mu) = \sqrt{A_1^2(\mu) + t^{-1}}$$

and $\tilde{v}_{\text{ap}}$ is the *second approximation* which is of the form

$$(5.1) \quad \tilde{v}_{\text{ap}} := t^{-\frac{5}{2}} \mathbf{1}_{\{|x| < t\}} \sum_{n=2}^{\infty} A_n(\mu) \cos(n(\alpha + \tilde{\Psi}(\mu) \log t - \beta)).$$

where choice of A_n will be specified later. Note that $\tilde{v}_{\text{ap}}(t) = O(t^{-1})$ in L_x^2 . Hereafter, we consider the case $|x| < t$ only because $\tilde{u}_{\text{ap}}$ and $\tilde{v}_{\text{ap}}$ are identically zero in the region $|x| \geq t$.

We first focus on the nonlinear part $N(\tilde{u}_{\text{ap}}) = \lambda |\tilde{u}_{\text{ap}}|^{2/3} \tilde{u}_{\text{ap}}$. As is the case of $d = 2$, $N(u) = \lambda |u|^{2/3} u$ is not polynomial in $(u, \bar{u})$, so we use the Fourier series expansion of $N(\tilde{u}_{\text{ap}})$ to decompose $N(\tilde{u}_{\text{ap}})$ into the resonant part and the rest, the *non-resonant part*. This decomposition is done as follows.

$$(5.2) \quad \begin{aligned} N(\tilde{u}_{\text{ap}}) &= \lambda t^{-\frac{5}{2}} A_1(\mu)^{\frac{5}{3}} |\cos(\alpha + \tilde{\Phi}(\mu) \log t - \beta)|^{\frac{2}{3}} \cos(\alpha + \tilde{\Phi}(\mu) \log t - \beta) \\ &= \lambda t^{-\frac{5}{2}} A_1(\mu)^{\frac{5}{3}} \sum_{n \geq 1} c_n \cos(n(\alpha + \tilde{\Phi}(\mu) \log t - \beta)) \\ &= \lambda t^{-\frac{5}{2}} A_1(\mu)^{\frac{5}{3}} c_1 \cos(\alpha + \tilde{\Phi}(\mu) \log t - \beta) \\ &\quad + \sum_{n \geq 2} \lambda c_n t^{-\frac{5}{2}} A_1(\mu)^{\frac{5}{3}} \cos(n(\alpha + \tilde{\Phi}(\mu) \log t - \beta)) \\ &=: N_r(\tilde{u}_{\text{ap}}) + N_{\text{nr}}(\tilde{u}_{\text{ap}}), \end{aligned}$$

where c_n are the Fourier coefficients for the function $|\cos \theta|^{2/3} \cos \theta$:

$$c_n = \frac{1}{\pi} \int_0^{2\pi} |\cos \theta|^{\frac{2}{3}} \cos \theta \cos n\theta d\theta.$$

Note that c_n are explicitly given by

$$\begin{cases} \frac{2(-1)^{\frac{n-1}{2}} \Gamma(\frac{11}{6}) \Gamma(\frac{3n-5}{6})}{\sqrt{\pi} \Gamma(-\frac{1}{3}) \Gamma(\frac{3n+11}{6})} & \text{if } n \text{ is odd,} \\ 0 & \text{if } n \text{ is even,} \end{cases}$$

see Masaki, Miyazaki and Uriya [20] for the detail. Since both of the resonant and non-resonant parts are $O(t^{-1})$ in L_x^2 , we need to cancel out those terms by the linear part, otherwise (2.2) fails. Thanks to the phase correction $\tilde{\Psi}$, we have the desired cancellation of the resonant part. Namely, we have

$$(\square + 1)\tilde{u}_{\text{ap}} = N_r(\tilde{u}_{\text{ap}}) + O(t^{-\frac{11}{5}}(\log t)), \quad \text{in } L^2$$

as $t \rightarrow \infty$. We then add a *second approximation* $\tilde{v}_{\text{ap}}$ of u , given in (5.1), in order to cancel the non-resonant term $N_{\text{nr}}(\tilde{u}_{\text{ap}})$ out.

To obtain the desired cancellation, we will choose A_n appropriately. More precisely, we choose them so that the leading term of n -th term of $(\square + 1)\tilde{v}_{\text{ap}}$ and n -th term of the Fourier expansion of $N_{\text{nr}}(\tilde{u}_{\text{ap}})$ coincide. By a computation, we have

$$(\square + 1)\tilde{v}_{\text{ap}} = t^{-\frac{5}{2}} \sum_{n=2}^{\infty} (1 - n^2) A_n(\mu) \cos(n(\alpha + \Phi(\mu) \log t - \beta)) + O(t^{-2}), \quad \text{in } L^2$$

as $t \rightarrow \infty$. Hence, we obtain the specific choice

$$(5.3) \quad A_n(\mu) = \frac{c_n \lambda}{1 - n^2} A_1^{\frac{5}{3}}(\mu).$$

With this choice, the leading term of the n -th term of $(\square + 1)\tilde{v}_{\text{ap}}$ and the n -th term of the Fourier expansion for $N_{\text{nr}}(\tilde{u}_{\text{ap}})$ successfully cancel out each other. Further, it turns out that the error term can be handled thanks to fast decay of A_n in n . Remark that the coefficients of A_n is order $O(|n|^{-14/3})$ as $|n| \rightarrow \infty$. The decay rate of the Fourier coefficients reflects the smoothness of the nonlinearity $\lambda|u|^{2/3}u$. Thus, we see that $A = \tilde{u}_{\text{ap}} + \tilde{v}_{\text{ap}}$ satisfies the conditions (2.1) and (2.2).

Acknowledgments. J.S. is partially supported by JSPS KAKENHI Grant Number JP17H02851.

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