

# 指標付き約数問題

## (Divisor Problem with Characters)

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Let  $\chi$  be a Dirichlet character mod  $k$  and let  $r_a(n, \chi)$  be the function defined by

$$r_a(n, \chi) = \sum_{d|n} \chi(d) d^a,$$

where  $a$  is a fixed real number. When  $\chi$  is identically 1, this function is a classical divisor function usually written by  $\sigma_a(n)$ . On the other hand, when  $a = 0$  and  $\chi$  is the Kronecker symbol corresponding to  $\mathbb{Q}(i)$ , then

$$r_a(n, \chi) = \frac{1}{4} r(n).$$

with  $r(n) = \#\{(x, y) \in \mathbb{Z}^2 \mid x^2 + y^2 = n\}$ . Hence we can also consider  $r_a(n, \chi)$  as a generalization of the counting function of the lattice points on a circle. We shall consider the sum of  $r_a(n, \chi)$  and the mean square of its error term.

Before stating our results, we shall recall some known results about the sum of  $\sigma_a(n)$ . Put

$$\Delta_0(x) = \sum'_{n \leq x} \sigma_0(n) - x(\log x + 2\gamma - 1) - 1/4,$$

where  $\gamma$  is Euler's constant and the prime on the summation means that the last term is to be halved if  $x$  is an integer. In 1956, Tong proved that

$$\int_1^X \Delta_0(x)^2 dx = \frac{1}{6\pi^2} \frac{\zeta^4(3/2)}{\zeta(3)} X^{3/2} + O(X \log^5 X).$$

and the  $O$ -term was improved to  $O(X \log^4 X)$  by Preissmann in 1988. For  $-1 < a < 0$ , we define

$$\Delta_a(x) = \sum_{n \leq x} ' \sigma_a(n) - \zeta(1-a)x - \frac{\zeta(1+a)}{1+a} x^{1+a} + \frac{1}{2} \zeta(-a).$$

The mean square formula of  $\Delta_a(x)$  for  $-1/2 < a < 0$  was first considered by Kiuchi, and improved by Meurman. In fact, Meurman [2] proved that

$$\begin{aligned} & \int_1^X |\Delta_a(x)|^2 dx \\ &= \begin{cases} \frac{\zeta(3/2-a)\zeta(3/2+a)\zeta^2(3/2)}{(6+4a)\pi^2\zeta(3)} X^{3/2+a} + O(X) & \text{for } -1/2 < a < 0 \\ \frac{\zeta^2(3/2)}{24\zeta(3)} X \log X + O(X) & \text{for } a = -1/2 \\ O(X) & \text{for } -1 < a < -1/2 \end{cases} \end{aligned}$$

For the case  $-1 < a < -1/2$ , the more precise formula had already been obtained by S. Chowla in 1932. In [1], he showed that

$$\int_1^X |\Delta_a(x)|^2 dx = \frac{\zeta^2(1-a)\zeta(-2a)}{12\zeta(2-2a)} X + O(X^{3/2+a} \log X)$$

for  $-1 < a < -1/2$ . Recently, the last formula was obtained independently by Yanagisawa in a somewhat general situation in [3]. He proved that

$$\begin{aligned} \int_1^X \Delta_a\left(\frac{h}{k}x\right) \Delta_b(x) dx &= \frac{1}{2\pi^2 k h} \left( \sum_{n=1}^{\infty} \frac{\sigma_{1+a}(hn) \sigma_{1+b}(kn)}{n^2} \right) X \\ &+ O(X^{(3+a+b)/2} \log X). \end{aligned}$$

for  $-1 < a < 0, -1 < b < 0, a+b < -1$  and  $h > 0, k > 0, (h, k) = 1$ .

The aim of this note is to show the corresponding mean square formula for  $r_a(n, \chi)$ . We assume throughout that  $\chi$  is a non-trivial Dirichlet character mod  $k$ . In our case, the error term of the sum function of  $r_a(n, \chi)$  is defined by

$$\Delta_a(x, \chi) = \sum_{n \leq x} ' r_a(n, \chi) - L(1-a, \chi)x + \frac{1}{2} L(-a, \chi).$$

**Theorem 1.** Let  $-1 < a, b < 0$  be real numbers and let  $k \ll \sqrt{X}$ . Suppose that  $\chi_1$  and  $\chi_2$  are primitive Dirichlet characters mod  $k$  with the same parity.

(i) For  $a + b > -1$ , we have

$$\begin{aligned} & \int_1^X \Delta_a(x, \chi_1) \Delta_b(x, \chi_2) dx \\ &= C_1 X^{(3+a+b)/2} + O(\min(k^2 X, kX(\log X)^2)) + O(k^{\max(\frac{5}{4} + \frac{a+b}{2}, 1)} X), \end{aligned}$$

where

$$\begin{aligned} C_1 &= \frac{\tau(\chi_1)\tau(\chi_2)k^{(-1+a+b)/2}}{2\pi^2(3+a+b)} \\ &\times \frac{\zeta\left(\frac{3-a-b}{2}\right)L\left(\frac{3+a-b}{2}, \bar{\chi}_1\right)L\left(\frac{3-a+b}{2}, \bar{\chi}_2\right)L\left(\frac{3+a+b}{2}, \bar{\chi}_1\bar{\chi}_2\right)}{L(3, \bar{\chi}_1\bar{\chi}_2)}, \end{aligned}$$

and

$$\tau(\chi) = \sum_{n=1}^{k-1} \chi(n) e^{2\pi i n/k}.$$

(ii) For  $a + b = -1$ , we have

$$\begin{aligned} & \int_1^X \Delta_a(x, \chi_1) \Delta_{-1-a}(x, \chi_2) dx \\ &= \begin{cases} C_2 X \log X + O(\min(k^2 X, kX(\log X)^2)) & \text{if } \chi_2 = \bar{\chi}_1 \\ O(\min(k^2 X, kX(\log X)^2)) & \text{otherwise} \end{cases} \end{aligned}$$

with

$$C_2 = \frac{\chi_1(-1)L(2+a, \bar{\chi}_1)L(1-a, \chi_1)}{24\zeta(3)} \prod_{p|k} \frac{p^2}{p^2 + p + 1}.$$

**Theorem 2.** Let  $-1 < a, b < 0$ ,  $a + b \geq -1$  and  $k \ll \sqrt{X}$ . Suppose that  $\chi_1$  and  $\chi_2$  are primitive Dirichlet characters mod  $k$  with the opposite parity. Then we have

$$\begin{aligned} & \int_1^X \Delta_a(x, \chi_1) \Delta_b(x, \chi_2) dx \\ &= O(\min(k^2 X, kX(\log X)^2)) + O(k^{\max(\frac{5}{4} + \frac{a+b}{2}, 1)} X). \end{aligned}$$

For the case  $a + b < -1$ , we have the following theorem.

**Theorem 3.** *Let  $-1 < a$ ,  $b < 0$ ,  $a + b < -1$  and  $k \ll \sqrt{X}$ . Let  $\chi_1$  and  $\chi_2$  be non-trivial Dirichlet characters mod  $k$ . Then we have*

$$\int_1^X \Delta_a(x, \chi_1) \Delta_b(x, \chi_2) dx = C_3 X + O(k^2 \log k X^{(3+a+b)/2} \log X)$$

where

$$C_3 = \frac{L(1-a, \chi_1) L(1-b, \chi_2) L(-a-b, \chi_1 \chi_2)}{12L(2-a-b, \chi_1 \chi_2)}.$$

In these theorems, the O-constants do not depend on the modulus  $k$ .

### Outline of the proof of Theorems 1 and 2.

For the proof of Theorems 1 and 2, we need the Voronoï formula for  $\Delta_a(x, \chi)$ . Let  $\phi(s)$  be the generating function of  $r_a(n, \chi)$ . It is easily seen that  $\phi(s) = \zeta(s) L(s-a, \chi)$ . Let

$$\delta_\chi = \begin{cases} 0 & \text{if } \chi \text{ is an even character,} \\ 1 & \text{if } \chi \text{ is an odd character,} \end{cases}$$

and  $W(\chi) = (-1)^{\delta_\chi} \tau(\chi) / \sqrt{k}$ . Then the functional equation of  $\phi(s)$  is given by

$$\phi(s) = W(\chi) k^{1/2+a-s} \pi^{2s-(1+a)} \frac{\Gamma((1-s)/2) \Gamma((1+a-s+\delta_\chi)/2)}{\Gamma(s/2) \Gamma((s-a+\delta_\chi)/2)} \tilde{\phi}(1+a-s)$$

where

$$\tilde{\phi}(s) = \zeta(s-a) L(s, \bar{\chi}).$$

The coefficients of  $\tilde{\phi}(s)$  are given by

$$\tilde{r}_a(n, \chi) = \sum_{d|n} \bar{\chi}\left(\frac{n}{d}\right) d^a.$$

This function plays an important role in Voronoï formula.

First we assume that  $-1/2 < a < 0$ . We have

$$\begin{aligned}
\Delta_a(x, \chi) &= -W(\chi) k^{a/2} x^{(a+1)/2} \sum_{n=1}^{\infty} \frac{\tilde{r}_a(n, \chi)}{n^{(a+1)/2}} \left\{ \cos \frac{\pi(a - \delta_\chi)}{2} Y_{1+a}(4\pi \sqrt{nx/k}) \right. \\
&\quad \left. + \sin \frac{\pi(a - \delta_\chi)}{2} J_{1+a}(4\pi \sqrt{nx/k}) + \frac{2}{\pi} \cos \frac{\pi(a - \delta_\chi)}{2} K_{1+a}(4\pi \sqrt{nx/k}) \right\} \\
&= \frac{W(\chi)}{\sqrt{2}\pi} k^{a/2+1/4} x^{a/2+1/4} \sum_{n=1}^{\infty} \frac{\tilde{r}_a(n, \chi)}{n^{3/4+a/2}} \cos(4\pi \sqrt{nx/k} - (1/4 + \delta_\chi/2)\pi) \\
&\quad - \frac{4a^2 + 8a + 3}{32\sqrt{2}\pi^2} W(\chi) k^{a/2+3/4} x^{a/2-1/4} \sum_{n=1}^{\infty} \frac{\tilde{r}_a(n, \chi)}{n^{5/4+a/2}} \\
&\quad \times \cos(4\pi \sqrt{nx/k} - (3/4 + \delta_\chi/2)\pi) + O(k^{a/2+5/4} x^{a/2-3/4}).
\end{aligned}$$

Here,  $Y_{1+a}(x)$ ,  $J_{1+a}(x)$  and  $K_{1+a}(x)$  denote the standard Bessel functions of order  $1+a$ . We also need the sum formula of  $\tilde{r}_a(n, \chi)$ . It is given by

$$\sum_{n \leq x} \tilde{r}_a(n, \chi) = \frac{L(1+a, \bar{\chi})}{1+a} x^{1+a} + \zeta(-a)L(0, \bar{\chi}) + \tilde{\Delta}_a(n, \chi)$$

with

$$\begin{aligned}
\tilde{\Delta}_a(x, \chi) &= -\overline{W(\chi)} k^{-a/2} x^{(a+1)/2} \sum_{n=1}^{\infty} \frac{r_a(n, \chi)}{n^{(a+1)/2}} \left\{ \cos \frac{\pi(a - \delta_\chi)}{2} Y_{1+a}(4\pi \sqrt{nx/k}) \right. \\
&\quad \left. + \sin \frac{\pi(a - \delta_\chi)}{2} J_{1+a}(4\pi \sqrt{nx/k}) + \frac{2}{\pi} \cos \frac{\pi(a + \delta_\chi)}{2} K_{1+a}(4\pi \sqrt{nx/k}) \right\} \\
&= \frac{\overline{W(\chi)}}{\sqrt{2}\pi} k^{-a/2+1/4} x^{a/2+1/4} \sum_{n=1}^{\infty} \frac{r_a(n, \chi)}{n^{3/4+a/2}} \cos(4\pi \sqrt{nx/k} - (1/4 + \delta_\chi/2)\pi) \\
&\quad - \frac{4a^2 + 8a + 3}{32\sqrt{2}\pi^2} \overline{W(\chi)} k^{-a/2+3/4} x^{a/2-1/4} \sum_{n=1}^{\infty} \frac{r_a(n, \chi)}{n^{5/4+a/2}} \\
&\quad \times \cos(4\pi \sqrt{nx/k} - (3/4 + \delta_\chi/2)\pi) + O(k^{-a/2+5/4} x^{a/2-3/4}).
\end{aligned}$$

Following Meurman, we get the truncated expression of  $\Delta_a(x, \chi)$  from the above formulas valid for  $-1 < a < 0$ .

**Lemma 1.** For  $-1 < a < 0, y \geq 1, X \geq y, Z \geq 2y$  and  $y \notin \mathbb{Z}$ , we have

$$\begin{aligned} \Delta_a(y, \chi) &= \Delta_a(y, X, \chi) + R_a(y, X, Z) \\ &\quad + O(k^{1+\varepsilon} y^{-1/4+a/2} + k^{3/4+a/2} y^{-1/2} + k^{5/4+a/2} y^{-3/4+a/2}), \end{aligned}$$

where

$$\begin{aligned} \Delta_a(y, X, \chi) &= \frac{W(\chi)}{\sqrt{2}\pi} k^{a/2+1/4} y^{a/2+1/4} \\ &\quad \times \int_1^2 \sum_{n \leq uX} \frac{\tilde{r}_a(n, \chi)}{n^{3/4+a/2}} \cos(4\pi \sqrt{ny/k} - (1/4 + \delta_\chi/2)\pi) du \end{aligned}$$

and

$$\begin{aligned} R_a(y, X, Z) &= cW(\chi) \sum_{n \leq Z} r_a(n, \chi) \int_1^2 \int_{uX}^\infty t^{-1} \sin\left(\frac{4\pi}{\sqrt{k}}(\sqrt{y} - \sqrt{n})\sqrt{t}\right) dt du \end{aligned}$$

with some constant  $c$  which is independent on  $k$ .

From this Lemma, we get Theorems 1 and 2. Note that if  $\chi_1$  and  $\chi_2$  have the opposite parity, there occurs no main term.

### Outline of the proof of Theorem 3

In this case the Voronoï formula does not work well. We use the Chowla-Walum's type formula instead. Namely we have

**Lemma 2.**

$$\Delta_a(x, \chi) = - \sum_{m \leq \sqrt{x}} \chi(m) m^a \psi\left(\frac{x}{m}\right) - x^a \sum_{n \leq \sqrt{x}} n^{-a} P\left(\frac{x}{n}\right) + O(k^{3/2} (\log k) x^{a/2})$$

where

$$\begin{aligned} \psi(x) &= x - [x] - 1/2 \\ P(x) &= - \sum_{n \leq x} \chi(n) - \frac{1}{k} \sum_{n \leq k} \chi(n)n. \end{aligned}$$

The main term comes from the product of the first sums of  $\Delta_a(x, \chi_1)$  and  $\Delta_b(x, \chi_2)$ . The remaining products give the error term. To show this, we need Yanagisawa's main lemma.

**Lemma 3.** *Let  $f(t)$  and  $g(t)$  be piecewise continuous functions of period  $A$  and of bounded variations. Suppose that*

$$|f(t)| \leq F, \quad |g(t)| \leq G,$$

and

$$\int_0^A f(t) dt = 0.$$

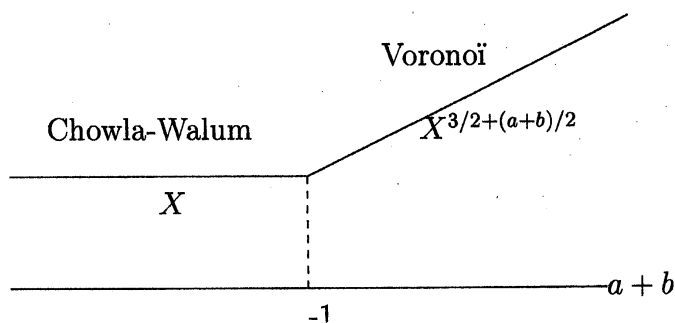
Then, for any  $n \leq \sqrt{X}$  and any sequence of points  $\{X_m\}$ , and  $\{Y_m\}$  with  $0 < Y_m - X_m \ll X$ , we have

$$\sum_{m \leq \sqrt{X}} \left| \int_{X_m}^{Y_m} f\left(\frac{x}{m}\right) g\left(\frac{x}{n}\right) dx \right| \ll GX \log X (F(A + \sigma_{-1}(n)) + V_f)$$

where  $V_f$  is the total variation of  $f$  in  $[0, A]$ .

The above summation is estimated as  $\sigma_{-1}(n)X \log X$  in [3], but it is not sufficient for our purpose. So we made the dependence on the modulus  $k$  explicit in Lemma 3. Note that  $\chi_1$  and  $\chi_2$  do not need to be primitive characters, because we don't use functional equation in this case.

As in the theorems of Meurman and Chowla-Yanagisawa, the mean square formula is effectively deduced by Voronoï formula when  $a + b \geq -1$  and by Chowla-Walum formula when  $a + b < -1$ . This phenomenon can be expressed symbolically as



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## References

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