

L^∞ -decay property for parabolic-elliptic Keller-Segel systems with porous-medium diffusion

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Abstract. This paper deals with the Keller-Segel system $(KS)_0$ of parabolic-elliptic type with porous-medium diffusion. In this type Sugiyama-Kunii [16] established the L^r -decay property ($1 \leq r < \infty$) of solutions to $(KS)_0$ with small initial data when $q \geq m + \frac{2}{N}$ (m denotes the intensity of diffusion and q denotes the nonlinearity). However, the L^∞ -decay property was not obtained yet. Therefore this paper gives the L^∞ -decay property of solutions to $(KS)_0$ with small initial data when $q > m + \frac{2}{N}$.

1. Introduction and results

In this paper we consider the following quasilinear degenerate Keller-Segel system of parabolic-elliptic type:

$$(KS)_0 \quad \begin{cases} \frac{\partial u}{\partial t} = \nabla \cdot (\nabla u^m - u^{q-1} \nabla v) & \text{in } \mathbb{R}^N \times (0, \infty), \\ 0 = \Delta v - v + u & \text{in } \mathbb{R}^N \times (0, \infty), \\ u(x, 0) = u_0(x), & x \in \mathbb{R}^N, \end{cases}$$

where $N \in \mathbb{N}$, $m \geq 1$, $q \geq 2$. The initial data satisfies

$$(1.1) \quad u_0 \geq 0, \quad u_0 \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N).$$

The minimal Keller-Segel system of parabolic-parabolic type, i.e., $(KS)_0$ with $m = 1$, $q = 2$ and the second equation replaced with

$$\frac{\partial v}{\partial t} = \Delta v - v + u,$$

was proposed by Keller-Segel [6], and power type was studied by Sugiyama-Kunii [16] (see also Sugiyama [13] and Ishida-Yokota [2], [3]). On the other hand, the system $(KS)_0$ of parabolic-elliptic type was considered by [16]. In particular, $(KS)_0$ with $m = 1$ and $q = 2$ is called the Nagai model, and investigated until now (see e.g., Nagai-Senba-Yoshida [11], Nagai [10], Sugiyama [12], [14], [15] and Kozono-Sugiyama [7]; see also T. Suzuki [18]). These models describe a part of cellular slime molds with the chemotaxis at the life cycle. Usually $u(x, t)$ shows the density of cellular slime molds and $v(x, t)$ shows the density of the semiochemical at place x and time t .

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The purpose of this paper is to give the L^∞ -decay property of solutions to $(KS)_0$ with small initial data when $q \geq m + \frac{2}{N}$. Substituting the second equation $\Delta v = v - u$ into the first equation in $(KS)_0$ implies

$$(E1) \quad \begin{aligned} \frac{\partial u}{\partial t} &= \Delta u^m - \nabla u^{q-1} \cdot \nabla v - u^{q-1} \Delta v \\ &= \Delta u^m - \nabla u^{q-1} \cdot \nabla v + u^q - u^{q-1} v. \end{aligned}$$

This is analogous to the following nonlinear degenerate heat equation:

$$(NLD) \quad \frac{\partial z}{\partial t} = \Delta z^m + z^q \quad \text{in } \mathbb{R}^N \times (0, \infty).$$

The studies for (NLD) and $(KS)_0$ in Table 1.1 are currently known.

		(NLD) ($q > m + \frac{2}{N}$)	$(KS)_0$ ($q \geq m + \frac{2}{N}$)
Decay property of $z(t)$ or $u(t)$	L^r	–	$(t+1)^{-\frac{N}{N(m-1)+2} \frac{r-1}{r}}$ (Sugiyama-Kunii [16])
	L^∞	$t^{-\frac{1}{q-1}}$ (Kawanago [5])	Unsolved (A)
Behavior of $z(t)$ or $u(t)$ as $t \rightarrow \infty$	L^r	–	Barenblatt sol. ($m > 1$) (Luckhaus-Sugiyama [8]) Heat kernel ($m = 1$) (Luckhaus-Sugiyama [9])
	L^∞	Barenblatt sol. ($m > 1$) (Kawanago [5]) Heat kernel ($m = 1$) (Kawanago [5])	Unsolved (B)

Table 1.1. The known results for (NLD) and $(KS)_0$ with small initial data.

Therefore our aim is to give an answer to the unsolved part (A) in Table 1.1.

Before stating our result we define global weak solutions to $(KS)_0$.

Definition 1.1. Let $T > 0$. A pair (u, v) of non-negative functions defined on $\mathbb{R}^N \times (0, T)$ is called a *weak solution* to $(KS)_0$ on $[0, T)$ if

- (a) $u \in L^\infty(0, T; L^p(\mathbb{R}^N))$ ($\forall p \in [1, \infty]$), $u^m \in L^2(0, T; H^1(\mathbb{R}^N))$,
- (b) $v \in L^\infty(0, T; H^1(\mathbb{R}^N))$,
- (c) (u, v) satisfies $(KS)_0$ in the distributional sense, i.e., for every $\varphi \in C_0^\infty(\mathbb{R}^N \times [0, T))$,

$$\begin{aligned} \int_0^T \int_{\mathbb{R}^N} (\nabla u^m \cdot \nabla \varphi - u^{q-1} \nabla v \cdot \nabla \varphi - u \varphi_t) dx dt &= \int_{\mathbb{R}^N} u_0(x) \varphi(x, 0) dx, \\ \int_0^T \int_{\mathbb{R}^N} (\nabla v \cdot \nabla \varphi + v \varphi - u \varphi) dx dt &= 0. \end{aligned}$$

In particular, if $T > 0$ can be taken arbitrarily, then (u, v) is called a *global weak solution* to $(KS)_0$.

We now state our main result in this paper.

Theorem 1.1. *Let $N \in \mathbb{N}$, $m \geq 1$, $q \geq 2$. Let m and q satisfy*

$$q > m + \frac{2}{N}.$$

Assume further that u_0 satisfy (1.1) and

$$(1.2) \quad \begin{cases} \|u_0\|_{L^{\frac{N}{2}(q-m)}} \leq \min\{\delta_{u, \frac{N}{2}(q-m)}, \delta_{u, r_3}, \delta_{u, r_0}\} & \text{when } q \geq m+1 (N \geq 3), N = 1, 2 \\ \|u_0\|_{L^{\frac{N}{2}}} \leq \min\{\delta_{u, \frac{N}{2}}, \delta_{u, r_3}, \delta_{u, r_0}\} & \text{when } q < m+1 (N \geq 3), \end{cases}$$

where

$$\delta_{u, r} = \min\left\{1, \frac{4m}{2^{q-2}rC'}, \left(\frac{4m(r+q-2)}{2^{q-2}(r+m-1)^2C''}\right)^{\frac{1}{q-m}}\right\},$$

$C' = C'(r, m, q, N)$, $C'' = C''(r, m, q, N)$, $r_3 = r_3(m, q, N)$ (defined in subsection 3.2) and $r_0 = \max\{N - m + 1, m - 3, N(q - m) - m + 1\}$ are positive constants. Then $(KS)_0$ has a non-negative weak solution (u, v) on $[0, \infty)$ which satisfies the following decay property:

$$(1.3) \quad \|u(t)\|_{L^\infty(\mathbb{R}^N)} \leq Kt^{-\frac{1}{q-1}} = Kt^{-\frac{N}{N(m-1)+2q_*}}, \quad \text{a.a. } t \in (0, \infty),$$

$$(1.4) \quad \|u(t)\|_{L^\infty(\mathbb{R}^N)} \leq K_\rho(t + \rho)^{-\frac{N}{N(m-1)+2}}, \quad \text{a.a. } t \in [5\rho, \infty),$$

where $q_* := \frac{N}{2}(q - m)$, $K = K(\|u_0\|_{L^{q_*}}, C_{r_3}, r_3, m, q, N) > 0$ is a constant, $\rho \in (0, 1]$ is arbitrary and $K_\rho = K_\rho(\rho, C_{r_3}, r_3, \|u_0\|_{L^1}, \|u_0\|_{L^{q_*}}, \|u_0\|_{L^{r_3}}, m, q, N)$ ($\rightarrow \infty$ as $\rho \rightarrow 0$) is a positive constant, where C_r is the constant given in Proposition 2.1.

The decay rate in Theorem 1.1 may be best possible, because of the following two reasons.

First Reason: As stated above, $(KS)_0$ can be rewritten as the equation (E1) like (NLD). From comparing the diffusion term Δu^m with the aggregation term u^q in (E1), $(KS)_0$ has the global solvability and the solution has L^r -decay property when $q \geq m + \frac{2}{N}$ and the initial data is sufficiently small ([16]). Kawanago [5] showed the L^∞ -decay property for (NLD) when $q > m + \frac{2}{N}$, that is, if the initial data is sufficiently small, then (NLD) has a global solution which satisfies

$$\|z(t)\|_{L^\infty(\mathbb{R}^N)} \leq M_0 t^{-\frac{1}{q-1}} = M_0 t^{-\frac{N}{N(m-1)+2q_*}},$$

where $q_* = \frac{N}{2}(q - m)$ and $M_0 > 0$ is some constant. Hence we expect that the solution to $(KS)_0$ has

$$\|u(t)\|_{L^\infty(\mathbb{R}^N)} \leq M_1 t^{-\frac{1}{q-1}} = M_1 t^{-\frac{N}{N(m-1)+2q_*}},$$

where M_1 is some constant.

Second Reason: Sugiyama-Kunii [16] showed the L^r -decay property of solutions to $(KS)_0$:

$$(1.5) \quad \|u(t)\|_{L^r} \leq C_r(1+t)^{-\alpha}, \quad r \in [1, \infty),$$

where

$$\alpha = \frac{N}{N(m-1)+2} \cdot \frac{r-1}{r}.$$

Giving an eye to the decay rate α , we have

$$\frac{N}{N(m-1)+2} \cdot \frac{r-1}{r} \rightarrow \frac{N}{N(m-1)+2} \quad (r \rightarrow \infty).$$

Hence we expect that the solution to $(\text{KS})_0$ has

$$\|u(t)\|_{L^\infty(\mathbb{R}^N)} \leq M_2 t^{-\frac{N}{N(m-1)+2}},$$

where M_2 is some constant.

One of the difficulties in showing the L^∞ -decay estimates is that the coefficient $C_r \rightarrow \infty$ as $r \rightarrow \infty$ in (1.5) (see the definition of C_r in Proposition 2.1 below), and hence the L^∞ -decay property is not obtained by the limiting process in (1.5). To evade this problem and obtain the L^∞ -decay property we establish the following two kinds of L^∞ - L^r estimates without assuming that the initial data is small (see Section 3):

$$(I) \quad \|u(t)\|_{L^\infty(\mathbb{R}^N)}^{r-(q_*+q-1)} \leq C(r) \left(\left(\frac{t}{2}\right) \|u(\frac{t}{2})\|_{L^r(\mathbb{R}^N)}^r + \left(\frac{t}{2}\right)^{1-\frac{r-q_*}{q-1}} \right),$$

$$(II) \quad \|u(t)\|_{L^\infty(\mathbb{R}^N)}^r \leq \tilde{C}(r) (t+\rho)^{-\frac{N}{N(m-1)+2}} \left(\|u(\frac{t}{2} - \frac{\rho}{2})\|_{L^r(\mathbb{R}^N)}^r + \|u_0\|_{L^1} (t+\rho)^{-\frac{N(r-1)}{N(m-1)+2}} \right),$$

where $q_* = \frac{N}{2}(q-m)$, $C(r)$, and $\tilde{C}(r)$ are positive constants. We can obtain the L^∞ -decay properties (1.3) and (1.4) by combining the L^r -decay estimate with (I) and (II), respectively. The condition $q > m + \frac{2}{N}$ is necessary to show that the coefficient $\tilde{C}(r)$ is bounded as $r \rightarrow \infty$. The proofs of (I) and (II) are based on R. Suzuki [17] in which he studied the following equation:

$$(E2) \quad \frac{\partial z}{\partial t} = \Delta z^m + a \cdot \nabla z^p + z^q \quad \text{in } \mathbb{R}^N \times (0, \infty),$$

where $m \geq 1$, $p, q > 1$, $a \in \mathbb{R}^N$, $a \neq 0$. He proved that the solution to (E2) has the following decay property when $q > m + \frac{2}{N}$: if the initial data is sufficiently small, then

$$\|z(t)\|_{L^\infty(\mathbb{R}^N)} \leq M_3 \min\{t^{-\frac{N}{N(m-1)+2q_*}}, t^{-\frac{N}{N(m-1)+2}}\}, \quad \text{a.a. } t > 0,$$

where $q_* = \frac{N}{2}(q-m)$, $M_3 > 0$ is some constant. Also from this, we can expect that the solution to $(\text{KS})_0$ has the L^∞ -decay properties (1.3) and (1.4). Moreover, he showed in [17] that the solution to (E2) behaves like the Barenblatt solution ($m > 1$) or the Heat kernel ($m = 1$) when $q > m + \frac{2}{N}$ and $p > m + \frac{1}{N}$.

Finally, we glance at the unsolved part (B) in Table 1.1. From the known results for the behavior of solutions ([5], [8], [9] and [17]), we conjecture that the solution to $(\text{KS})_0$ has a similar behavior in the case where $q > m + \frac{2}{N}$ and the initial data is small. This conjecture will be discussed in our forthcoming paper.

This paper is organized as follows. In Section 2 we recall the L^r -decay of solutions to $(\text{KS})_0$. First we deal with the case where $N \geq 2$ in Section 3, because the approximation is different between more than one dimension and $1D$. Section 3 consists of two subsections. Section 3.1 gives the L^∞ -bound of solutions to $(\text{KS})_0$. Section 3.2 is the main part of this paper, where the L^∞ -decay of solutions to $(\text{KS})_0$ is obtained. Finally we consider the case where $N = 1$ in Section 4.

2. L^r -decay property

First we state the result on the global existence and L^r -decay property of solutions to $(KS)_0$. This proposition is stated in [16, Theorem 3].

Proposition 2.1 (global existence of weak solutions to $(KS)_0$). *Let $N \in \mathbb{N}$, $m \geq 1$, $q \geq 2$. Suppose that m and q satisfy the super-critical condition, i.e.,*

$$q \geq m + \frac{2}{N}$$

Let the initial data satisfy (1.1) and the smallness condition (1.2) in Theorem 1.1. Then $(KS)_0$ has a non-negative global weak solution (u, v) which has the mass conservation law:

$$(2.1) \quad \|u(t)\|_{L^1(\mathbb{R}^N)} = \|u_0\|_{L^1(\mathbb{R}^N)}, \quad t \geq 0.$$

Moreover, $t \mapsto \|u(t)\|_{L^r(\mathbb{R}^N)}$ ($1 \leq r < \infty$) is a non-increasing function with the following decay property:

$$(2.2) \quad \|u(t)\|_{L^r(\mathbb{R}^N)} \leq C_r(1+t)^{-\alpha}, \quad r \in [1, \infty), \quad t \geq 0,$$

where

$$(2.3) \quad \alpha = \frac{N}{N(m-1)+2} \cdot \frac{r-1}{r},$$

$$(2.4) \quad C_r = \max \left\{ \frac{(r+m-1)^2}{r} \cdot \frac{1}{2m(m-1+\frac{2}{N})} (c(N)\|u_0\|_{L^1})^{\frac{N}{N(m-1)+2} \cdot \frac{r-1}{r}}, \|u_0\|_{L^r} \right\}.$$

Remark 2.1. The non-negativity of the solutions is obtained from the standard argument and the comparison principle (see [16]).

Remark 2.2. In [16], they assume the smallness only $\|u_0\|_{L^{\frac{N}{2}(q-m)}} (N \geq 1)$. However from the approximation to the nonlinear term in the first equation in $(KS)_0$, when $m + \frac{2}{N} \leq q < m+1$ ($N \geq 3$), we should assume the smallness of $\|u_0\|_{L^{\frac{N}{2}}}$ (see [4]).

Remark 2.3. In [16], it seems difficult to prove the L^∞ -bound of the approximate solution without assuming that $u_0 = 0$. Indeed, they assume the smallness $\|u_0\|_{L^{\frac{N(q-m)}{2}}} \leq \delta_{u,r} = C_0 r^{-\frac{1}{q-m}}$ to obtain the L^r -estimate. If $r \rightarrow \infty$ in this assumption, then it should be $\|u_0\|_{L^{\frac{N(q-m)}{2}}} = 0$. To overcome the difficulty we give a proof by using Moser's iteration technique (cf. R. Suzuki [17, Section 3.1]).

3. The case where $N \geq 2$

In this section we establish two kinds of “ L^∞ - L^r estimates” of solutions to $(KS)_0$. The first one is for the L^∞ -bound (Proposition 3.1) and the second is for L^∞ -decay property (Proposition 3.5). In the end of this section we prove Theorem 1.1 ($N \geq 2$). Now we introduce the approximate problem:

$$(KS)_\varepsilon \quad \begin{cases} \frac{\partial u_\varepsilon}{\partial t} = \nabla \cdot (\nabla(u_\varepsilon + \varepsilon)^m - (u_\varepsilon + \varepsilon^{\frac{m}{q-2}})^{q-2} u_\varepsilon \nabla v_\varepsilon) & \text{in } \mathbb{R}^N \times (0, T), \quad \cdots (1)_\varepsilon \\ 0 = \Delta v_\varepsilon - v_\varepsilon + u_\varepsilon & \text{in } \mathbb{R}^N \times (0, T), \quad \cdots (2)_\varepsilon \\ u_\varepsilon(x, 0) = u_{0\varepsilon}(x), \quad v_\varepsilon(x, 0) = v_{0\varepsilon}(x), & x \in \mathbb{R}^N, \end{cases}$$

where $N \geq 2$, $m \geq 1$, $q \geq 2$ and $\varepsilon \in (0, 1)$. The initial data $u_{0\varepsilon} \in C_0^\infty(\mathbb{R}^N)$ is given as $u_{0\varepsilon} := (\rho_\varepsilon * u_0) \zeta_\varepsilon$, where ρ_ε is a mollifier such that

$$0 \leq \rho_\varepsilon \in C_0^\infty(\mathbb{R}^N), \quad \text{supp } \rho_\varepsilon \subset \overline{B(0, \varepsilon)}, \quad \int_{\mathbb{R}^N} \rho_\varepsilon(x) dx = 1,$$

and ζ_ε is a cut-off function, i.e., $\zeta_\varepsilon(x) := \zeta(\varepsilon x)$, where ζ is a fixed function in $C_0^\infty(\mathbb{R}^N)$ such that

$$0 \leq \zeta \leq 1, \quad \zeta(x) = \begin{cases} 1 & (|x| \leq 1), \\ 0 & (|x| \geq 2). \end{cases}$$

Remark 3.1. Let $T > 0$. Let u_ε be a solution to $(\text{KS})_\varepsilon$ on $[0, T]$. Then the following continuity holds:

$$(3.1) \quad \|u_\varepsilon(t)\|_{L^r(\mathbb{R}^N)} \in C([0, T]) \quad (\forall r \in [1, \infty)).$$

Indeed, reading the standard argument to construct the local (approximate) solution again (see [16, Proposition 8, Lemmas 11 and 12], Amann [1, Theorem IV.1.5.1]), we see that $u_\varepsilon \in C([0, T]; L^\alpha(\mathbb{R}^N))$ for every $\alpha \in (N, \infty)$. This fact together with the mass conservation law (2.1) implies the continuity (3.1). This continuity will be used in Lemma 3.3.

Remark 3.2. If u_0 satisfies the smallness condition as in Theorem 1.1. then the approximate solution u_ε has the same L^r -decay as (2.2) and $t \mapsto \|u_\varepsilon(t)\|_{L^r(\mathbb{R}^N)}$ is a non-increase function.

3.1. L^∞ -bounds

The next proposition shows the L^∞ -bound of the solution u to $(\text{KS})_0$. Indeed, (3.3) (in Proposition 3.1) implies that $\|u(t)\|_{L^\infty(\mathbb{R}^N)} \leq K_0$ a.a. $t \in (\rho, T)$ for every $\rho > 0$.

Proposition 3.1 (L^∞ -estimate of solutions to $(\text{KS})_0$). *Let $N \geq 2$, $m \geq 1$, $q \geq 2$, $\varepsilon \in (0, 1)$ and $T > 0$. Let (u, v) be a weak solution to $(\text{KS})_0$ on $[0, T]$. Assume that m and q satisfy*

$$(3.2) \quad q \geq m + \frac{2}{N}$$

and u_0 satisfies (1.1) and the smallness condition (1.2) in Theorem 1.1. Then the following estimate holds:

$$(3.3) \quad \|u(t)\|_{L^\infty(\mathbb{R}^N)} \leq K_1 t^{-\frac{N}{N(m-1)+2q_*}}, \quad \text{a.a. } t \in (0, T),$$

where $q_ = \frac{N}{2}(q - m)$, $K_1 = K_1(\|u_0\|_{L^{q_*}}, C_{r_1}, m, q, N) > 0$ and C_{r_1} is the same constant as in Proposition 2.1.*

The proof of this proposition employs the similar method to R. Suzuki [17, Section 4]. For this purpose we prepare three lemmas.

Lemma 3.2. Let $N \geq 2$, $m \geq 1$, $q \geq 2$, $\varepsilon \in (0, 1)$, $T > 0$ and $0 \leq t_1 < t_2 \leq T$. Let $(u_\varepsilon, v_\varepsilon)$ be a unique solution to $(KS)_\varepsilon$ on $[0, T]$. Let $\psi(t) \in C^1([t_1, t_2])$ with $0 \leq \psi \leq 1$, $\psi(t_1) = 0$, $\psi(t_2) = 1$. Assume that m and q satisfy (3.2). Then for $r > q$,

$$\begin{aligned}
 (3.4) \quad & \|u_\varepsilon(t_2)\|_{L^{r-q+1}(\mathbb{R}^N)}^{r-q+1} + \frac{4m(r-q+1)(r-q)}{(r-q+m)^2} \int_{t_1}^{t_2} \psi(t) \|\nabla u_\varepsilon^{\frac{r-q+m}{2}}(t)\|_{L^2(\mathbb{R}^N)}^2 dt \\
 & + \varepsilon^{m-1} \frac{4m(r-q)}{r-q+1} \int_{t_1}^{t_2} \psi(t) \|\nabla u_\varepsilon^{\frac{r-q+1}{2}}(t)\|_{L^2(\mathbb{R}^N)}^2 dt \\
 & \leq \int_{t_1}^{t_2} \psi'(t) \|u_\varepsilon(t)\|_{L^{r-q+1}(\mathbb{R}^N)}^{r-q+1} dt \\
 & + 2^{q-2}(r-q) \left(\int_{t_1}^{t_2} \|u_\varepsilon(t)\|_{L^r(\mathbb{R}^N)}^r dt + \varepsilon^m \int_{t_1}^{t_2} \|u_\varepsilon(t)\|_{L^{r-q+2}(\mathbb{R}^N)}^{r-q+2} dt \right).
 \end{aligned}$$

Proof. Let $r > 2$. Multiplying the first approximate equation $(1)_\varepsilon$ by u_ε^{r-1} and integrating it over $\mathbb{R}^N$, we obtain

$$\begin{aligned}
 (3.5) \quad & \frac{1}{r} \frac{d}{dt} \|u_\varepsilon(t)\|_{L^r(\mathbb{R}^N)}^r \\
 & \leq -\frac{4m(r-1)}{(r+m-1)^2} \|\nabla u_\varepsilon^{\frac{r+m-1}{2}}(t)\|_{L^2(\mathbb{R}^N)}^2 - \frac{4m(r-1)\varepsilon^{m-1}}{r^2} \|\nabla u_\varepsilon^{\frac{r}{2}}(t)\|_{L^2(\mathbb{R}^N)}^2 \\
 & \quad + \int_{\mathbb{R}^N} (u_\varepsilon + \varepsilon^{\frac{m}{q-2}})^{q-2} u_\varepsilon \nabla v_\varepsilon \cdot \nabla u_\varepsilon^{r-1} dx.
 \end{aligned}$$

Multiplying (3.5) by $\psi(t)$ and integrating it by parts over (t_1, t_2) , we see that

$$\begin{aligned}
 (3.6) \quad & \|u_\varepsilon(t_2)\|_{L^r(\mathbb{R}^N)}^r - \int_{t_1}^{t_2} \psi'(t) \|u_\varepsilon(t)\|_{L^r(\mathbb{R}^N)}^r dt \\
 & \leq -\frac{4mr(r-1)}{(r+m-1)^2} \int_{t_1}^{t_2} \psi(t) \|\nabla u_\varepsilon^{\frac{r+m-1}{2}}(t)\|_{L^2(\mathbb{R}^N)}^2 dt \\
 & \quad - \frac{4m(r-1)\varepsilon^{m-1}}{r} \int_{t_1}^{t_2} \psi(t) \|\nabla u_\varepsilon^{\frac{r}{2}}(t)\|_{L^2(\mathbb{R}^N)}^2 dt + r \int_{t_1}^{t_2} \psi(t) \mathbf{I}_2 dt.
 \end{aligned}$$

We denoted by $\mathbf{I}_2$ the third term on the right-hand side of (3.5). We make an estimation of $\mathbf{I}_2$. Letting

$$F(s) := \int_0^s (\tau + \varepsilon^{\frac{m}{q-2}})^{q-2} \tau^{r-1} d\tau, \quad \tau \geq 0, \quad s \geq 0, \quad \varepsilon \in (0, 1)$$

and noting that

$$0 \leq F(s) \leq 2^{q-2} \left[\frac{s^{r+q-2}}{r+q-2} + \frac{\varepsilon^m s^r}{r} \right],$$

we find by $(2)_\varepsilon$ that

$$\begin{aligned}
 (3.7) \quad I_2 &= -(r-1) \int_{\mathbb{R}^N} F(u_\varepsilon) \Delta v_\varepsilon dx \\
 &= -(r-1) \int_{\mathbb{R}^N} (v_\varepsilon - u_\varepsilon) F(u_\varepsilon) dx \\
 &\leq (r-1) \int_{\mathbb{R}^N} u_\varepsilon F(u_\varepsilon) dx \\
 &\leq \frac{2^{q-2}(r-1)}{r+q-2} \int_{\mathbb{R}^N} u_\varepsilon^{r+q-1} dx + \frac{2^{q-2}\varepsilon^m(r-1)}{r} \int_{\mathbb{R}^N} u_\varepsilon^{r+1} dx.
 \end{aligned}$$

Hence it follows from (3.6), (3.7) and $0 \leq \psi \leq 1$ that

$$\begin{aligned}
 (3.8) \quad &\|u_\varepsilon(t_2)\|_{L^r(\mathbb{R}^N)}^r + \frac{4mr(r-1)}{(r+m-1)^2} \int_{t_1}^{t_2} \psi(t) \|\nabla u_\varepsilon^{\frac{r+m-1}{2}}(t)\|_{L^2(\mathbb{R}^N)}^2 dt \\
 &\quad + \frac{\varepsilon^{m-1}4m(r-1)}{r} \int_{t_1}^{t_2} \psi(t) \|\nabla u_\varepsilon^{\frac{r}{2}}(t)\|_{L^2(\mathbb{R}^N)}^2 dt \\
 &\leq \int_{t_1}^{t_2} \psi'(t) \|u_\varepsilon(t)\|_{L^r(\mathbb{R}^N)}^r dt \\
 &\quad + 2^{q-2}(r-1) \int_{t_1}^{t_2} \left(\frac{r}{r+q-2} \|u_\varepsilon(t)\|_{L^{r+q-1}(\mathbb{R}^N)}^{r+q-1} + \varepsilon^m \|u_\varepsilon(t)\|_{L^{r+1}(\mathbb{R}^N)}^{r+1} \right) dt.
 \end{aligned}$$

Replacing r with $r-q+1$ in (3.8), we obtain (3.4) for $r > q$. $\square$

Lemma 3.3. *Let $N \geq 2$, $m \geq 1$, $q \geq 2$, $\varepsilon \in (0, 1)$ and $T > 0$. Let $(u_\varepsilon, v_\varepsilon)$ be a unique solution to $(KS)_\varepsilon$ on $[0, T]$. Put $I = [\tau, \tau + s]$ and $I' = [\tau - \sigma, \tau + s]$ with $0 < \sigma < \tau < \tau + s < T$. Put $q_* := \frac{N}{2}(q-m)$, $h := \sup_{t \in [0, T]} \|u_\varepsilon(t)\|_{q_*}^{q_*}$ and $r_* > q_* + q - 1$ is some constant. Assume further that m and q satisfy (3.2) and $\delta > 0$ satisfies*

$$(3.9) \quad \sigma \delta^{q-1} \leq 1.$$

Then for $r \geq r_*$,

$$\begin{aligned}
 (3.10) \quad &\mu_0(Y_{I, k(r-q+1)+m-1} + Z_{I, k(r-q+1)})^{\frac{1}{k}} \\
 &\leq \left(\frac{4}{\sigma} \delta^{-q+1} + 2^{q-1}(r-q) \right) Y_{I', r} + 2^{q-1} \varepsilon^m (r-q) Z_{I', r-q+2},
 \end{aligned}$$

where $k := 1 + \frac{2}{N}$, $\mu_0 = \mu_0(h, m, q, N)$ and

$$Y_{I, r} := \int_I \int_{\mathbb{R}^N} u_\varepsilon^r dx dt + \frac{(s+\sigma)h}{\delta^{q_*}} \delta^r, \quad Z_{I, r} := \int_I \int_{\mathbb{R}^N} u_\varepsilon^r dx dt.$$

Proof. Let $r > q$. From (3.1) we can take $\tilde{t} \in I$ such that

$$\max_{t \in I} \int_{\mathbb{R}^N} u_\varepsilon^{r-q+1}(t) dx = \int_{\mathbb{R}^N} u_\varepsilon^{r-q+1}(\tilde{t}) dx.$$

Let

$$\tilde{\psi}(t) := \frac{t - \tau + \sigma}{\tilde{t} - \tau + \sigma}, \quad \tilde{t}_1 := \tau - \sigma, \quad \tilde{t}_2 := \tilde{t}$$

and note that $0 \leq \tilde{\psi} \leq 1$, $\tilde{\psi}(\tilde{t}_1) = 0$, $\tilde{\psi}(\tilde{t}_2) = 1$, $0 \leq \tilde{\psi}'(t) = \frac{1}{\tilde{t} - \tau + \sigma} \leq \frac{1}{\sigma}$ and $[\tilde{t}_1, \tilde{t}_2] \subset I'$. Then we can substitute $\tilde{\psi}$, $\tilde{t}_1$ and $\tilde{t}_2$ into ψ , t_1 and t_2 in (3.4) and thus, we have

(3.11)

$$\begin{aligned} & \max_{t \in I} \int_{\mathbb{R}^N} u_\varepsilon^{r-q+1}(t) dx \\ & \leq \frac{1}{\sigma} \int_{I'} \int_{\mathbb{R}^N} u_\varepsilon^{r-q+1} dx dt + 2^{q-2}(r-q) \int_{I'} \left(\|u_\varepsilon(t)\|_{L^r(\mathbb{R}^N)}^r + \varepsilon^m \|u_\varepsilon(t)\|_{L^{r-q+2}(\mathbb{R}^N)}^{r-q+2} \right) dt. \end{aligned}$$

Next letting

$$\hat{\psi}(t) := \begin{cases} 1, & t \in [\tau, \tau + s], \\ -\sigma^{-2}(t - \tau)^2 + 1, & t \in [\tau - \sigma, \tau], \end{cases} \quad \hat{t}_1 := \tau - \sigma, \quad \hat{t}_2 := \tau + s$$

and noting that $0 \leq \hat{\psi} \leq 1$, $\hat{\psi}(\hat{t}_1) = 0$, $\hat{\psi}(\hat{t}_2) = 1$, $0 \leq \hat{\psi}'(t) \leq \frac{2}{\sigma}$ and $I \subset [\hat{t}_1, \hat{t}_2] \subset I'$, we can substitute $\hat{\psi}$, $\hat{t}_1$ and $\hat{t}_2$ into ψ , t_1 and t_2 in (3.4). Hence we see that

(3.12)

$$\begin{aligned} & \nu_0 \int_I \|\nabla u_\varepsilon^{\frac{r+m-q}{2}}(t)\|_{L^2(\mathbb{R}^N)}^2 dt + \varepsilon^{m-1} \nu_1 \int_I \|\nabla u_\varepsilon^{\frac{r-q+1}{2}}(t)\|_{L^2(\mathbb{R}^N)}^2 dt \\ & \leq \frac{2}{\sigma} \int_{I'} \int_{\mathbb{R}^N} u_\varepsilon^{r-q+1} dx dt + 2^{q-2}(r-q) \int_{I'} \left(\|u_\varepsilon(t)\|_{L^r(\mathbb{R}^N)}^r + \varepsilon^m \|u_\varepsilon(t)\|_{L^{r-q+2}(\mathbb{R}^N)}^{r-q+2} \right) dt, \end{aligned}$$

where $\nu_0 := \min\{1, \inf_{r \geq r_*} \frac{4m(r-q+1)(r-q)}{(r+m-q)^2}\}$, $\nu_1 := \min\{1, \inf_{r \geq r_*} \frac{4m(r-q)}{r-q+1}\}$ and $r_* > \frac{N}{2}(q-m) + q - 1$ is some constant. Combining (3.11) with (3.12), we have

$$\begin{aligned} (3.13) \quad & \max_{t \in I} \int_{\mathbb{R}^N} u_\varepsilon^{r-q+1}(t) dx \\ & + \nu_0 \int_I \|\nabla u_\varepsilon^{\frac{r+m-q}{2}}(t)\|_{L^2(\mathbb{R}^N)}^2 dt + \varepsilon^{m-1} \nu_1 \int_I \|\nabla u_\varepsilon^{\frac{r-q+1}{2}}(t)\|_{L^2(\mathbb{R}^N)}^2 dt \\ & \leq \frac{3}{\sigma} \int_{I'} \int_{\mathbb{R}^N} u_\varepsilon^{r-q+1} dx dt + 2^{q-1}(r-q) \left(\int_{I'} \|u_\varepsilon(t)\|_{L^r}^r dt + \varepsilon^m \int_{I'} \|u_\varepsilon(t)\|_{L^{r-q+2}}^{r-q+2} dt \right). \end{aligned}$$

We estimate the first term on the right-hand side of (3.13). Set

$$\mathbf{E}_\delta(t) := \{x \in \mathbb{R}^N; u_\varepsilon(x, t) \geq \delta\}, \quad q_* := \frac{N}{2}(q-m), \quad h := \sup_{t \in [0, T]} \|u_\varepsilon(t)\|_{L^{q_*}(\mathbb{R}^N)}^{q_*}.$$

Noting that $|I'| = s + \sigma$, we see that for $r \geq \max\{q, r_*\} = r_*(> q_* + q - 1)$,

$$\begin{aligned} (3.14) \quad & \int_{I'} \int_{\mathbb{R}^N} u_\varepsilon^{r-q+1} dx dt = \left(\int_{I'} \int_{\mathbf{E}_\delta(t)} + \int_{I'} \int_{\mathbb{R}^N \setminus \mathbf{E}_\delta(t)} \right) u_\varepsilon^{r-q+1} dx dt \\ & \leq \delta^{-q+1} \int_{I'} \int_{\mathbb{R}^N} u_\varepsilon^r dx dt + \delta^{r-q_*-q+1} h(s + \sigma). \end{aligned}$$

To estimate the left-hand side of (3.13), we use the Sobolev type inequality in [17, Lemma 2.9]:

$$(3.15) \quad \left[\int_I \int_{\mathbb{R}^N} |f|^{\tilde{\alpha}} dx dt \right]^{\frac{1}{k}} \leq C_0^{\frac{1}{k}} \left[\max_{t \in I} \int_{\mathbb{R}^N} |f|^\alpha dx + \int_I \int_{\mathbb{R}^N} |\nabla f|^2 dx dt \right],$$

where $\alpha \geq 0$, $\tilde{\alpha} = 2(\frac{\alpha}{N} + 1)$, $k = 1 + \frac{2}{N}$, $f \in C(I; L^\alpha(\mathbb{R}^N)) \cap L^2(I; H^1(\mathbb{R}^N))$ and C_0 is a positive constant depending only on N . Applying (3.15) with $f = u_\varepsilon^{\frac{r+m-q}{2}}$ and $\alpha = \frac{2(r-q+1)}{r-q+m}$ or $f = u_\varepsilon^{\frac{r-q+1}{2}}$ and $\alpha = 2$, we find that for $r \geq q - 1$,

$$(3.16) \quad \left\{ \frac{1}{C_0} \int_I \int_{\mathbb{R}^N} u_\varepsilon^{k(r-q+1)+m-1} dx dt \right\}^{\frac{1}{k}} \leq \max_{t \in I} \int_{\mathbb{R}^N} u_\varepsilon^{r-q+1}(t) dx + \int_I \|\nabla u_\varepsilon^{\frac{r+m-q}{2}}(t)\|_{L^2(\mathbb{R}^N)}^2 dt,$$

$$(3.17) \quad \left\{ \frac{1}{C_0} \int_I \int_{\mathbb{R}^N} u_\varepsilon^{k(r-q+1)} dx dt \right\}^{\frac{1}{k}} \leq \max_{t \in I} \int_{\mathbb{R}^N} u_\varepsilon^{r-q+1}(t) dx + \int_I \|\nabla u_\varepsilon^{\frac{r-q+1}{2}}(t)\|_{L^2(\mathbb{R}^N)}^2 dt.$$

Let $r \geq \max\{r_*, q - 1\} = r_*$. Plugging (3.16)–(3.17) into (3.14) to left- and right-hand sides of (3.13), respectively, we have

$$(3.18) \quad \begin{aligned} & \frac{\nu_0}{2C_0^{1/k}} \left\{ \int_I \int_{\mathbb{R}^N} u_\varepsilon^{k(r-q+1)+m-1} dx dt \right\}^{\frac{1}{k}} + \varepsilon^{m-1} \frac{\nu_1}{2C_0^{1/k}} \left\{ \int_I \int_{\mathbb{R}^N} u_\varepsilon^{k(r-q+1)} dx dt \right\}^{\frac{1}{k}} \\ & \leq \left[\frac{3}{\sigma} \delta^{-q+1} + 2^{q-1}(r-q) \right] \int_{I'} \int_{\mathbb{R}^N} u_\varepsilon^r dx dt + \frac{3(s+\sigma)}{\sigma} \delta^{r-q_*-q+1} h \\ & \quad + 2^{q-1} \varepsilon^m (r-q) \int_{I'} \int_{\mathbb{R}^N} u_\varepsilon^{r-q+2} dx dt. \end{aligned}$$

Adding $\frac{s+\sigma}{\sigma} \delta^{r-q_*-q+1} h$ to the both sides of (3.18), we obtain

$$(3.19) \quad \begin{aligned} & \frac{\nu_0}{2C_0^{1/k}} \left\{ \int_I \int_{\mathbb{R}^N} u_\varepsilon^{k(r-q+1)+m-1} dx dt \right\}^{\frac{1}{k}} + \frac{s+\sigma}{\sigma} \delta^{r-q_*-q+1} h \\ & \quad + \varepsilon^{m-1} \frac{\nu_1}{2C_0^{1/k}} \left\{ \int_I \int_{\mathbb{R}^N} u_\varepsilon^{k(r-q+1)} dx dt \right\}^{\frac{1}{k}} \\ & \leq \left[\frac{3}{\sigma} \delta^{-q+1} + 2^{q-1}(r-q) \right] \int_{I'} \int_{\mathbb{R}^N} u_\varepsilon^r dx dt + \frac{4}{\sigma} \delta^{-q+1} (s+\sigma) h \delta^{r-q_*} \\ & \quad + 2^{q-1} \varepsilon^m (r-q) \int_{I'} \int_{\mathbb{R}^N} u_\varepsilon^{r-q+2} dx dt \\ & \leq \left[\frac{4}{\sigma} \delta^{-q+1} + 2^{q-1}(r-q) \right] \left\{ \int_{I'} \int_{\mathbb{R}^N} u_\varepsilon^r dx dt + \frac{(s+\sigma)h}{\delta^{q_*}} \delta^r \right\} \\ & \quad + 2^{q-1} \varepsilon^m (r-q) \int_{I'} \int_{\mathbb{R}^N} u_\varepsilon^{r-q+2} dx dt. \end{aligned}$$

Since $\sigma \delta^{q-1} \leq 1$ and

$$k(r-q+1) + m - 1 = k(r-q_*-q+1) + q_* + q - 1,$$

it follows that

$$(3.20) \quad \frac{(s+\sigma)h}{\sigma} \delta^{r-q_*-q+1} = \left\{ \frac{(s+\sigma)h}{\delta^{q_*}} \delta^{k(r-q+1)+m-1} \right\}^{\frac{1}{k}} \left(\frac{1}{\sigma \delta^{q-1}} \right)^{\frac{1}{k}} \left(\frac{s+\sigma}{\sigma} h \right)^{1-\frac{1}{k}} \\ \geq h^{1-\frac{1}{k}} \left\{ \frac{(s+\sigma)h}{\delta^{q_*}} \delta^{k(r-q+1)+m-1} \right\}^{\frac{1}{k}}.$$

Taking (3.20) in the left-hand side of (3.19) and using the inequality $(A+B)^{\frac{1}{k}} \leq A^{\frac{1}{k}} + B^{\frac{1}{k}}$ ($A, B > 0$), we have

$$\mu_0 \left\{ \int_I \int_{\mathbb{R}^N} u_\varepsilon^{k(r-q+1)+m-1} dx dt + \frac{(s+\sigma)h}{\delta^{q_*}} \delta^{k(r-q+1)+m-1} \right. \\ \left. + \varepsilon^{m-1} \int_I \int_{\mathbb{R}^N} u_\varepsilon^{k(r-q+1)} dx dt \right\}^{\frac{1}{k}} \\ \leq \left[\frac{4}{\sigma} \delta^{-q+1} + 2^{q-2}(r-q) \right] \cdot \left\{ \int_{I'} \int_{\mathbb{R}^N} u_\varepsilon^r dx dt + \frac{(s+\sigma)h}{\delta^{q_*}} \delta^r \right\} \\ + 2^{q-1} \varepsilon^m (r-q) \int_{I'} \int_{\mathbb{R}^N} u_\varepsilon^{r-q+2} dx dt,$$

where $\mu_0 := \min\left\{\frac{\nu_0}{2C_0^{1/k}}, \frac{\nu_1}{2C_0^{1/k}}, h^{1-\frac{1}{k}}\right\}$. Thus we obtain (3.10). $\square$

Lemma 3.4. Let $N \geq 2$, $m \geq 1$, $q \geq 2$, $\varepsilon \in (0, 1)$, $T > 0$ and $0 < \chi < \tau < \tau + s < T$. Let $(u_\varepsilon, v_\varepsilon)$ be a unique solution to $(KS)_\varepsilon$ on $[0, T)$. Assume that m and q satisfy (3.2) and δ satisfies

$$\chi \delta^{q-1} \leq 1.$$

Then the following estimate holds:

$$(3.21) \quad \|u_\varepsilon\|_{L^\infty(\tau, \tau+s; L^\infty(\mathbb{R}^N))}^{r_1-(q_*+q-1)} \\ \leq [2B(2k)^{\frac{1}{k-1}}]^{\frac{k}{k-1}} \left\{ (1+\varepsilon^m) \int_{\tau-\chi}^{\tau+s} \int_{\mathbb{R}^N} u_\varepsilon^{r_1} dx dt + \left(s + \frac{\chi}{2}\right) h \delta^{r_1-q_*} \right\},$$

where $k = 1 + \frac{2}{N}$, $q_* = \frac{N}{2}(q-m)$, $r_1 = r_1(m, q, N) \geq 1$, $h := \sup_{t \in [0, T]} \|u_\varepsilon(t)\|_{q_*}^{q_*}$ and $B = B(h, r_1, \chi, \delta, m, q, N) > 0$ are constants.

Proof. Let $q_* := \frac{N}{2}(q-m)$, $k := 1 + \frac{2}{N}$, $\lambda_0 := q_* + q - 1$, $\Lambda_0 := \frac{N}{2} + q - 1$ and let $r_* > \lambda_0$ be some constant. First let the sequence $\{\lambda_n\}_n \subset \mathbb{R}$ be defined by

$$\begin{cases} \lambda_n = (\lambda_{n-1} - q + 1)k + m - 1, \\ \lambda_1 = r_1 := \max\{r_*, \lambda_0, \Lambda_0\}. \end{cases}$$

Thus

$$(3.22) \quad \lambda_n = \lambda_0 + (r_1 - \lambda_0)k^{n-1}.$$

Since $k = 1 + \frac{2}{N} > 1$, it follows that

$$\lambda_{n+1} > \lambda_n, \quad r_1 \leq \lambda_n \leq r_1 k^{n-1} \quad \text{and} \quad \lim_{n \rightarrow \infty} \lambda_n = \infty.$$

Next define the sequence $\{\Lambda_n\}_n \subset \mathbb{R}$ as

$$\begin{cases} \Lambda_n - q + 2 = (\Lambda_{n-1} - q + 1)k, \\ \Lambda_1 - q + 2 = r_1, \end{cases}$$

and then,

$$\Lambda_n = \Lambda_0 + (r_1 - \Lambda_0)k^{n-1}.$$

Since $k = 1 + \frac{2}{N} > 1$, it follows that

$$\Lambda_{n+1} > \Lambda_n, \quad r_1 \leq \Lambda_n \leq r_1 k^{n-1} \quad \text{and} \quad \lim_{n \rightarrow \infty} \Lambda_n = \infty.$$

Let $I_n := [\tau - 2^{-n+1}\chi, \tau + s]$ and $\delta > 0$ such that $\chi\delta^{q-1} \leq 1$. Then (3.9) holds for δ :

$$\{(\tau - 2^{-n}\chi) - (\tau - 2^{-n+1}\chi)\}\delta^{q-1} = (2^{-n}\chi)\delta^{q-1} \leq 1 \quad (n \geq 1)$$

and thus, we can put $I = I_{n+1}$ and $I' = I_n$ in (3.10). Setting

$$\mathbf{J}_n := \int_{I_n} \int_{\mathbb{R}^N} u_\varepsilon^{\lambda_n} dx dt + \frac{(s + 2^{-n}\chi)h}{\delta^{q*}} \delta^{\lambda_n} + \varepsilon^m \int_{I_n} \int_{\mathbb{R}^N} u_\varepsilon^{\Lambda_n - q + 2} dx dt,$$

we see from (3.10) that

$$(3.23) \quad \mu_0 \mathbf{J}_{n+1}^{\frac{1}{k}} \leq \left\{ \frac{4}{2^{-n}\chi\delta^{q-1}} + 2^{q-1}(\lambda_n - q) + 2^{q-1}(\Lambda_n - q) \right\} \mathbf{J}_n.$$

Now we evaluate the coefficients in (3.23). Noting that $2^{-n}\chi\delta^{q-1} \leq 1$, $\lambda_n \leq r_1 k^{n-1}$ and $\Lambda_n \leq r_1 k^{n-1}$, we find that

$$\begin{aligned} (3.24) \quad & \frac{4}{2^{-n}\chi\delta^{q-1}} + 2^{q-1}(\lambda_n - q) + 2^{q-1}(\Lambda_n - q) \\ & \leq \frac{1}{2^{-n}\chi\delta^{q-1}} \left\{ 4 + 2^{q-1}(\lambda_n - q) + 2^{q-1}(\Lambda_n - q) \right\} \\ & \leq \frac{2^{q-1}}{2^{-n}\chi\delta^{q-1}} (\lambda_n + \Lambda_n) \\ & \leq \frac{2^q r_1}{\chi\delta^{q-1}} 2^n k^{n-1}. \end{aligned}$$

From (3.23) and (3.24) it follows that

$$(3.25) \quad \mathbf{J}_{n+1}^{\frac{1}{k}} \leq \frac{2^q r_1}{\mu_0 \chi \delta^{q-1}} 2^n k^{n-1} \mathbf{J}_n =: B \cdot 2^n k^{n-1} \mathbf{J}_n.$$

Therefore we obtain

$$\begin{aligned}
 (3.26) \quad & (J_{n+1})^{\frac{1}{k^n}} \\
 & \leq (B \cdot 2^n k^{n-1})^{\frac{1}{k^{n-1}}} (B \cdot 2^{n-1} k^{n-2})^{\frac{1}{k^{n-2}}} \times \cdots \times (B \cdot 2) J_1 \\
 & = (2B)^{\frac{1}{k^{n-1}} + \frac{1}{k^{n-2}} + \cdots + 1} (2k)^{\frac{n-1}{k^{n-1}} + \frac{n-2}{k^{n-2}} + \cdots + \frac{1}{k}} J_1.
 \end{aligned}$$

From the definition of J_n and (3.22) we see that

$$\begin{aligned}
 \liminf_{n \rightarrow \infty} (J_{n+1})^{\frac{1}{k^n}} & \geq \liminf_{n \rightarrow \infty} \left(\int_{\tau-2^{-n}\chi}^{\tau+s} \int_{\mathbb{R}^N} u_\varepsilon^{\lambda_{n+1}} dx dt \right)^{\frac{r_1 - \lambda_0}{\lambda_{n+1} - \lambda_0}} \\
 & \geq \liminf_{n \rightarrow \infty} \|u_\varepsilon\|_{L^{\lambda_{n+1}}(\tau, \tau+s; L^{\lambda_{n+1}}(\mathbb{R}^N))}^{\frac{\lambda_{n+1}}{\lambda_{n+1} - \lambda_0} \cdot r_1 - \lambda_0} \\
 & = \|u_\varepsilon\|_{L^\infty(\tau, \tau+s; L^\infty(\mathbb{R}^N))}^{r_1 - \lambda_0}.
 \end{aligned}$$

Hence it follows from (3.26) that

$$\begin{aligned}
 & \|u_\varepsilon\|_{L^\infty(\tau, \tau+s; L^\infty(\mathbb{R}^N))}^{r_1 - \lambda_0} \\
 & \leq \liminf_{n \rightarrow \infty} (J_{n+1})^{\frac{1}{k^n}} \\
 & \leq \limsup_{n \rightarrow \infty} (2B)^{\frac{1}{k^{n-1}} + \frac{1}{k^{n-2}} + \cdots + 1} (2k)^{\frac{n-1}{k^{n-1}} + \frac{n-2}{k^{n-2}} + \cdots + \frac{1}{k}} J_1 \\
 & = (2B)^{\frac{k}{k-1}} (2k)^{\frac{k}{(k-1)^2}} \left((1 + \varepsilon^m) \int_{\tau-\chi}^{\tau+s} \int_{\mathbb{R}^N} u_\varepsilon^{r_1} dx dt + \left(s + \frac{\chi}{2}\right) h \delta^{r_1 - q_*} \right).
 \end{aligned}$$

Therefore we obtain (3.21). $\square$

Now we prove Proposition 3.1. From Lemmas 3.2–3.4 we can obtain L^∞ - L^r estimate without assuming that the initial data is small. In the proof of Proposition 3.1 we assume the smallness condition of the initial data to apply the L^r -decay property of u_ε .

Proof of Proposition 3.1. Put $q_* := \frac{N}{2}(q - m)$, $k := 1 + \frac{2}{N}$ and let $r_* > q_* + q - 1$ be some constant. Let $r_1 := \max\{m + q - 2, r_*, q_* + q - 1, \frac{N}{2} + q - 1\}$ and $0 < \chi < t < T$. From Lemma 3.4, u_ε satisfies (3.21). Moreover, (3.21) implies that for a.a. $0 < t < T$,

$$\begin{aligned}
 (3.27) \quad & \|u_\varepsilon(t)\|_{L^\infty(\mathbb{R}^N)}^{r_1 - (q_* + q - 1)} \\
 & \leq (2B)^{\frac{k}{k-1}} (2k)^{\frac{k}{(k-1)^2}} \left((1 + \varepsilon^m) \int_{t-\chi}^t \int_{\mathbb{R}^N} u_\varepsilon^{r_1} dx ds + \frac{\chi}{2} h \delta^{r_1 - q_*} \right),
 \end{aligned}$$

where $B = \frac{2^q r_1}{\mu_0 \chi \delta^{q-1}} > 0$, $h := \sup_{t \in [0, T]} \|u_\varepsilon(t)\|_{L^{q_*}(\mathbb{R}^N)}^{q_*} = \|u_{0\varepsilon}\|_{L^{q_*}}^{q_*}$ and $\mu_0 = \mu_0(m, q, N, h)$ is the same constant as in the proof of Lemma 3.3. Let $0 < t < T$. Taking χ and δ such that $\chi = \delta^{-(q-1)} = \frac{t}{2}$ in (3.27), and noting that $t \mapsto \|u_\varepsilon(t)\|_{L^{r_1}(\mathbb{R}^N)}$ is a non-increasing function

on $[0, T)$ and using the L^r -decay property (see Proposition 2.1 and Remark 3.2), we see that

$$\begin{aligned}
\|u_\varepsilon(t)\|_{L^\infty(\mathbb{R}^N)}^{r_1-(q_*+q-1)} &\leq C_1 \left\{ (1+\varepsilon^m) \int_{\frac{t}{2}}^t \int_{\mathbb{R}^N} u_\varepsilon^{r_1} dx ds + \frac{h}{2} \left(\frac{t}{2}\right)^{1-\frac{r_1-q_*}{q-1}} \right\} \\
&\leq C_1 \left\{ (1+\varepsilon^m) \frac{t}{2} \int_{\mathbb{R}^N} u_\varepsilon^{r_1} \left(\frac{t}{2}\right) dx + \frac{h}{2} \left(\frac{t}{2}\right)^{1-\frac{r_1-q_*}{q-1}} \right\} \\
&\leq C_1 \left\{ (1+\varepsilon^m) C_{r_1} \frac{t}{2} \left(\frac{t}{2} + 1\right)^{-\frac{N(r_1-1)}{N(m-1)+2}} + \frac{h}{2} \left(\frac{t}{2}\right)^{1-\frac{r_1-q_*}{q-1}} \right\} \\
&\leq C_1 2^{\frac{r_1-q_*-q+1}{q-1}} \left\{ (1+\varepsilon^m) C_{r_1} + \frac{h}{2} \right\} t^{-\frac{r_1-q_*-q+1}{q-1}},
\end{aligned}$$

where

$$C_1 = \left(\frac{2^{q+1} r_1}{\mu_0} \right)^{\frac{k}{k-1}} (2k)^{\frac{k}{(k-1)^2}}$$

and C_{r_1} is the same constant as in (2.2). Thus we obtain

$$\begin{aligned}
(3.28) \quad \|u_\varepsilon(t)\|_{L^\infty(\mathbb{R}^N)} &\leq K_0(\varepsilon) t^{-\frac{1}{q-1}} \\
&= K_0(\varepsilon) t^{-\frac{N}{N(m-1)+2q_*}}, \quad \text{a.a. } t \in (0, T),
\end{aligned}$$

where

$$K_0(\varepsilon) = 2^{\frac{1}{q-1}} \left\{ C_1 (1+\varepsilon^m) C_{r_1} + \frac{h}{2} \right\}^{\frac{1}{r_1-(q_*+q-1)}}.$$

It follows from (3.28) that for a.a. $t \in (0, T)$,

$$\begin{aligned}
(3.29) \quad \|u_\varepsilon(t)\|_{L^\infty(\mathbb{R}^N)} &\leq K_0(\varepsilon) t^{-\frac{N}{N(m-1)+2q_*}} \\
&\leq K_0(1) t^{-\frac{N}{N(m-1)+2q_*}}.
\end{aligned}$$

This inequality and $\|u_{0\varepsilon}\|_{L^r} \leq \|u_0\|_{L^r}$ ($1 \leq r \leq \infty$) show that the right-hand side of this inequality is independent of ε . Hence we see that

$$\begin{aligned}
\|u(t)\|_{L^\infty(\mathbb{R}^N)} &\leq \liminf_{\varepsilon \rightarrow 0} \|u_\varepsilon(t)\|_{L^\infty(\mathbb{R}^N)} \\
&\leq \liminf_{\varepsilon \rightarrow 0} K_0(\varepsilon) t^{-\frac{N}{N(m-1)+2q_*}} \\
&= K_1 t^{-\frac{N}{N(m-1)+2q_*}},
\end{aligned}$$

where $K_1 := K_0(1) > 0$ is a constant which depends on $\|u_0\|_{L^{q_*}}$, C_{r_1} , r_1 , m , q and N . Therefore we obtain the desired inequality (3.3). $\square$

Remark 3.3. The estimate (3.3) holds for some $r \geq r_1$. In fact, by recalling the definitions of λ_n and Λ_n , we see that if $\lambda_1 = \Lambda_1 = r$, then (3.3) holds with $r_1 = r$.

3.2. L^∞ -decay property

In this subsection we prove the L^∞ -decay property of solutions to $(\text{KS})_0$.

Proposition 3.5. (L^∞ -decay property) *Let $N \geq 2$, $m \geq 1$, $q \geq 2$ and $\rho \in (0, 1]$. Let (u, v) be a global weak solution to $(\text{KS})_0$ on $[0, \infty)$. Assume further that m and q satisfy*

$$(3.30) \quad q > m + \frac{2}{N}$$

and u_0 satisfies (1.1) and the smallness condition as in Theorem 1.1. Then the solution u has the following decay property:

$$(3.31) \quad \|u(t)\|_{L^\infty(\mathbb{R}^N)} \leq K_\rho(t + \rho)^{-\frac{N}{N(m-1)+2}}, \quad \text{a.a. } t \in [5\rho, \infty),$$

where $K_\rho = K_\rho(\rho, r, C_r, \|u_0\|_{L^1}, \|u_0\|_{L^{q}}, \|u_0\|_{L^r}, m, q, N)$ with $q_* = \frac{N}{2}(q - m)$ and $r \geq r_3 = r_3(m, q, N)$ are positive constants and C_r is the same constant as in Proposition 2.1.*

The proof is based on [17, Sections 5–7]. To this end we need three lemmas.

Lemma 3.6. *Let $N \geq 2$, $m \geq 1$, $q \geq 2$, $\varepsilon \in (0, 1)$, $\rho \in (0, 1]$ and r_1 is the same constant as in Section 3.1. Let $(u_\varepsilon, v_\varepsilon)$ be a unique solution to $(\text{KS})_\varepsilon$ on $[0, \infty)$. Assume that m and q satisfy (3.2) and u_0 satisfies the smallness condition as in Theorem 1.1. Then for $r \geq r_1$ and a.a. $t \in [2\rho, \infty)$,*

$$(3.32) \quad \|u_\varepsilon(t)\|_{L^\infty(\mathbb{R}^N)}^{r-(q_*+q-1)} \leq C'_\rho \|u_\varepsilon(t - \rho)\|_{L^r(\mathbb{R}^N)}^{r\{1 - \frac{q-1}{r-q_*}(1 + \frac{N}{2})\}},$$

where $q_ = \frac{N}{2}(q - m)$ and $C'_\rho = C'_\rho(\rho, \varepsilon, r, \|u_{0\varepsilon}\|_{L^r}, \|u_{0\varepsilon}\|_{L^{q*}}, m, q, N) > 0$ is a constant.*

Proof. Let $\rho \in (0, 1]$, $r \geq r_1$ (see Section 3.1), $t \geq 2\rho$, $q_* = \frac{N}{2}(q - m)$ and $k = 1 + \frac{2}{N}$. Since $t \mapsto \|u_\varepsilon(t)\|_{L^r(\mathbb{R}^N)}$ is a non-increasing function, we can take χ and δ such that

$$\chi = \rho, \quad \delta = \left(\|u_\varepsilon(t - \rho)\|_{L^r(\mathbb{R}^N)}^{\frac{r}{r-q_*}} \right) \left(\|u_{0\varepsilon}\|_{L^r(\mathbb{R}^N)}^{\frac{r}{r-q_*}} \right)^{-1} (\leq 1)$$

in (3.27) with $r_1 = r$ (see Remark 3.3). Hence it follows that for a.a. $t \geq 2\rho$,

$$\begin{aligned} & \|u_\varepsilon(t)\|_{L^\infty(\mathbb{R}^N)}^{r-(q_*+q-1)} \\ & \leq C_\rho \left(\|u_\varepsilon(t - \rho)\|_{L^r(\mathbb{R}^N)}^{\frac{r}{r-q_*}} \right)^{-\frac{(q-1)k}{k-1}} \\ & \quad \times \left\{ (1 + \varepsilon^m) \int_{t-\rho}^t \|u_\varepsilon(s)\|_{L^r(\mathbb{R}^N)}^r ds + \frac{\rho h}{2} \left(\frac{\|u_\varepsilon(t - \rho)\|_{L^r(\mathbb{R}^N)}^r}{\|u_{0\varepsilon}\|_{L^r(\mathbb{R}^N)}^r} \right)^r \right\} \\ & \leq C_\rho \left(\|u_\varepsilon(t - \rho)\|_{L^r(\mathbb{R}^N)}^{\frac{r}{r-q_*}} \right)^{-\frac{(q-1)k}{k-1}} \left(\rho(1 + \varepsilon^m) + \frac{\rho h}{2} \|u_{0\varepsilon}\|_{L^r(\mathbb{R}^N)}^{-r} \right) \|u_\varepsilon(t - \rho)\|_{L^r(\mathbb{R}^N)}^r \\ & = C_\rho \left(\rho(1 + \varepsilon^m) + \frac{\rho h}{2} \|u_{0\varepsilon}\|_{L^r(\mathbb{R}^N)}^{-r} \right) \left(\int_{\mathbb{R}^N} u_\varepsilon(t - \rho)^r dx \right)^{1 - \frac{q-1}{r-q_*}(1 + \frac{N}{2})}, \end{aligned}$$

where

$$C_\rho = \left(\frac{2^{q+1}r}{\rho\mu_0} \|u_{0\varepsilon}\|_{L^r(\mathbb{R}^N)}^{\frac{r(q-1)}{r-q_*}} \right)^{\frac{k}{k-1}} (2k)^{\frac{k}{(k-1)^2}}$$

and μ_0 is the same constant as in the proof of Lemma 3.3. Therefore we obtain (3.32), where $C'_\rho = C_\rho(\rho(1 + \varepsilon^{m-1}) + \frac{\rho h}{2} \|u_{0\varepsilon}\|_{L^r}^{-r})$. $\square$

Lemma 3.7. *Let $N \geq 2$, $m \geq 1$, $q \geq 2$, $\varepsilon \in (0, 1)$, $\rho \in (0, 1]$ and $t \geq 2\rho$. Let $(u_\varepsilon, v_\varepsilon)$ be a unique solution to $(KS)_\varepsilon$ on $[0, \infty)$. Assume that m and q satisfy (3.2). Put*

$$(3.33) \quad G(s) := (r-1) \int_0^s (\tau + \varepsilon^{\frac{m}{q-2}})^{q-2} d\tau,$$

$$(3.34) \quad w_\varepsilon(x, t) := u_\varepsilon e^{-\int_{2\rho}^t G(\|u_\varepsilon(s)\|_{L^\infty(\mathbb{R}^N)}) ds}.$$

Then w_ε satisfies the following:

$$(3.35) \quad \|w_\varepsilon(t)\|_{L^1(\mathbb{R}^N)} \leq \|u_{0\varepsilon}\|_{L^1}, \quad t \geq 2\rho,$$

$$(3.36) \quad \frac{d}{dt} \int_{\mathbb{R}^N} w_\varepsilon^{r-m+1}(t) dx + \mu_1 \int_{\mathbb{R}^N} |\nabla w_\varepsilon^{\frac{r}{2}}(t)|^2 dx \leq 0, \quad r > m, \quad t \geq 2\rho,$$

$$(3.37) \quad t \mapsto \|w_\varepsilon(t)\|_{L^r(\mathbb{R}^N)} \quad (1 \leq r < \infty) \text{ is a non-increasing function on } [2\rho, \infty),$$

where $\mu_1 = \mu_1(m)$ is a positive constant.

Proof. First we prove (3.35). From the definition of w_ε and the mass conservation law, we see that for $t \geq 2\rho$,

$$\|w_\varepsilon(t)\|_{L^1(\mathbb{R}^N)} \leq \|u_\varepsilon(t)\|_{L^1(\mathbb{R}^N)} = \|u_{0\varepsilon}\|_{L^1}.$$

Thus we obtain (3.35). Next we prove (3.36). Let $r > 1$ and $t \geq 2\rho$. Differentiating w_ε about t , we see by the first approximate equation $(1)_\varepsilon$ (see $(KS)_\varepsilon$ in the top of Section 3) that

$$(3.38) \quad \begin{aligned} \frac{dw_\varepsilon}{dt} &= e^{-\int_{2\rho}^t G(\|u_\varepsilon(s)\|_{L^\infty}) ds} \\ &\quad \times \left(\nabla \cdot (\nabla(u_\varepsilon + \varepsilon))^m - (u_\varepsilon + \varepsilon^{\frac{m}{q-2}})^{q-2} u_\varepsilon \nabla v_\varepsilon - u_\varepsilon G(\|u_\varepsilon(t)\|_{L^\infty}) \right). \end{aligned}$$

Multiplying (3.38) by w_ε^{r-1} and integrating it over $\mathbb{R}^N$, we have

$$(3.39) \quad \begin{aligned} &\frac{1}{r} \frac{d}{dt} \|w_\varepsilon(t)\|_{L^r(\mathbb{R}^N)}^r \\ &= \left(e^{-r \int_{2\rho}^t G(\|u_\varepsilon(s)\|_{L^\infty}) ds} \right) \times \left(\int_{\mathbb{R}^N} \nabla \cdot (\nabla(u_\varepsilon + \varepsilon))^m - (u_\varepsilon + \varepsilon^{\frac{m}{q-2}})^{q-2} u_\varepsilon \nabla v_\varepsilon u_\varepsilon^{r-1} dx \right. \\ &\quad \left. - \int_{\mathbb{R}^N} u_\varepsilon^r G(\|u_\varepsilon(t)\|_{L^\infty}) dx \right) \\ &=: \left(e^{-r \int_{2\rho}^t G(\|u_\varepsilon(s)\|_{L^\infty}) ds} \right) \times (I_4 - I_5). \end{aligned}$$

By a similar argument from (3.5) to (3.7) in Lemma 3.2, it follows that

$$(3.40) \quad \mathbf{I}_4 \leq -\frac{4m(r-1)}{(r+m-1)^2} \|\nabla u_\varepsilon^{\frac{r+m-1}{2}}(t)\|_{L^2(\mathbb{R}^N)}^2 - \frac{4m(r-1)\varepsilon^{m-1}}{r^2} \|\nabla u_\varepsilon^{\frac{r}{2}}(t)\|_{L^2(\mathbb{R}^N)}^2 \\ + (r-1) \int_{\mathbb{R}^N} u_\varepsilon F(u_\varepsilon) dx,$$

where

$$F(s) := \int_0^s (\tau + \varepsilon^{\frac{m}{q-2}})^{q-2} \tau^{r-1} d\tau.$$

Recalling the definition of the function G , we see that

$$(3.41) \quad (r-1) \int_{\mathbb{R}^N} u_\varepsilon F(u_\varepsilon) dx - \mathbf{I}_5 \\ = (r-1) \int_{\mathbb{R}^N} \left\{ u_\varepsilon \int_0^{u_\varepsilon} (\tau + \varepsilon^{\frac{m}{q-2}})^{q-2} \tau^{r-1} d\tau - u_\varepsilon^r \int_0^{\|u_\varepsilon(t)\|_{L^\infty}} (\tau + \varepsilon^{\frac{m}{q-2}})^{q-2} d\tau \right\} dx \\ \leq (r-1) \int_{\mathbb{R}^N} \left\{ u_\varepsilon^r \left(\int_0^{u_\varepsilon} - \int_0^{\|u_\varepsilon(t)\|_{L^\infty}} \right) (\tau + \varepsilon^{\frac{m}{q-2}})^{q-2} d\tau \right\} dx \\ \leq 0.$$

Hence it follows from (3.39)–(3.41) that

$$(3.42) \quad \frac{d}{dt} \|w_\varepsilon(t)\|_{L^r(\mathbb{R}^N)}^r \leq \left(-e^{-r \int_{2\rho}^t G(\|u_\varepsilon(s)\|_{L^\infty}) ds} \right) \cdot 4mr(r-1) \\ \times \left(\frac{1}{(r+m-1)^2} \|\nabla u_\varepsilon^{\frac{r+m-1}{2}}(t)\|_{L^2(\mathbb{R}^N)}^2 + \frac{\varepsilon^{m-1}}{r^2} \|\nabla u_\varepsilon^{\frac{r}{2}}(t)\|_{L^2(\mathbb{R}^N)}^2 \right).$$

Since

$$\|\nabla u_\varepsilon^{\frac{r+m-1}{2}}\|_{L^2(\mathbb{R}^N)}^2 = \left(e^{(r+m-1) \int_{2\rho}^t G(\|u_\varepsilon(s)\|_{L^\infty}) ds} \right) \cdot \|\nabla w_\varepsilon^{\frac{r+m-1}{2}}\|_{L^2(\mathbb{R}^N)}^2$$

by the definition of w_ε , we see from (3.42) that

$$(3.43) \quad \frac{d}{dt} \|w_\varepsilon(t)\|_{L^r(\mathbb{R}^N)}^r \leq -e^{(m-1) \int_{2\rho}^t G(\|u_\varepsilon(s)\|_{L^\infty}) ds} \frac{4mr(r-1)}{(r+m-1)^2} \|\nabla w_\varepsilon^{\frac{r+m-1}{2}}(t)\|_{L^2(\mathbb{R}^N)}^2.$$

Replacing r by $r-m+1$ in (3.43) and setting $\mu_1 := \inf_{r \geq m} \frac{4m(r-m+1)(r-m)}{r^2}$, we obtain (3.36) for $r > m$. Finally we prove (3.37). From (3.43) we see that for $r \geq 1$,

$$\frac{d}{dt} \|w_\varepsilon(t)\|_{L^r(\mathbb{R}^N)}^r \leq 0, \quad t \geq 2\rho,$$

so $t \mapsto \|w_\varepsilon(t)\|_{L^r(\mathbb{R}^N)}$ ($1 \leq r < \infty$) is a non-increasing function on $[2\rho, \infty)$. $\square$

The next lemma gives the L^∞ -estimate of w_ε . The lemma similar to Lemma 3.8 is proved in [17, Section 6], where they considered the following function $\tilde{w}_\varepsilon$ instead of w_ε :

$$\tilde{w}_\varepsilon(x, t) := u_\varepsilon \exp\left(-\int_{2\rho}^t \|u_\varepsilon(s)\|_{L^\infty(\mathbb{R}^N)}^{q-1} ds\right).$$

The proof starts with (3.36) and uses (3.37) with $r = \frac{2N}{N-1}$, $r = 2$ and (3.35). Thus the next lemma is proved by using not the definition of w_ε but the property of w_ε .

Lemma 3.8. *Let $N \geq 2$, $m \geq 1$, $q \geq 2$, $\varepsilon \in (0, 1)$ and $\rho \in (0, 1]$. Let $(u_\varepsilon, v_\varepsilon)$ be a unique solution to $(KS)_\varepsilon$ on $[0, \infty)$. Assume that m and q satisfy (3.2) and u_0 satisfies (1.1) and the smallness condition as in Theorem 1.1. Put G and w_ε as in (3.33) and (3.34). Assume further that $\delta' > 0$ satisfies*

$$t^{\frac{1}{2}} \delta'^{\frac{1}{N} + \frac{m-1}{2}} \leq 1.$$

Then

$$(3.44) \quad \|w_\varepsilon(t)\|_{L^\infty(\mathbb{R}^N)} \leq C_3((t+\rho)\delta'^{m-1})^{-\frac{N}{2}} \left(\int_{\mathbb{R}^N} w_\varepsilon^r \left(\frac{t}{2} - \frac{\rho}{2} \right) dx + \|u_{0\varepsilon}\|_{L^1} \delta'^{r-1} \right), \quad t \geq 5\rho,$$

where $C_3 = C_3(\|u_{0\varepsilon}\|_{L^1}, m, q, N)$ is a positive constant.

Proof of Proposition 3.5. Let $\rho \in (0, 1]$, $r \geq r_1$ (see Section 3.1) and $t \geq 5\rho$. We use the same notation as (3.33) and (3.34). Recalling the definition of w_ε , we see that

$$(3.45) \quad \int_{\mathbb{R}^N} w_\varepsilon^r \left(\frac{t}{2} - \frac{\rho}{2} \right) dx \leq \int_{\mathbb{R}^N} u_\varepsilon^r \left(\frac{t}{2} - \frac{\rho}{2} \right) dx,$$

$$(3.46) \quad \|u_\varepsilon(t)\|_{L^\infty(\mathbb{R}^N)} \leq \exp \left(\int_{2\rho}^t G(\|u_\varepsilon(s)\|_{L^\infty}) ds \right) \|w_\varepsilon(t)\|_{L^\infty(\mathbb{R}^N)}.$$

It follows from (3.46), (3.44) in Lemma 3.8 and (3.45) that

$$(3.47) \quad \begin{aligned} & \|u_\varepsilon(t)\|_{L^\infty(\mathbb{R}^N)}^r \\ & \leq C_3 e^{r \int_{2\rho}^t G(\|u_\varepsilon(s)\|_{L^\infty}) ds} ((t+\rho)\delta'^{m-1})^{-\frac{N}{2}} \left(\int_{\mathbb{R}^N} u_\varepsilon^r \left(\frac{t}{2} - \frac{\rho}{2} \right) dx + \|u_{0\varepsilon}\|_{L^1} \delta'^{r-1} \right). \end{aligned}$$

Take $\delta' = (t+\rho)^{-\frac{N}{N(m-1)+2}}$ in (3.47). It follows from the L^r -decay property of u_ε (see (2.2) in Proposition 2.1 and Remark 3.2) that

$$\begin{aligned} \|u_\varepsilon(t)\|_{L^\infty(\mathbb{R}^N)}^r & \leq C_3 e^{r \int_{2\rho}^t G(\|u_\varepsilon(s)\|_{L^\infty}) ds} \\ & \times (t+\rho)^{-\frac{N}{N(m-1)+2}} \left(\int_{\mathbb{R}^N} u_\varepsilon^r \left(\frac{t}{2} - \frac{\rho}{2} \right) dx + \|u_{0\varepsilon}\|_{L^1} (t+\rho)^{-\frac{N(r-1)}{N(m-1)+2}} \right) \\ & \leq C_3 C_r^r e^{r \int_{2\rho}^t G(\|u_\varepsilon(s)\|_{L^\infty}) ds} \\ & \times (t+\rho)^{-\frac{N}{N(m-1)+2}} \left(\left(\frac{t}{2} - \frac{\rho}{2} + 1 \right)^{-\frac{N(r-1)}{N(m-1)+2}} + \|u_{0\varepsilon}\|_{L^1} (t+\rho)^{-\frac{N(r-1)}{N(m-1)+2}} \right) \\ & = C_4 e^{r \int_{2\rho}^t G(\|u_\varepsilon(s)\|_{L^\infty}) ds} (t+\rho)^{-\frac{Nr}{N(m-1)+2}}, \end{aligned}$$

where

$$C_4 = C_3 C_r^r (2^{\frac{N(r-1)}{N(m-1)+2}} + \|u_{0\varepsilon}\|_{L^1}),$$

C_3 and C_r are the same constants as in Lemma 3.8 and Proposition 2.1, respectively. Hence we have

$$(3.48) \quad \|u_\varepsilon(t)\|_{L^\infty(\mathbb{R}^N)} \leq C_4^{\frac{1}{r}} \exp \left(\int_{2\rho}^t G(\|u_\varepsilon(s)\|_{L^\infty(\mathbb{R}^N)}) ds \right) (t+\rho)^{-\frac{Nr}{N(m-1)+2}}, \quad t \geq 5\rho.$$

Here we estimate the function G . From (3.32) in Lemma 3.6 and the L^r -decay property (2.2), it follows that a.a. $t \geq 2\rho$,

$$\begin{aligned}
 (3.49) \quad & \int_{2\rho}^t G(\|u_\varepsilon(s)\|_{L^\infty(\mathbb{R}^N)}) ds \\
 &= (r-1) \int_{2\rho}^t \int_0^{\|u_\varepsilon(s)\|_{L^\infty}} (\tau + \varepsilon^{\frac{m}{q-2}})^{q-2} d\tau ds \\
 &= \frac{r-1}{q-1} \int_{2\rho}^t \left\{ (\|u_\varepsilon(s)\|_{L^\infty(\mathbb{R}^N)} + \varepsilon^{\frac{m}{q-2}})^{q-1} - \varepsilon^{\frac{m(q-1)}{q-2}} \right\} ds \\
 &\leq \frac{r-1}{q-1} \int_{2\rho}^t \left\{ \left((C'_\rho \|u_\varepsilon(s-\rho)\|_{L^r(\mathbb{R}^N)})^{r\{1-\frac{q-1}{r-q_*}(1+\frac{N}{2})\}} \right)^{\frac{1}{r-(q_*+q-1)}} + \varepsilon^{\frac{m}{q-2}} \right\}^{q-1} - \varepsilon^{\frac{m(q-1)}{q-2}} \right\} ds \\
 &\leq \frac{r-1}{q-1} \int_{2\rho}^t \left\{ (C_5(s-\rho+1)^{-\beta} + \varepsilon^{\frac{m}{q-2}})^{q-1} - \varepsilon^{\frac{m(q-1)}{q-2}} \right\} ds,
 \end{aligned}$$

where

$$\beta = \frac{N(r-1)}{N(m-1)+2} \left\{ 1 - \frac{q-1}{r-q_*} \left(1 + \frac{N}{2} \right) \right\} \frac{1}{r-(q_*+q-1)},$$

C'_ρ is the same constant as in (3.32) and $C_5 = C_5(C'_\rho, C_r, r, m, q, N)$ is a positive constant. From (3.29) (see the proof of Proposition 3.1), (3.48) and (3.49) we see that a.a. $t \geq 1$,

$$\begin{aligned}
 (3.50) \quad & \|u(t)\|_{L^\infty(\mathbb{R}^N)} \\
 &\leq \liminf_{\varepsilon \rightarrow 0} \|u_\varepsilon(t)\|_{L^\infty(\mathbb{R}^N)} \\
 &\leq \liminf_{\varepsilon \rightarrow 0} \left\{ C_4^{\frac{1}{r}} \exp \left(\int_{2\rho}^t G(\|u_\varepsilon(s)\|_{L^\infty(\mathbb{R}^N)}) ds \right) (t+\rho)^{-\frac{N}{N(m-1)+2}} \right\} \\
 &\leq \liminf_{\varepsilon \rightarrow 0} \left[C_4^{\frac{1}{r}} \exp \left(\frac{r-1}{q-1} \int_{2\rho}^t \left\{ (C_5(s-\rho+1)^{-\beta} + \varepsilon^{\frac{m}{q-2}})^{q-1} - \varepsilon^{\frac{m(q-1)}{q-2}} \right\} ds \right) \right. \\
 &\quad \left. \times (t+\rho)^{-\frac{N}{N(m-1)+2}} \right] \\
 &= C_4^{\frac{1}{r}} \exp \left(\int_{2\rho}^t C_6(s-\rho+1)^{-\beta(q-1)} ds \right) (t+\rho)^{-\frac{N}{N(m-1)+2}} \\
 &\leq C_4^{\frac{1}{r}} \exp \left(\int_{2\rho}^\infty C_6(s-\rho+1)^{-\beta(q-1)} ds \right) (t+\rho)^{-\frac{N}{N(m-1)+2}},
 \end{aligned}$$

where $C_6 = \frac{C_5(r-1)}{q-1}$. When $q > m + \frac{2}{N}$, we have

$$\begin{aligned}
 -\beta(q-1) &= -\frac{N(q-1)}{N(m-1)+2} \left\{ 1 - \frac{q-1}{r-q_*} \left(1 + \frac{N}{2} \right) \right\} \frac{r-1}{r-(q_*+q-1)} \\
 &\rightarrow -\frac{N(q-1)}{N(m-1)+2} < -1 \quad (r \rightarrow \infty).
 \end{aligned}$$

Hence there exists r_2 such that $-\beta(q-1) < -1$ for $r \geq r_2$. It follows that for $r \geq r_2$,

$$(3.51) \quad \int_{2\rho}^{\infty} C_6(s - \rho + 1)^{-\beta(q-1)} ds = \frac{C_6(\rho + 1)^{-\beta(q-1)+1}}{\beta(q-1) - 1}.$$

Therefore we see from (3.50) and (3.51) that for $r \geq r_3 := \max\{r_1, r_2\}$,

$$\|u(t)\|_{L^\infty(\mathbb{R}^N)} \leq K_\rho(t + \rho)^{-\frac{N}{N(m-1)+2}},$$

where $K_\rho = C_4^{\frac{1}{r}} \exp(\frac{C_6(\rho+1)^{-\beta(q-1)+1}}{\beta(q-1)-1})$. This is the required decay property. $\square$

Proof of Theorem 1.1 when $N \geq 2$. From Propositions 3.1 and 3.5 with $r = r_3$ (see the proof of Proposition 3.5) we see that

$$\|u(t)\|_{L^\infty(\mathbb{R}^N)} \leq \begin{cases} Kt^{-\frac{N}{N(m-1)+2q_*}}, & \text{a.a. } t \in (0, \infty), \\ K_\rho(t + \rho)^{-\frac{N}{N(m-1)+2}}, & \text{a.a. } t \in (5\rho, \infty), \end{cases}$$

where $q_* = \frac{N}{2}(q - m)$, $\rho \in (0, 1]$, $K = K(\|u_0\|_{L^{r_3}}, C_{r_3}, r_3, m, q, N) > 0$ and $K_\rho = K_\rho(\rho, \|u_0\|_{L^1}, \|u_0\|_{L^{q_*}}, \|u_0\|_{L^{r_3}}, C_{r_3}, r_3, m, q, N) > 0$ are constants, where C_r is the same constant as in Proposition 2.1. Thus we obtain (1.3) and (1.4). $\square$

4. The case where $N = 1$

In this section we consider the case where $N = 1$. First we introduce the approximate problem when $N = 1$.

$$(KS)_{\varepsilon, N=1} \quad \begin{cases} \frac{\partial u_\varepsilon}{\partial t} = \frac{\partial^2}{\partial x^2}(u_\varepsilon + \varepsilon)^m - \frac{\partial}{\partial x} \left(u_\varepsilon^{q-1} \frac{\partial v_\varepsilon}{\partial x} \right) & \text{in } \mathbb{R} \times (0, T), \quad \cdots (1)_{\varepsilon, N=1} \\ 0 = \frac{\partial^2 v_\varepsilon}{\partial x^2} - v_\varepsilon + u_\varepsilon & \text{in } \mathbb{R} \times (0, T), \quad \cdots (2)_{\varepsilon, N=1} \\ u_\varepsilon(x, 0) = u_{0\varepsilon}(x), & x \in \mathbb{R}, \end{cases}$$

where $m \geq 1$, $q \geq 2$ and $\varepsilon \in (0, 1)$. The initial data $u_{0\varepsilon} \in C_0^\infty(\mathbb{R})$ is given as $u_{0\varepsilon} := (\rho_\varepsilon * u_0) \zeta_\varepsilon$; ρ_ε is the mollifier and ζ_ε is the standard cut function.

Note that the nonlinear term in the first equation of $(KS)_{\varepsilon, N=1}$ is different from the approximate nonlinear term in the case where $N \geq 2$ (see $(KS)_\varepsilon$ in Section 3). The reason is that the condition $q \geq m + \frac{2}{N}$ gives $q \geq 3$ when $N = 1$. This condition relates with $\|\nabla u_\varepsilon(t)\|_{L^\infty(\mathbb{R}^N)}$ (see [16, Proposition 9]). Differentiating the nonlinear term $\nabla(u^{q-1} \nabla v)$ in $(KS)_0$ about x formally to obtain the estimate of $\|\nabla u_\varepsilon(t)\|_{L^\infty(\mathbb{R}^N)}$, we see that

$$\begin{aligned} \frac{\partial}{\partial x_j} \nabla(u^{q-1} \nabla v) &= (q-1)(q-2) \sum_{i=1}^N u^{q-3} \frac{\partial u}{\partial x_j} \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_i} \\ &\quad + (q-1) \sum_{i=1}^N u^{q-2} \frac{\partial}{\partial x_j} \left(\frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_i} \right) + \frac{\partial}{\partial x_j} (u^{q-1} \Delta v), \quad j = 1, \dots, N. \end{aligned}$$

Therefore if $q \geq 3$, then it is not necessary to approximate the nonlinear term to non-degenerate type (see Remark 2.2).

We obtain the following two propositions by the proofs parallel to Propositions 3.1 and 3.5.

Proposition 4.1 (L^∞ -estimate when $N = 1$). *Let $m \geq 1$, $q \geq 2$, and $T > 0$. Let (u, v) be a weak solution to $(KS)_0$ on $[0, T]$. Assume further that m and q satisfy*

$$q \geq m + 2$$

and u_0 satisfies (1.1) and the smallness condition (1.2) in Theorem 1.1. Then the following estimate holds:

$$\|u(t)\|_{L^\infty(\mathbb{R})} \leq K_2 t^{-\frac{1}{m-1+2q_{**}}}, \quad \text{a.a. } t \in (0, T),$$

*where $q_{**} = \frac{q-m}{2}$, $K_2 = K_2(\|u_0\|_{L^{q_{**}}}, C_r, m, q, N)$, $r \geq r' = r'(m, q, N)$ are positive constants and C_r is the same constant as in Proposition 2.1.*

Proposition 4.2 (L^∞ -decay property when $N = 1$). *Let $m \geq 1$, $q \geq 2$ and $\rho \in (0, 1]$. Let (u, v) be a global weak solution to $(KS)_0$. Assume further that m and q satisfy*

$$q > m + 2$$

and u_0 satisfies (1.1) and the smallness condition (1.2) in Theorem 1.1. Then the solution u has the following decay property:

$$\|u(t)\|_{L^\infty(\mathbb{R})} \leq K'_\rho (t + \rho)^{-\frac{1}{m+1}}, \quad \text{a.a. } t \in [5\rho, \infty),$$

where $K'_\rho = K'_\rho(\rho, \|u_0\|_{L^1}, \|u_0\|_{L^{\frac{q-m}{2}}}, \|u_0\|_{L^r}, r, C_r, m, q, N)$, $r \geq r'' = r''(m, q, N)$ are positive constants, where C_r is the same constant as in Proposition 2.1.

To prove Proposition 4.2 we need that $t \mapsto \|w_\varepsilon(t)\|_{L^\infty(\mathbb{R})}$ is a non-increasing function on $[2\rho, \infty)$, where

$$w_\varepsilon(x, t) := u_\varepsilon \exp\left(-\int_{2\rho}^t \|u_\varepsilon(s)\|_{L^\infty(\mathbb{R}^N)}^{q-1} ds\right)$$

(because we use (3.37) with $r = \frac{2N}{N-1} = \infty$ for $N = 1$ as stated in the front of Lemma 3.8). This property of w_ε is proved as follows. From a similar proof to Lemma 3.7 we see that $t \mapsto \|w_\varepsilon(t)\|_{L^r(\mathbb{R})}$ ($1 \leq r < \infty$) is a non-increasing function on $[2\rho, \infty)$. Let $t \geq s \geq 2\rho$. Then we have

$$(4.1) \quad \|w_\varepsilon(t)\|_{L^r(\mathbb{R})} \leq \|w_\varepsilon(s)\|_{L^r(\mathbb{R})} \quad (1 \leq r < \infty).$$

It follows from Proposition 4.1 that

$$\|w_\varepsilon(t)\|_{L^\infty(\mathbb{R})} \leq \|u_\varepsilon(t)\|_{L^\infty(\mathbb{R})} \leq K_2 t^{-\frac{1}{m-1+2q_{**}}}, \quad \text{a.a. } t \geq 2\rho,$$

and hence $w_\varepsilon(t) \in L^\infty(\mathbb{R})$ (a.a. $t \geq 2\rho$). Letting $r \rightarrow \infty$ in (4.1), we have

$$\|w_\varepsilon(t)\|_{L^\infty(\mathbb{R})} \leq \|w_\varepsilon(s)\|_{L^\infty(\mathbb{R})} \quad \text{a.a. } t \geq s \geq 2\rho.$$

Therefore we see that $t \mapsto \|w_\varepsilon(t)\|_{L^\infty(\mathbb{R})}$ is a non-increasing function on $[2\rho, \infty)$.

Proof of Theorem 1.1 when $N = 1$. Combining Propositions 4.1 and 4.2 with $r = \tilde{r} := \max\{r', r''\}$, we obtain

$$\|u(t)\|_{L^\infty(\mathbb{R})} \leq \begin{cases} \tilde{K} t^{-\frac{1}{m-1+2q_{**}}}, & \text{a.a. } t \in (0, \infty), \\ \tilde{K}_\rho (t + \rho)^{-\frac{1}{m+1}}, & \text{a.a. } t \in [5\rho, \infty), \end{cases}$$

where $q_{**} = \frac{q-m}{2}$, $\rho \in (0, 1]$, $\tilde{K} = \tilde{K}(K_2, \tilde{r})$ and $\tilde{K}_\rho = \tilde{K}_\rho(K'_\rho, \tilde{r})$ are positive constants. Thus we obtain (1.3) and (1.4). $\square$

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