

Linear evolution equations in a reflexive Banach space

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§1. INTRODUCTION

In this paper we discuss the construction of an evolution system associated with the well posed problem in the sense of Hadamard for the time-dependent differential equation in a Banach space X

$$(DE)_s \quad \begin{cases} (d/dt)u(t) = A(t)u(t) \text{ for } t \in [s, T] \\ u(s) = x, \end{cases}$$

where $s \in [0, T]$, $u(\cdot)$ stands for an X -valued unknown function on the interval $[s, T]$ and $\{A(t) : t \in [0, T]\}$ is a given family of linear operators in X .

Assume for the moment that there exist a dense subspace Y of X and an injective bounded linear operator C_1 on X such that $Y \subset D(A(t))$ for $t \in [0, T]$ and the following conditions hold:

- 1) For $s \in [0, T]$ and $x \in C_1(Y)$, there exists a unique solution $u(t; s, x)$ such that $u(t; s, x) \in Y$ for $t \in [s, T]$.
- 2) For $x \in C_1(Y)$, $u(t; s, x)$ is continuous for $0 \leq s \leq t \leq T$.
- 3) If $\{u(t; s, x_n)\}$ is a sequence of solutions with $x_n \rightarrow 0$ in the C_1^{-1} -graph norm as $n \rightarrow \infty$ then $u(t; s, x_n)$ converges to zero uniformly with respect to t and s .

Here we note that in the special case where $A(t) = A$, $s = 0$, $Y = D(A)$ and $C_1 = R(c : A)^n$ ($n \in \mathbb{N} \cup \{0\}$ and $c \in \rho(A)$), the concept of the above well posed problem is equal to that of the well posed problem in the sense of Hadamard in the autonomous case (see [5,8]), which several authors [1,4,9,10,11,12] recently have studied via the theory of integrated semigroups or C -semigroups.

Now we turn to the above well posed problem. We define a linear subspace $D(s)$ of X and a linear operator $U(t, s)$ on $D(s)$ by

$$\begin{cases} D(s) = \{x \in X : \text{the } (DE)_s \text{ has a unique solution } u(t; s, x)\} \\ U(t, s)x = u(t; s, x) \text{ for } x \in D(s). \end{cases}$$

Then, from the uniqueness of the solutions it follows that $U(t, s) : D(s) \rightarrow D(t)$ and $U(t, r)U(r, s) = U(t, s)$ on $D(s)$ for $0 \leq s \leq r \leq t \leq T$. Formally, the two parameter family $\{U(t, s) : 0 \leq s \leq t \leq T\}$ may have the properties

$$(1.1) \quad (\partial/\partial t)U(t, s) = A(t)U(t, s)$$

(this property is useful to show the existence of the solutions),

$$(1.2) \quad (\partial/\partial s)U(t, s) = -U(t, s)A(s)$$

(this property is useful to show the uniqueness of the solutions).

We define $\{V_1(t, s) : 0 \leq s \leq t \leq T\}$ by

$$V_1(t, s)y = U(t, s)C_1y \quad (= u(t; s, C_1y)) \quad \text{for } y \in Y.$$

Since Y is dense in X one can see by the condition 3) that $V_1(t, s)$ is extended to a bounded linear operator on X , which we denote by the same symbol. Then, the two parameter family $\{V_1(t, s) : 0 \leq s \leq t \leq T\}$ has the properties

- (i) for $x \in X$, $(t, s) \rightarrow V_1(t, s)x$ is continuous for $0 \leq s \leq t \leq T$,
- (ii) $V_1(t, s)(Y) \subset Y$ for $0 \leq s \leq t \leq T$,
- (iii) $(\partial/\partial t)V_1(t, s)y = A(t)V_1(t, s)y$ for $y \in Y$, and $V_1(s, s) = C_1$.

We also consider the following important property to show the uniqueness of the solutions:

$$(iv) \quad (\partial/\partial s)V_2(t, s)y = -V_2(t, s)A(s)y \text{ for } y \in Y, \text{ and } V_2(s, s) = C_2.$$

Multiplying (1.2) by the injective bounded linear operator C_2 from the left-hand side, and then defining $V_2(t, s)$ by $C_2U(t, s)$ we obtain the property (iv).

Moreover, the following relation between $V_1(t, s)$ and $V_2(t, s)$ holds:

$$(v) \quad C_2 V_1(t, s) = V_2(t, s) C_1 \text{ for } 0 \leq s \leq t \leq T.$$

In §2 we will construct a pair of evolution systems $(\{V_1(t, s)\}, \{V_2(t, s)\})$ having the properties (i) - (v) in order to investigate the well posed problem in the sense of Hadamard for the time-dependent differential equation $(DE)_s$. As an application we also consider the second order differential equation in a reflexive Banach space X

$$(DE)_s^2 \quad \begin{cases} u''(t) = Au(t) + B(t)u(t) & \text{for } t \in [s, T] \\ u(s) = x, \quad u'(s) = y, \end{cases}$$

where A is the infinitesimal generator of a cosine family and $\{B(t) : t \in [0, T]\}$ is a given family of linear operators in X .

§2. CONSTRUCTION OF EVOLUTION SYSTEMS

Let X and Y be Banach spaces with norm $\|\cdot\|$ and $\|\cdot\|_Y$ respectively. We write $B(Y, X)$ for the set of all bounded linear operators on Y to X and denote $B(X, X)$ by $B(X)$. For each $i = 1, 2$, let C_i be an injective operator in $B(X)$.

Throughout this paper we will assume that

(H_1) Y is reflexive,

(H_2) Y is densely and continuously imbedded in X , that is, Y is a dense subspace of X and there is a constant L such that $\|y\| \leq L\|y\|_Y$ for $y \in Y$,

(H_3) $C_1(Y) \subset Y$ and $C_1(Y)$ is $\|\cdot\|_Y$ -dense in Y .

We will make the following assumptions on a family $\{A(t) : t \in [0, T]\}$ of closed linear operators in X :

(A_1) There are constants $M_1 \geq 0$ and $\omega_1 \geq 0$ such that

$$(\omega_1, \infty) \subset \rho(A(t)) \text{ for } t \in [0, T] \text{ and} \\ \left\| \lambda^m \left(\prod_{i=1}^m R(\lambda : A(t_i)) \right) C_1 \right\| \leq M_1 \text{ for } \lambda > \omega_1$$

and every finite sequence $\{t_i\}_{i=1}^m$ such that $0 \leq t_1 \leq \dots \leq t_m \leq T$ and m with $0 \leq m/\lambda \leq T$.

(A₂) There are constants $M_2 \geq 0$ and $\omega_2 \geq \omega_1$ such that

$$\left(\prod_{i=1}^m R(\lambda : A(t_i)) \right) C_1(Y) \subset Y \text{ and}$$

$$\left\| \lambda^m \left(\prod_{i=1}^m R(\lambda : A(t_i)) \right) C_1 \right\|_Y \leq M_2 \text{ for } \lambda > \omega_2$$

and every finite sequence $\{t_i\}_{i=1}^m$ such that $0 \leq t_1 \leq \dots \leq t_m \leq T$ and m with $0 \leq m/\lambda \leq T$.

(A₃) There are constants $M_3 \geq 0$ and $\omega_3 \geq \omega_1$ such that

$$\left\| C_2 \left(\lambda^m \left(\prod_{i=1}^m R(\lambda : A(t_i)) \right) \right) \right\| \leq M_3 \text{ for } \lambda > \omega_3$$

and every finite sequence $\{t_i\}_{i=1}^m$ such that $0 \leq t_1 \leq \dots \leq t_m \leq T$ and m with $0 \leq m/\lambda \leq T$.

(A₄) For $t \in [0, T]$, $D(A(t)) \supset Y$ and $D(C_1^{-1}A(t)C_1) \supset Y$, and the function $t \rightarrow A(t)$ is continuous in the $B(Y, X)$ norm $\|\cdot\|_{Y \rightarrow X}$ and $M_4 = \sup\{\|C_1^{-1}A(t)C_1\|_{Y \rightarrow X} : t \in [0, T]\} < \infty$.

The main result of this paper is given by

THEOREM 2.1. *If the family $\{A(t) : t \in [0, T]\}$ of closed linear operators in X satisfies (A₁) – (A₄) then there exists a unique pair $(\{V_1(t, s)\}, \{V_2(t, s)\})$ of strongly continuous families of bounded linear operators defined on the triangle $\Delta = \{(t, s) : 0 \leq s \leq t \leq T\}$ with the following properties:*

- (a) For $i = 1, 2$, $V_i(s, s) = C_i$ on $[0, T]$ and $C_2 V_1(t, s) = V_2(t, s) C_1$ on Δ .
- (b) $V_1(t, s)(Y) \subset Y$ for $0 \leq s \leq t \leq T$.
- (c) For $y \in Y$ and $y^* \in Y^*$, $(t, s) \rightarrow \langle y^*, V_1(t, s)y \rangle$ is continuous on Δ .

$$(d) \quad \langle x^*, V_1(t, s)y - V_1(r, s)y \rangle = \int_r^t \langle x^*, A(\tau)V_1(\tau, s)y \rangle d\tau$$

for $y \in Y, x^* \in X^*$ and $0 \leq s \leq r \leq t \leq T$. In particular, $(\partial/\partial t)V_1(t, s)y$ exists for almost every $t \in [s, T]$ and equals $A(t)V_1(t, s)y$.

$$(e) \quad V_2(t, r)y - V_2(t, s)y = - \int_s^r V_2(t, \tau)A(\tau)y d\tau$$

for $y \in Y$ and $0 \leq s \leq r \leq t \leq T$.

Remarks. 1) In the case where $A(t) \in C_1^{-1}A(t)C_1$ for $t \in [0, T]$, the condition (A_3) is automatically satisfied with $C_2 = C_1$ if the condition (A_1) is satisfied.

2) In the case where $C_1 = C_2 = I$ (the identity operator on X), Theorem 2.1 is [6, Theorem 5.1].

Before proving Theorem 2.1 we prepare three lemmas. Let $s \in [0, T)$ and let $\lambda > 0$ be such that $\lambda\omega_3 < 1$. Set

$$P_{\lambda, k}(s) = \prod_{i=1}^k J_{\lambda}(s + i\lambda) \text{ for } 0 \leq k \leq [(T - s)/\lambda],$$

where $[\]$ denotes the Gaussian bracket and $J_{\lambda}(t) = (1 - \lambda A(t))^{-1} = \lambda^{-1}R(\lambda^{-1} : A(t))$ for $t \in [0, T]$.

Now we define $A_{k, l}$ and $B_{k, l}$ by

$$\begin{cases} A_{k, l}x = P_{\lambda, k}(s)C_1x - P_{\mu, l}(s)C_1x \text{ for } x \in X, \\ B_{k, l}y = \mu(A(s + k\lambda) - A(s + l\mu))P_{\mu, l}(s)C_1y \text{ for } y \in Y. \end{cases}$$

Here we note by the conditions (A_2) and (A_4) that $B_{k, l}$ is well defined because $P_{\mu, l}(s)C_1(Y) \subset Y \subset D(A(t))$ for $t \in [0, T]$.

Using the resolvent identity we obtain by a standard argument

LEMMA 2.2. Let $s \in [0, T)$ and $\lambda, \mu > 0$ be such that $\lambda\omega_3, \mu\omega_3 < 1$. Then, for $y \in Y$ we have

$$(2.1) \quad A_{k, l}y = J_{\mu}(s + k\lambda)(\alpha A_{k-1, l-1}y + \beta A_{k, l-1}y + B_{k, l}y)$$

for $0 \leq k \leq [(T-s)/\lambda]$ and $0 \leq l \leq [(T-s)/\mu]$, where $\alpha = \frac{\mu}{\lambda}$ and $\beta = \frac{\lambda-\mu}{\lambda}$.

Let $s \in [0, T)$ and $\lambda, \mu > 0$ be such that $\lambda\omega_3, \mu\omega_3 < 1$. Let k and j be nonnegative integers. We denote by $H(m, k)$ the set of all operators Q obtained by multiplying k operators $J_\mu(t_i)$ ($i = 1, \dots, k$) in the family $\{J_\mu(s + i\lambda) : i = 1, \dots, m\}$ such that $Q = \prod_{i=1}^k J_\mu(t_i)$ for $0 \leq s + \lambda \leq t_1 \leq \dots \leq t_k \leq s + m\lambda \leq T$; $H(m, 0) = H(0, k) = \{ \text{the identity operator} \}$. By $H(m, k, j)$ we denote the set of all sums of j operators Q_i ($i = 1, \dots, j$) in $H(m, k)$, where in j operators $Q_1, \dots, Q_j$, same operators are allowed to appear repeatedly.

Using the relation (2.1) and then taking account of the definition $H(\cdot, \cdot, \cdot)$ we obtain by a routine calculation the following crucial estimate:

LEMMA 2.3. Let $s \in [0, T)$ and let $\lambda, \mu > 0$ such that $\lambda\omega_3, \mu\omega_3 < 1$. Then, for $y \in Y$ we have

$$\begin{aligned} A_{m,n}y \in & \sum_{i=0}^{(m-1) \wedge n} \alpha^i \beta^{n-i} H\left(m, n, \binom{n}{i}\right) A_{m-i,0}y \\ & + \sum_{i=m}^n \alpha^m \beta^{i-m} H\left(m, i, \binom{i-1}{m-1}\right) A_{0,n-i}y \\ & + \sum_{j=0}^{n-1} \sum_{i=0}^{(m-1) \wedge j} \alpha^i \beta^{j-i} H\left(m, j+1, \binom{j}{i}\right) B_{m-i,n-j}y \end{aligned}$$

for $0 \leq m \leq [(T-s)/\lambda]$ and $0 \leq n \leq [(T-s)/\mu]$, where $\alpha = \frac{\mu}{\lambda}$, $\beta = \frac{\lambda-\mu}{\lambda}$, $l \wedge k = \min(l, k)$ and $\binom{j}{i}$ is the binomial coefficient.

LEMMA 2.4. (I) Let $s \in [0, T)$ and let $\lambda > \mu > 0$ be such that $\lambda\omega_3 < 1$. Then, there exists a positive constant K , depending only on M_i ($i = 1, 2, 3, 4$), such that

(2.2)

$$\begin{aligned} \|C_2 P_{\lambda,m}(s) C_1 y - C_2 P_{\mu,n}(s) C_1 y\| \leq & K \|y\|_Y \left\{ 2((n\mu - m\lambda)^2 + T(\lambda - \mu))^{1/2} \right. \\ & \left. + T(\rho(|n\mu - m\lambda|) + \rho(\delta)) + \frac{T^2}{\delta^2} \rho(T)(\lambda - \mu) \right\} \end{aligned}$$

for $1 \leq m \leq [(T-s)/\lambda]$, $1 \leq n \leq [(T-s)/\mu]$, $y \in Y$ and $\delta > 0$, where $\rho(r) = \sup\{\|A(t) - A(s)\|_{Y \rightarrow X} : t, s \in [0, T], |t - s| \leq r\}$ for $r \geq 0$.

(II) Let $0 \leq r \leq s \leq T$ and let $\lambda > 0$ be such that $\lambda\omega_3 < 1$. Then there exists a positive constant K , depending only on $M_i (i = 2, 3)$, such that

$$(2.3) \quad \|C_2 P_{\lambda,m}(s)C_1 y - C_2 P_{\lambda,m}(r)C_1 y\| \leq KT\|y\|_Y \rho(s-r)$$

for $1 \leq m \leq [(T-s)/\lambda]$ and $y \in Y$.

PROOF: By virtue of Lemma 2.3 we can show (2.2) in the same way as in the proof of [2, Theorem 2.1]. To prove (2.3), let $0 \leq r \leq s \leq T$ and let $\lambda > 0$ be such that $\lambda\omega_3 < 1$. For $1 \leq k \leq [(T-s)/\lambda]$ we define A_k and B_k by

$$\begin{cases} A_k x = P_{\lambda,k}(s)C_1 x - P_{\lambda,k}(r)C_1 x & \text{for } x \in X, \\ B_k y = \lambda(A(s+k\lambda) - A(r+k\lambda))P_{\lambda,k}(s)C_1 y & \text{for } y \in Y. \end{cases}$$

Then, by a simple computation we have

$$\begin{aligned} A_k y &= (J_\lambda(s+k\lambda) - J_\lambda(r+k\lambda))P_{\lambda,k-1}(s)C_1 y \\ &\quad + J_\lambda(r+k\lambda)(P_{\lambda,k-1}(s)C_1 y - P_{\lambda,k-1}(r)C_1 y) \\ &= J_\lambda(r+k\lambda)(A_{k-1}y + B_k y) \end{aligned}$$

for $y \in Y$. By solving this we find

$$A_m y = \sum_{i=1}^m \left(\prod_{k=i}^m J_\lambda(r+k\lambda) \right) B_i y$$

for $y \in Y$ and $1 \leq m \leq [(T-s)/\lambda]$. Therefore, we obtain the desired estimate (2.3) by the conditions (A₂) and (A₃). Q.E.D.

PROOF OF THEOREM 2.1: Let $s, r \in [0, T)$ and let $\lambda > \mu > 0$ be such that $\lambda\omega_3 < 1$. Let m and n be integers such that $0 \leq s + m\lambda, r + n\mu \leq T$ and let $y \in Y$. If $s \leq r$ then $0 \leq s + n\mu \leq T$, so that $P_{\mu,n}(s)$ is well defined. Similarly, $P_{\lambda,m}(r)$ is well defined if $s \geq r$. Therefore, $C_2 P_{\lambda,m}(s)C_1 y - C_2 P_{\mu,n}(r)C_1 y$ can be written as

$$\begin{cases} C_2 P_{\lambda,m}(s)C_1 y - C_2 P_{\mu,n}(s)C_1 y + (C_2 P_{\mu,n}(s)C_1 y - C_2 P_{\mu,n}(r)C_1 y) & \text{if } s \leq r \\ C_2 P_{\lambda,m}(s)C_1 y - C_2 P_{\lambda,m}(r)C_1 y + (C_2 P_{\lambda,m}(r)C_1 y - C_2 P_{\mu,n}(r)C_1 y) & \text{if } s \geq r. \end{cases}$$

Applying Lemma 2.4 to this we see that there exists a positive constant K , depending only on $M_i (i = 1, 2, 3, 4)$, such that

$$\begin{aligned} & \|C_2 P_{\lambda, m}(s) C_1 y - C_2 P_{\mu, n}(r) C_1 y\| \\ & \leq K \|y\|_Y \left\{ 2((n\mu - m\lambda)^2 + T(\lambda - \mu))^{1/2} + T(\rho(|n\mu - m\lambda|)) \right. \\ & \quad \left. + \rho(\delta) + \rho(|r - s|) + \frac{T^2}{\delta^2} \rho(T)(\lambda - \mu) \right\} \end{aligned}$$

for $\delta > 0$ and $y \in Y$. Since $C_1(Y)$ is dense in X and $\|C_2 P_{\lambda_n, n}(s_n)\| \leq M_3$ for $n \geq 1$ it follows that

$$(2.4) \quad V_2(t, s)x = \lim_{n \rightarrow \infty} C_2 \left(\prod_{i=1}^n J_{\lambda_n}(s_n + i\lambda_n) \right) x$$

exists for $x \in X$ if $\{s_n\}$ is a sequence of nonnegative numbers with $\lim_{n \rightarrow \infty} s_n = s$ and $\{\lambda_n\}$ is a sequence such that $0 \leq s_n + n\lambda_n \leq T$ and $s_n + n\lambda_n \rightarrow t - s > 0$ as $n \rightarrow \infty$. Here we have used the fact that $\rho(\delta) \rightarrow 0$ as $\delta \rightarrow 0+$. We note that the limit is independent of $\{s_n\}$ and $\{\lambda_n\}$.

Let $\{s_n\}$ be a sequence of nonnegative numbers such that $\lim_{n \rightarrow \infty} s_n = s$ and let $\{\lambda_n\}$ be a sequence such that $0 \leq s_n + n\lambda_n \leq T$ and $s_n + n\lambda_n \rightarrow t - s > 0$ as $n \rightarrow \infty$. We then define $V_1^{(n)}(t, s)$ on X by

$$V_1^{(n)}(t, s) = \begin{cases} C_1 & \text{for } t = s, \\ \left(\prod_{i=1}^n J_{\lambda_n}(s_n + i\lambda_n) \right) C_1 & \text{for } s < t. \end{cases}$$

Then, by the condition (A₂) we have

$$V_1^{(n)}(t, s)(Y) \subset Y \text{ and } \|V_1^{(n)}(t, s)\|_Y \leq M_2 \text{ for } 0 \leq s \leq t \leq T \text{ and } n \geq 1.$$

We now show that for $y \in Y$ and $y^* \in Y^*$, $\langle y^*, V_1^{(n)}(t, s)y \rangle$ is convergent. Let $\{n_k\}$ be any subsequence of $\{n\}$. Since Y is reflexive there exists a subsequence $\{n'_k\}$ of $\{n_k\}$ and $y(t, s) \in Y$, depending upon $\{n'_k\}$, such that

$$\langle y^*, V_1^{(n'_k)}(t, s)y \rangle \rightarrow \langle y^*, y(t, s) \rangle$$

for $y^* \in Y^*$ as $n \rightarrow \infty$. In particular, for $x^* \in X^*$ we have

$$\langle C_2^* x^*, V_1^{(n'_k)}(t, s)y \rangle \rightarrow \langle C_2^* x^*, y(t, s) \rangle = \langle x^*, C_2 y(t, s) \rangle$$

as $n \rightarrow \infty$, since $C_2^* x^*|_Y \in Y^*$. On the other hand, by (2.4) we obtain for $x^* \in X^*$,

$$\langle C_2^* x^*, V_1^{(n'_k)}(t, s)y \rangle = \langle x^*, C_2 V_1^{(n'_k)}(t, s)y \rangle \rightarrow \langle x^*, V_2(t, s)C_1 y \rangle$$

as $n \rightarrow \infty$. Hence $C_2 y(t, s) = V_2(t, s)C_1 y$, so that $y(t, s)$ is independent of $\{n'_k\}$. Therefore it is proved that

$$\lim_{n \rightarrow \infty} \langle y^*, V_1^{(n)}(t, s)y \rangle = \langle y^*, C_2^{-1} V_2(t, s)C_1 y \rangle$$

for $y \in Y$. By this together with the fact that $x^*|_Y \in Y^*$ we have for $x^* \in X^*$,

$$\langle x^*, C_2^{-1} V_2(t, s)C_1 y \rangle = \lim_{n \rightarrow \infty} \langle x^*, V_1^{(n)}(t, s)y \rangle \text{ for } y \in Y.$$

Hence

$$\|C_2^{-1} V_2(t, s)C_1 y\| \leq M_1 \|y\|$$

for $y \in Y$ and $0 \leq s \leq t \leq T$. Since Y is dense in X we see by the closed graph theorem that $C_2^{-1} V_2(t, s)C_1 \in B(X)$ and $\|C_2^{-1} V_2(t, s)C_1\| \leq M_1$ for $0 \leq s \leq t \leq T$.

We now define $V_1(t, s)$ on X by

$$V_1(t, s) = C_2^{-1} V_2(t, s)C_1 \text{ for } 0 \leq s \leq t \leq T.$$

Then, it follows from the fact which has been proved above that $\|V_1(t, s)\| \leq M_1$, $V_1(t, s)(Y) \subset Y$, $\|V_1(t, s)\|_Y \leq M_2$ and $C_2 V_1(t, s) = V_2(t, s)C_1$ for $0 \leq s \leq t \leq T$. Moreover, we have

$$\lim_{n \rightarrow \infty} \left\langle y^*, \left(\prod_{i=1}^n J_{\lambda_n}(s_n + i\lambda_n) \right) C_1 y \right\rangle = \langle y^*, V_1(t, s)y \rangle$$

for $y \in Y$ and $y^* \in Y^*$ if $\{s_n\}$ is a sequence of nonnegative numbers such that $\lim_{n \rightarrow \infty} s_n = s$ and $\{\lambda_n\}$ is a sequence such that $0 \leq s_n + n\lambda_n \leq T$ and $s_n + n\lambda_n \rightarrow t - s > 0$ as $n \rightarrow \infty$.

To prove that for $x \in X$, $(t, s) \rightarrow V_1(t, s)x$ is continuous on Δ , since Y is dense in X and $\|V_1(t, s)\| \leq M_1$ on Δ it suffices to show that

$$(2.5) \quad \|V_1(t, s)y - V_1(\tau, s)y\| \leq K(t - \tau)\|y\|_Y$$

for $y \in Y$ and $0 \leq s \leq \tau \leq t \leq T$,

$$(2.6) \quad \|V_1(t, s+h)y - V_1(t, s)y\| \leq Kh\|y\|_Y$$

for $y \in Y$ and $0 \leq s \leq s+h \leq t \leq T$.

To prove (2.5), let $y \in Y$ and $0 \leq s \leq \tau \leq t \leq T$ and let $\lambda > 0$ be such that $\lambda\omega_3 < 1$. If n and m be integers such that $m < n \leq [(T-s)/\lambda]$ then

$$(2.7) \quad \begin{aligned} & \langle x^*, P_{\lambda, n}(s)C_1y - P_{\lambda, m}(s)C_1y \rangle \\ &= \left\langle x^*, \sum_{k=m}^{n-1} (P_{\lambda, k+1}(s)C_1y - P_{\lambda, k}(s)C_1y) \right\rangle \\ &= \left\langle x^*, \lambda \sum_{k=m}^{n-1} A(s + (k+1)\lambda)P_{\lambda, k+1}(s)C_1y \right\rangle \text{ for } x^* \in X^*, \end{aligned}$$

from which it follows that

$$\begin{aligned} & |\langle x^*, P_{\lambda, n}(s)C_1y - P_{\lambda, m}(s)C_1y \rangle| \\ & \leq \|x^*\| \lambda(n-m) \cdot \sup\{\|A(t)\|_{Y \rightarrow X} : t \in [0, T]\} \cdot M_2\|y\|_Y \end{aligned}$$

for $x^* \in X^*$. Setting $n = [(t-s)/\lambda]$ and $m = [(\tau-s)/\lambda]$, and then letting $\lambda \rightarrow \infty$ we obtain the desired estimate (2.5).

To prove (2.6) let $y \in Y$ and $0 \leq s < s+h < t \leq T$, and choose a sequence $\{k(n)\}$ of integers such that $k(n)h/n \leq t - (s+h)$ and $k(n)h/n \rightarrow t - (s+h)$ as $n \rightarrow \infty$. Then, since

$$(2.8) \quad \begin{aligned} & \left(\prod_{i=1}^{k(n)} J_{h/n}(s+h+ih/n) \right) y - \left(\prod_{i=1}^{n+k(n)} J_{h/n}(s+ih/n) \right) y \\ &= \sum_{j=1}^n \left\{ \left(\prod_{i=j+1}^{n+k(n)} J_{h/n}(s+ih/n) \right) y - \left(\prod_{i=j}^{n+k(n)} J_{h/n}(s+ih/n) \right) y \right\} \\ &= -(h/n) \sum_{j=1}^n \left(\prod_{i=j}^{n+k(n)} J_{h/n}(s+ih/n) \right) A(s+jh/n)y, \end{aligned}$$

it follows from the conditions (A₁) and (A₄) that

$$|\langle x^*, P_{h/n, k(n)}(s+h)C_1y - P_{h/n, n+k(n)}(s)C_1y \rangle| \leq hM_1M_4\|y\|_Y\|x^*\|$$

for $x^* \in X^*$. Passing to the limit as $n \rightarrow \infty$ we obtain (2.6).

The strongly continuity of $V_2(t, s)$ immediately follows from the strongly continuity of $V_1(t, s)$ and the relation that $C_2V_1(t, s) = V_2(t, s)C_1$, since $C_1(X)$ is dense in Y and $\|V_2(t, s)\| \leq M_3$ on Δ .

Since Y is reflexive, using the strongly continuity of $V_1(t, s)$ together with the facts that $V_1(t, s)(Y) \subset Y$ and $\|V_1(t, s)\|_Y \leq M_2$ on Δ we see by a standard argument that for $y \in Y$ and $y^* \in Y^*$, $(t, s) \rightarrow \langle y^*, V_1(t, s)y \rangle$ is continuous for $0 \leq s \leq t \leq T$.

To prove that $\{V_1(t, s) : 0 \leq s \leq t \leq T\}$ has the property (d), let $y \in Y$, $x^* \in X^*$ and $0 \leq s \leq r < t \leq T$. Setting $n = [(t-s)/\lambda]$ and $m = [(r-s)/\lambda]$ in (2.7) we have

$$\begin{aligned} & \langle x^*, P_{\lambda, [(t-s)/\lambda]}(s)C_1y - P_{\lambda, [(r-s)/\lambda]}(s)C_1y \rangle \\ &= \left\langle x^*, \sum_{k=[(r-s)/\lambda]}^{[(t-s)/\lambda]-1} \int_{s+k\lambda}^{s+(k+1)\lambda} A(s + ([(\tau-s)/\lambda] + 1)\lambda) P_{\lambda, [(\tau-s)/\lambda]+1}(s)C_1y \, d\tau \right\rangle \\ &= \int_{s+[(r-s)/\lambda]\lambda}^{s+[(t-s)/\lambda]\lambda} \langle \tilde{A}(s + ([(\tau-s)/\lambda] + 1)\lambda)^* x^*, P_{\lambda, [(\tau-s)/\lambda]+1}(s)C_1y \rangle d\tau, \end{aligned}$$

where $\tilde{A}(t)^* : X^* \rightarrow Y^*$ denotes the adjoint of the restriction $\tilde{A}(t)$ of $A(t)$ to Y . The condition (A₄) implies that $t \rightarrow \tilde{A}(t)^*$ is continuous in the $B(X^*, Y^*)$ norm; thus passing to the limit as $\lambda \rightarrow \infty$ we see by Lebesgue's convergence theorem that

$$\langle x^*, V_1(t, s)y - V_1(r, s)y \rangle = \int_r^t \langle \tilde{A}(\tau)^* x^*, V_1(\tau, s)y \rangle d\tau.$$

This shows that the property (d) is satisfied.

We next show that $\{V_2(t, s) : 0 \leq s \leq t \leq T\}$ has the property (e). Let $0 \leq s < s+h < t \leq T$ and choose a sequence $\{k(n)\}$ of integers such that

$k(n)h/n \leq t - (s + h)$ and $k(n)h/n \rightarrow t - (s + h)$ as $n \rightarrow \infty$. By (2.8) we have

$$\begin{aligned} & C_2 P_{h/n, k(n)}(s + h)y - C_2 P_{h/n, n+k(n)}(s)y \\ &= - \sum_{j=1}^n \int_{s+(j-1)h/n}^{s+jh/n} C_2 P_{h/n, n+k(n)-j+1}(s + (j-1)h/n) A(s + jh/n)y \, dr \\ &= - \int_s^{s+h} C_2 P_{h/n, n+k(n)-r(n)}(s + r(n)h/n) A(s + (r(n) + 1)h/n)y \, dr \end{aligned}$$

for $y \in Y$, where $r(n) = [(r - s)/(h/n)]$. Letting $n \rightarrow \infty$ in this equality we see that the property (e) is satisfied.

Suppose that $(\{W_1(t, s)\}, \{W_2(t, s)\})$ is a pair of strongly continuous families of bounded linear operators defined on the triangle Δ with the properties (a) - (e). Then, by the properties (d) and (e) we see that for $y \in Y$, the function $r \rightarrow V_2(t, r)W_1(r, s)y$ is Lipschitz continuous and $(\partial/\partial r)V_2(t, r)W_1(r, s)y = 0$ for almost every $r \in [s, T]$. Integrating this from s to t we obtain $C_2 W_1(t, s)y = V_2(t, s)C_1 y$ for $y \in Y$. By the property (a), $W_2(t, s)$ is equal to $V_2(t, s)$ on the dense subspace $C_1(Y)$ of X , so that $(\{V_1(t, s)\}, \{V_2(t, s)\})$ is the only pair of strongly continuous families of bounded linear operators defined on the triangle Δ with the properties (a) - (e). Q.E.D.

Definition 2.1. A function $u(\cdot; s, x)$ on $[s, T]$ is a *strong solution of $(DE)_s$* if

- (i) $u(\cdot; s, x) \in A^{1,1}(s, T; X)$,
- (ii) $u(\cdot; s, x)$ satisfies $(DE)_s$ almost everywhere.

Here we denote by $A^{k,p}(a, b; X)$ the space of all absolutely continuous functions $u : [a, b] \rightarrow X$ for which $d^j u/dt^j$ exist (and are defined almost everywhere) for $j = 1, 2, \dots, k$ such that $d^j u/dt^j$, $j = 1, 2, \dots, k-1$, are all absolutely continuous and $d^k u/dt^k \in L^p(a, b; X)$.

Existence and uniqueness of the strong solutions of the time-dependent differential equation $(DE)_s$ is provided by

THEOREM 2.5. *If the family $\{A(t) : t \in [0, T]\}$ of closed linear operators in X satisfies the conditions $(A_1) - (A_4)$ then, for every initial data $x \in C_1(Y)$ the $(DE)_s$ has a unique strong solution satisfying $u(t; s, x) \in Y$ for $t \in [s, T]$ and $\sup\{\|u(t; s, x)\|_Y : t \in [s, T]\} < \infty$.*

PROOF: By Theorem 2.1 there exists a unique pair $(\{V_1(t, s)\}, \{V_2(t, s)\})$ of strongly continuous families of bounded linear operators defined on the triangle $\Delta = \{(t, s) : 0 \leq s \leq t \leq T\}$ with the properties (a) - (e). Let $x \in C_1(Y)$ and set $u(t; s, x) = V_1(t, s)C_1^{-1}x$ for $0 \leq s \leq t \leq T$. Then, it is easy to see that $u(t; s, x)$ is a strong solution of $(DE)_s$ satisfying $u(t; s, x) \in Y$ for $t \in [s, T]$ and $\sup\{\|u(t; s, x)\|_Y : t \in [s, T]\} < \infty$. To prove the uniqueness of the solutions, let $v(t; s, x)$ be a strong solution of $(DE)_s$ satisfying $v(t; s, x) \in Y$ for $t \in [s, T]$ and $\sup\{\|v(t; s, x)\|_Y : t \in [s, T]\} < \infty$. Then, we deduce from the property (e) that $r \rightarrow V_2(t, r)(u(r; s, x) - v(r; s, x))$ is absolutely continuous on $[s, T]$ and

$$(\partial/\partial r)V_2(t, r)(u(r; s, x) - v(r; s, x)) = 0$$

for almost every $r \in [s, T]$. Integrating this equality from s to t we have

$$C_2(u(t; s, x) - v(t; s, x)) = 0,$$

which shows that $u(t; s, x) = v(t; s, x)$ for $t \in [s, T]$, since C_2 is injective. Q.E.D.

We next consider the second order differential equation in a reflexive Banach space X

$$(DE)_s^2 \quad \begin{cases} u''(t) = Au(t) + B(t)u(t) & \text{for } t \in [s, T] \\ u(s) = x, \quad u'(s) = y, \end{cases}$$

where A is the infinitesimal generator of a cosine family and $\{B(t) : t \in [0, T]\}$ is a family of linear operators in X satisfying the following conditions:

(B₁) $D(A) \subset D(B(t))$ for $t \in [0, T]$.

(B₂) There are constants $M \geq 0$ and $\omega \geq 0$ such that $\{\lambda^2 : \lambda > \omega\} \subset \rho(A)$, for $t \in [0, T]$ $B(t)R(\lambda^2 : A)$ is strongly infinitely differentiable in $\lambda > \omega$ and satisfies

$$\|(1/n!)(\lambda - \omega)^{n+1}(d/d\lambda)^n B(t)R(\lambda^2 : A)x\| \leq M\|x\|$$

for $x \in X, \lambda > \omega$ and $n = 0, 1, \dots$.

(B₃) $\lim_{t \rightarrow s} \sup\{\|B(t)x - B(s)x\| : x \in D(A), \|x\| + \|Ax\| \leq 1\} = 0$.

(B₄) There exists $\lambda_0 > \omega$ such that $(\lambda_0^2 - A)B(t)R(\lambda_0^2 : A) = B(t) + P(t)$, where $\{P(t) : t \in [0, T]\}$ is a strongly continuous family of bounded linear operators on X .

Definition 2.2. A function $u(\cdot; s, x, y)$ on $[s, T]$ is a *strong solution of $(DE)_s^2$* if

(i) $u(\cdot; s, x, y) \in A^{2,1}(s, T; X)$,

(ii) $u(\cdot; s, x, y)$ satisfies $(DE)_s^2$ almost everywhere.

Without proof we state the existence and uniqueness theorem of the strong solutions of the second order differential equation $(DE)_s^2$ which is obtained by applying Theorem 2.5 with $A(t) = \begin{pmatrix} 0 & 1 \\ A + B(t) & 0 \end{pmatrix}$ and $C_1 = C_2 = \begin{pmatrix} 0 & 1 \\ A - \lambda_0^2 & 0 \end{pmatrix}^{-1}$.

THEOREM 2.6. Assume that A is the infinitesimal generator of a cosine family and $\{B(t) : t \in [0, T]\}$ is a family of linear operators in X satisfying the conditions (B₁) - (B₄). Then, for every initial data $x \in D(A)$ and $y \in D(A)$ the $(DE)_s^2$ has a unique strong solution $u(t; s, x, y)$ such that $u(t; s, x, y) \in D(A)$ for $t \in [s, T]$ and $\sup\{\|Au(t; s, x, y)\| : t \in [s, T]\} < \infty$.

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