

Invariant subspaces and other animals

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An “invariant subspace” T for a linear operator T on a vector space is, precisely, a linear subspace Y for which

$$0.1 \quad T(Y) \subseteq Y \subseteq X .$$

The point of an invariant subspace is the *restriction* operator

$$0.2 \quad T_Y : Y \rightarrow Y ,$$

where of course $T_Y(y) = Ty$ for each $y \in Y$. The relationship between T and its restriction T_Y involves also the induced *quotient*

$$0.3 \quad T'_Y : X/Y \rightarrow X/Y ,$$

where $T'_Y(x + Y) = (Tx) + Y$ for each $x \in X$. Now the “three space property” of invertibility says that if any two of the three operators T , T_Y and T'_Y is invertible then so is the third. Recalling that invertibility is the same as one one and onto, this follows from the six implications ([1] Theorems 3.11.1, 3.11.2)

$$0.4 \quad T_Y, T'_Y \text{ one one} \implies T \text{ one one} \implies T_Y \text{ one one} ;$$

$$0.5 \quad T_Y, T'_Y \text{ onto} \implies T \text{ onto} \implies T_Y \text{ onto} ;$$

$$0.6 \quad T \text{ one one}, T_Y \text{ onto} \implies T'_Y \text{ one one} ;$$

$$0.7 \quad T \text{ onto}, T'_Y \text{ one one} \implies T_Y \text{ onto} .$$

All this remains valid for bounded operators on Banach spaces, when of course we only consider closed invariant subspaces. In terms of the *spectrum*

$$\sigma(T) = \{\lambda \in \mathbf{C} : T - \lambda I \text{ not invertible}\} ,$$

the spectrum of each of the operators T , T_Y and T'_Y is contained in the union of the other two. Equivalently

$$0.8 \quad \sigma(T) \subseteq \sigma(T_Y) \cup \sigma(T'_Y) \subseteq \sigma(T) \cup (\sigma(T_Y) \cap \sigma(T'_Y)) .$$

This leads to a new kind of invariant subspace ([3] (2.3)):

1. Definition An invariant subspace $Y \subseteq X$ is called *spectrally invariant* for T if

$$1.1 \quad \sigma(T_Y) \cap \sigma(T'_Y) = \emptyset ,$$

in which case also

$$1.2 \quad \sigma(T) = \sigma(T_Y) \cup \sigma(T'_Y) .$$

Of course (0.2) is a consequence of (1.1) and (0.8). For bounded operators on Banach spaces, spectrally invariant subspaces are both *reducing* and *hyperinvariant*: there is a projection $P = P^2 \in B(X)$ for which

$$1.3 \quad ST = TS \implies SP = PS$$

with

$$1.4 \quad Y = P(X) .$$

Naturally the projection comes from the splitting of the spectrum via functional calculus ([1] Definition 9.7.1):

$$1.5 \quad P = f(T) \equiv \frac{1}{2\pi i} \oint_{\sigma(T)} (zI - T)^{-1} dz$$

with the function f given by the characteristic function of the restriction spectrum,

$$1.6 \quad f = \chi_K \text{ where } K = \sigma(T_Y) .$$

Since both the range $P(X)$ and its complement $P^{-1}(0)$ are invariant under T it is clear that $P(X)$ is a reducing subspace for T ; since by (1.5) the range of P is invariant under everything which commutes with T it is also hyperinvariant. It is also clear that the restriction and the quotient of T with respect to $P(X)$ are the same as with respect to Y : with a little more work it turns out that Y and $P(X)$ are the same.

Intermediate between the invariant and the hyperinvariant are two further kinds of invariant subspace ([3] Definition 1):

2. Definition The invariant subspace $Y \subseteq X$ is called *holomorphically invariant* for T if

$$2.1 \quad f \in \text{Holo}(\sigma(T)) \implies f(T)Y \subseteq Y ,$$

and *comm square invariant* for T if

$$2.2 \quad S \in \text{comm}^2(T) \implies SY \subseteq Y .$$

Evidently

$$\begin{aligned} &\text{spectrally invariant} \implies \text{hyperinvariant} \implies \text{comm square invariant} \\ &\implies \text{holomorphically invariant} \implies \text{invariant} ; \end{aligned}$$

we claim that none of these implications is reversible. Our counterexamples will all be built from the *forward* and the *backward shifts* u and v , and the *standard weight* w , where for each $x = (x_1, x_2, x_3, \dots) \in E = \ell_p$ with $p = 2$ and each $n \in \mathbf{N}$

$$2.3 \quad (ux)_1 = 0, (ux)_{n+1} = x_n ; (vx)_n = x_{n+1} ; (wx)_n = (1/n)x_n .$$

The spectrum σ , the onto spectrum τ^{right} and the eigenvalues π^{left} are given by

$$2.4 \quad \tau^{\text{right}}(v) = \partial \mathbf{D} \subseteq \mathbf{D} = \sigma(v) = \sigma(u) = \tau^{\text{right}}(u) ,$$

$$2.5 \quad \sigma(w) = \mathbf{O} \cup \mathbf{N}^{-1} ; \sigma(wu) = \mathbf{O} \equiv \{0\}$$

and

$$2.6 \quad \pi^{\text{left}}(u) = \emptyset ; \pi^{\text{left}}(v) = \text{int } \mathbf{D} ,$$

where $\mathbf{D} = \{|z| \leq 1\} \subseteq \mathbf{C}$ is the closed unit disc. The eigenvalues of the backward shift v all have one dimensional eigenspaces:

$$|\lambda| < 1 \implies 1 - \lambda u \text{ invertible and } v - \lambda = v(1 - \lambda u) ,$$

giving

$$2.7 \quad v^{-1}(0) = (1 - uv)(E) = \mathbf{C}\delta_1 = \{(\lambda, 0, 0, \dots) : \lambda \in \mathbf{C}\}$$

and

$$2.8 \quad (v - \lambda)^{-1}(0) = (1 - \lambda u)^{-1}v^{-1}(0) = (1 - \lambda u)^{-1}(1 - uv)(E) .$$

In fact our examples are on the direct sum $X = E \oplus E$ of two copies of $E = \ell_p = \ell_2$, and appear as *operator matrices*.

Not every invariant subspace is holomorphically invariant ([3] Example 1):

3. Example With

$$3.1 \quad U = \begin{pmatrix} u & 1-uv \\ 0 & v \end{pmatrix}, \quad V = \begin{pmatrix} v & 0 \\ 1-uv & u \end{pmatrix}, \quad P = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

and

$$3.2 \quad Y = P(X) \subseteq X,$$

$$3.3 \quad U(Y) \subseteq Y$$

but not

$$3.4 \quad U^{-1}Y = V(Y) \subseteq Y.$$

Not every comm square invariant subspace is hyperinvariant ([3] Example 2):

4. Example With

$$4.1 \quad \mathbf{u} = \begin{pmatrix} u & 0 \\ 0 & u \end{pmatrix}, \quad \mathbf{v} = \begin{pmatrix} v & 0 \\ 0 & v \end{pmatrix},$$

$$4.2 \quad P = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}; \quad Q = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},$$

we have

$$4.3 \quad \mathbf{u}P - P\mathbf{u} = \mathbf{v}P - P\mathbf{v} = O$$

and

$$4.4 \quad \mathbf{u}Q - Q\mathbf{u} = \mathbf{v}Q - Q\mathbf{v} = O,$$

but

$$4.5 \quad PQ \neq QP,$$

so that

$$4.6 \quad P \in \text{comm}(\mathbf{v}) \setminus \text{comm}^2(\mathbf{v})$$

and

$$4.7 \quad P \in \text{comm}(\mathbf{u}) \setminus \text{comm}^2(\mathbf{u})$$

and

$$4.8 \quad Y = P(X) \text{ invariant under } \mathbf{v}, \mathbf{u}, P \text{ but not } Q.$$

Not every holomorphically invariant subspace is comm square invariant ([3] Example 3). This is the most delicate of our examples: we need in particular to see that not everything in the double commutant need be a holomorphic function [2]:

5. Example With

$$5.1 \quad T = \begin{pmatrix} u & 0 \\ 0 & 1-u \end{pmatrix}, \quad S = \begin{pmatrix} v & 0 \\ 0 & 1-v \end{pmatrix}, \quad P = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

we have

$$5.2 \quad P \in \text{comm}^2(S) \setminus \text{Holo}(S)$$

and

$$5.3 \quad P \in \text{comm}^2(T) \setminus \text{Holo}(T) .$$

Also

$$\begin{aligned} W &= (S - \lambda I)^{-1}(0) = \begin{pmatrix} (v - \lambda)^{-1}(0) \\ (1 - v - \lambda)^{-1}(0) \end{pmatrix} \\ &= \begin{pmatrix} (1 - \lambda u)^{-1} & 0 \\ 0 & (1 - (1 - \lambda)u)^{-1} \end{pmatrix} \begin{pmatrix} (1 - uv)E \\ (1 - uv)E \end{pmatrix} \end{aligned}$$

is (hyper)invariant under S , and

$$5.4 \quad Y = \begin{pmatrix} (1 - \lambda u)^{-1} & 0 \\ 0 & (1 - (1 - \lambda)u)^{-1} \end{pmatrix} Y' \text{ where } Y' = \begin{pmatrix} 1 - uv \\ 1 - uv \end{pmatrix} E$$

is (holomorphically) invariant under S but not invariant under P .

Indeed since the diagonal elements of S do not have disjoint spectrum, P cannot ([2] Theorem 1) be a holomorphic function of S :

$$\sigma(v) \cap \sigma(v - 1) \neq \emptyset \implies P \notin \text{Holo}(S) .$$

On the other hand

$$\begin{aligned} \begin{pmatrix} a & m \\ n & b \end{pmatrix} \in \text{comm}(S) &\implies m(1 - v) - vm = (1 - v)n - nv = 0 \\ &\implies m = n = 0 \iff \begin{pmatrix} a & m \\ n & b \end{pmatrix} \in \text{comm}(P) \end{aligned}$$

and there is ([2] Theorem 2) implication

$$\begin{aligned} x = vx + xv &\implies (1 - v)x(1 - u^n v^n) = 0 \quad (n \in \mathbf{N}) \\ &\implies x = xuv = xu^2 v^2 = xu^3 v^3 = \dots \implies x = 0 . \end{aligned}$$

Not every hyperinvariant reducing subspace is spectrally invariant ([3] Example 4):

6. Example *With*

6.1

$$R = \begin{pmatrix} v & 0 \\ 0 & wu \end{pmatrix},$$

the null space $R^{-1}(0)$ is hyperinvariant and reducing for R , but not spectrally invariant.

Alternatively

6.2

$$W = (S - \lambda I)^{-1}(0)$$

is hyperinvariant and reducing for S but not spectrally invariant.

Neither hyperinvariance nor reducing implies the other ([3] Example 5):

7. Example *The subspace*

$$P(X) = E \oplus O$$

is comm square invariant and reducing but not hyperinvariant for $\mathbf{u}$ and for $\mathbf{v}$, and is hyperinvariant but not reducing for Q .

Alternatively, on ℓ_∞ the closure of the range of w is hyperinvariant but ([1] Theorem 5.10.2) uncomplemented.

We remark that each of the operators $\mathbf{u}$ and $\mathbf{v}$ satisfies the condition (1.2) but not the disjointness (1.1).

References

- [1] R.E.Harte, *Invertibility and singularity*, Dekker 1988
- [2] R.E. Harte, *Block diagonalization in Banach algebras*, Proc. Amer. Math. Soc. **129** (2000) 181-190
- [3] S.V. Djordjević, R.E. Harte and D.A. Larson, *Partially hyperinvariant subspaces*, Operators and Matrices (to appear).