

Singular Values of Differences of Positive Semidefinite Matrices¹

Xingzhi Zhan

Graduate School of Information Sciences, Tohoku University

1 Introduction

Let M_n be the space of $n \times n$ complex matrices. For simplicity we treat matrices here, but all our results hold for compact operators on a Hilbert space. Suppose $A, B \in M_n$ are positive semidefinite. We shall study the relations between the singular values of

$$A - B \quad \text{and} \quad \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix},$$

and those of

$$A - |z|B, \quad A + zB, \quad \text{and} \quad A + |z|B$$

where z is a complex number.

A norm $||| \cdot |||$ on M_n is called *unitarily invariant* if $|||UAV||| = |||A|||$ for all A and all unitary U, V . Every unitarily invariant norm is a symmetric gauge function of the singular values. See [3, 7]. We always denote the singular values of A by $s_1(A) \geq \cdots \geq s_n(A)$, and put $s(A) \equiv (s_1(A), \dots, s_n(A))$. Familiar examples of unitarily invariant norms are the Ky Fan k -norms defined by $||A||_{(k)} = \sum_1^k s_j(A)$ and the Schatten p -norms: $||A||_p = (\sum_1^n s_j^p(A))^{1/p}$, $p \geq 1$. Note that $|| \cdot ||_\infty$ is just the operator (spectral) norm and $|| \cdot ||_2$ is the Frobenius norm.

¹This paper is to appear in SIAM J. Matrix Anal. Appl.

A unitarily invariant norm may be considered as defined on M_n for all orders n by the rule

$$|||A||| = ||| \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix} |||,$$

i.e., adding or deleting zero singular values does not affect the value of the corresponding symmetric gauge function.

Given a real vector $x = (x_i) \in \mathbb{R}^n$, rearrange its components as $x_{[1]} \geq \cdots \geq x_{[n]}$. For $x = (x_i), y = (y_i) \in \mathbb{R}^n$, if

$$\sum_1^k x_{[i]} \leq \sum_1^k y_{[i]}, \quad k = 1, 2, \dots, n,$$

then we say x is *weakly majorized* by y , denoted $x \prec_w y$. If the components of x and y are nonnegative and

$$\prod_1^k x_{[i]} \leq \prod_1^k y_{[i]}, \quad k = 1, 2, \dots, n$$

we say x is *weakly log-majorized* by y , denoted $x \prec_{w \log} y$. See [6] for a discussion of this topic.

Denote the block diagonal matrix $\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$ by $A \oplus B$. Bhatia and Kittaneh [4, Remark 5] observed that if $A, B \in M_n$ are positive semidefinite then

$$|||A - B||| \leq |||A \oplus B||| \tag{1.1}$$

for every unitarily invariant norm. By the Fan dominance principle [3, 7], (1.1) is equivalent to $s(A - B) \prec_w s(A \oplus B)$. We shall show that in fact each singular value of $A - B$ is not greater than the corresponding singular value of $A \oplus B$.

In another paper, Bhatia and Kittaneh [5, Thm 1] proved that for positive semidefinite $A, B \in M_n$ and any complex number z

$$|||A - |z|B||| \leq |||A + zB||| \leq |||A + |z|B||| \tag{1.2}$$

for all unitarily invariant norms. Again (1.2) is equivalent to

$$s(A - |z|B) \prec_w s(A + zB) \prec_w s(A + |z|B).$$

We shall prove that the corresponding weak log-majorizations hold. Since weak log-majorization implies weak majorization [6,7], our result strengthens (1.2).

2 Main Results

Our first result sharpens (1.1).

Theorem 1 *Let $A, B \in M_n$ be positive semidefinite. Then*

$$s_j(A - B) \leq s_j(A \oplus B), \quad j = 1, 2, \dots, n. \quad (2.1)$$

The following result sharpens (1.2).

Theorem 2 *Let $A, B \in M_n$ be positive semidefinite. Then for any complex number z*

$$s(A - |z|B) \prec_{wlog} s(A + zB) \prec_{wlog} s(A + |z|B). \quad (2.2)$$

The special case $z = i = \sqrt{-1}$ of Theorem 2 says

$$s(A - B) \prec_{wlog} s(A + iB) \prec_{wlog} s(A + B). \quad (2.3)$$

It has been proved in [2] that for positive A, B and $p > 1$

$$s(A^p + B^p) \prec_w s((A + B)^p). \quad (2.4)$$

When $p \geq 2$, the above relation is refined as follows:

$$s(A^p + B^p) \prec_w s((A^2 + B^2)^{p/2}) \prec_w s(|A + iB|^p) \prec_{wlog} s((A + B)^p). \quad (2.5)$$

The first relation in (2.5) follows from (2.4) and the third relation follows from (2.3). To see the second relation let $T = A + iB$. This is the *Cartesian decomposition*. From $A^2 + B^2 = (T^*T + TT^*)/2$ we get

$$s(A^2 + B^2) \prec_w s(|A + iB|^2).$$

Note that $f(t) = t^{p/2}$ is convex and increasing on $[0, \infty)$. By a majorization principle [3, 7], applying this f to the preceding weak majorization yields the second relation in (2.5).

From (2.3) and the results in [1] and [2] it follows that for $0 < p \leq 1$,

$$\begin{aligned} s(A^p - B^p) \prec_w s(|A - B|^p) \prec_{wlog} s(|A + iB|^p) &\prec_{wlog} s((A + B)^p) \\ &\prec_w s(A^p + B^p). \end{aligned}$$

One might wonder whether the weak majorization (2.4) can be replaced by the stronger log-majorization. The answer is no, even for $p = 2$. Consider the example

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

We have $\det(A^2 + B^2) = 2 > 1 = \det[(A + B)^2]$.

Recently we have generalized Theorem 1 and the second majorization result in Theorem 2 to the case of τ -measurable operators affiliated with a semifinite von Neumann algebra.

Acknowledgement This work was done while the author was at the Graduate School of Information Sciences, Tohoku University as a postdoctoral fellow of

the Japan Society for the Promotion of Science. He thanks JSPS for the support and Professor F. Hiai for helpful discussions.

REFERENCES

1. T. Ando, Comparison of norms $|||f(A) - f(B)|||$ and $|||f(|A - B|)|||$, Math. Z., 197(1988) 403-409.
2. T. Ando and X. Zhan, Norm inequalities related to operator monotone functions, Math. Ann., 315(1999) 771-780.
3. R. Bhatia, Matrix Analysis, Springer, 1997.
4. R. Bhatia and F. Kittaneh, On the singular values of a product of operators, SIAM J. Matrix Anal. Appl., 11(1990) 272-277.
5. R. Bhatia and F. Kittaneh, Norm inequalities for positive operators, Letters Math. Phys., 43(1998) 225-231.
6. F. Hiai, Log-majorizations and norm inequalities for exponential operators, in *Linear Operators*, Banach Center Publications, Vol. 38, 1997, pp.119-181
7. R. A. Horn and C. R. Johnson, Topics in Matrix Analysis, Cambridge University Press, New York, 1991.