

A survey of real transverse sections of holomorphic foliations

Toshikazu Ito and Bruno Scárdua

Introduction

Let M be a closed, connected, smooth submanifold of real dimension $2n-1$ in the complex space $\mathbf{C}^n$ of dimension $n \geq 2$. Given a holomorphic one form ω in $\mathbf{C}^n$, for each $p \in \mathbf{C}^n$ with $\omega(p) \neq 0$ we define a $(n-1)$ -dimensional linear subspace $P_\omega(p) = \{v \in T_p \mathbf{C}^n \mid \omega(p) \cdot v = 0\}$. If $\omega(p) = 0$, we set $P_\omega(p) = \{0\}$ and we shall say that the distribution P_ω defined by ω is singular at p . We denote by $Sing(\omega) = \{p \in \mathbf{C}^n \mid \omega(p) = 0\}$ the singular set of ω . We have the following definition of transversality.

Definition We shall say that M is transverse to P_ω if for every $p \in M$ we have $T_p M + P_\omega(p) = T_p \mathbf{R}^{2n}$ as real linear spaces.

In particular, since $P_\omega(p) = \{0\}$ for any singular point p , we conclude that $Sing(\omega) \cap M = \emptyset$. In this note, we survey an existence or a non-existence of M such that M is transverse to P_ω .

1 Facts and Known results

In this section, we review the case of holomorphic vector field Z in $\mathbf{C}^n$, $n \geq 2$.

Given complex numbers $\lambda_1, \dots, \lambda_n \in \mathbf{C}^*$, we denote by $\mathcal{H}(\lambda_1, \dots, \lambda_n)$ the convex hull of the subset $\{\lambda_1, \dots, \lambda_n\}$ in $\mathbf{C}$. Let $Z = \sum_{j=1}^n \lambda_j z_j \partial / \partial z_j$ be a linear vector field on $\mathbf{C}^n$, $n \geq 2$. We denote by $S^{2n-1}(1) = \{z \in \mathbf{C}^n \mid \|z\|^2 = 1\}$ the $(2n-1)$ -dimensional sphere. We have a well-known fact.

Fact (1) If the origin $0 \in \mathbf{C}$ does not belong to $\mathcal{H}(\lambda_1, \dots, \lambda_n)$, then $S^{2n-1}(1)$ is transverse to Z .

(2) If the origin 0 belongs to $\mathcal{H}(\lambda_1, \dots, \lambda_n)$, then $S^{2n-1}(1)$ is not transverse to Z .

This Fact suggests to us the following properties.

Theorem ([4]) If the origin 0 belongs to $\mathcal{H}(\lambda_1, \dots, \lambda_n)$, then there is no smooth embedding φ of a closed connected smooth manifold M of dimension $2n-1$ to $\mathbf{C}^n$ such that $\varphi(M)$ is transverse to $Z = \sum_{j=1}^n \lambda_j z_j \partial / \partial z_j$.

We have a Poincaré-Bendixson type theorem for holomorphic vector fields.

Theorem (A. Douady and T. Ito [3]) Let N denote a subset of $\mathbb{C}^n$ holomorphic and diffeomorphic to the $2n$ -dimensional closed disc $\overline{D^{2n}(1)}$ consisting of all z in $\mathbb{C}^n$ with $\|z\| \leq 1$.

Let Z be a holomorphic vector field in some neighborhood of N . If the boundary $M = \partial N$ of N is transverse to Z , then

- (1) Z has only one singular point, say p , in N .
- (2) the index of Z at p is equal to one.
- (3) each solution L of Z which crosses M tends to p , that is, p is in the closure of L . Further, the restriction $\mathcal{F}(Z) \big|_{N-\{p\}}$ of the foliation $\mathcal{F}(Z)$ defined by the solutions of Z to $N-\{p\}$ is $\mathbb{C}^\omega$ -diffeomorphic to the foliation $\mathcal{F}(Z) \big|_M \times (0, 1]$ of $N-\{p\}$, where $\mathcal{F}(Z) \big|_M$ denotes the restriction of $\mathcal{F}(Z)$ to M .

Theorem (M. Brunella[1]) Soit $\Omega \subset \mathbb{C}^n$, $n \geq 2$, un ouvert borné avec frontière lisse et fortement convexe, et soit v un champ de vecteurs holomorphe défini au voisinage de $\overline{\Omega}$ et transverse à $\partial\Omega$. Il existe un difféomorphisme $\Phi : \overline{\Omega} \rightarrow \overline{D^{2n}(1)}$ qui envoie le feuilletage holomorphe singulier engendré par v dans un feuilletage $\mathcal{G}$ singulier à l'origine et transverse aux sphères $S^{2n-1}(\lambda)$, $\lambda \in (0, 1]$.

Theorem (M. Brunella and P. Sad[2]) Let $\Omega \subset \mathbb{C}^2$ be a generalized bidisc and let $\mathcal{F}$ be a holomorphic foliation defined in a neighborhood of $\overline{\Omega}$ and transverse to $\partial\Omega$. Then there exists a locally injective holomorphic map ϕ which sends a neighborhood of $\overline{\Omega}$ to a neighborhood of 0 in $\mathbb{C}^n$ such that $\mathcal{F} = \phi^*(L_\lambda)$ for some $\lambda \in \mathbb{C} \setminus \mathbb{R}$, where L_λ is a linear hyperbolic foliation in $\mathbb{C}^2$ defined by $x dy + \lambda y dx = 0$.

2 Existence or non-existence of real transverse sections

Let $\omega = \sum_{j=1}^n f_j(z) dz_j$ be a holomorphic one form on $\mathbb{C}^n$, $n \geq 2$. We denote by P_ω the distribution defined by ω in $T\mathbb{C}^n$.

Theorem 1 ([5]) Let M be a real 2-dimensional closed, connected, smooth manifold. If a smooth embedding φ of M to $\mathbb{C}^n$ is transverse to P_ω , then M is a torus.

We can construct a torus T^2 which is transverse to a holomorphic vector field Z . Let $Z = z_1 \partial / \partial z_1 + \lambda z_2 \partial / \partial z_2$ be a linear vector field on $\mathbb{C}^2$ and $T^2(r_1, r_2) = S^1(r_1) \times S^1(r_2) = \{|z_1| = r_1\} \times \{|z_2| = r_2\}$ a torus in $\mathbb{C}^2$.

Proposition 1 ([6]) The 2-dimensional torus $T^2(r_1, r_2)$ is transverse to Z if and only if the imaginary part of λ is different from zero.

We have the following non-existence theorems of transverse sections.

Theorem 2 ([6]) Let $Z = z_1 \partial / \partial z_1 + \lambda z_2 \partial / \partial z_2$, $\lambda \in \mathbb{R}$ be a linear vector field or $Z = z_1 \partial / \partial z_1 + (nz_2 + z_1^n) \partial / \partial z_2$, $n \in \mathbb{N}$ a holomorphic vector field of Dulac's normal form and M a closed, connected 2-dimensional smooth manifold. Then there is no smooth embedding φ of M to $\mathbb{C}^2$ such that $\varphi(M)$ is transverse to Z .

Theorem 3 ([5]) There exists no holomorphic foliation $\mathcal{F}$ of codimension one in a neighborhood

of the polydisc Δ^4 in $\mathbf{C}^4$ with the property that $\mathcal{F}$ is transverse to the product of spheres $S_1^3(1) \times S_2^3(1) = \{(z_1, z_2, z_3, z_4) \in \mathbf{C}^4 \mid |z_1|^2 + |z_2|^2 = 1, |z_3|^2 + |z_4|^2 = 1\} \subset \mathbf{C}^2 \times \mathbf{C}^2$.

Theorem 4 ([5]) Let $\omega = \sum_{j=1}^n h_j(z) dz_j$ be a homogeneous integrable one form on $\mathbf{C}^n$, $n \geq 3$.

The sphere $S^{2n-1}(1)$ of dimension $2n-1$ is not transverse to the foliation $\mathcal{F}(\omega)$ defined by $\omega = 0$.

We are very interested in the following properties.

Proposition 2 ([5]) Take $\omega = z_1 z_2 z_3 (\sum_{j=1}^3 \lambda_j \frac{dz_j}{z_j})$ on $\mathbf{C}^3$, where the non-zero complex numbers

$\lambda_1, \lambda_2, \lambda_3$ are satisfied with the following properties: $\lambda_i/\lambda_j \notin \mathbf{R}$, ($i \neq j$) and $\lambda_1 + \lambda_2 + \lambda_3 \neq 0$.

We get the following statements (i) $\sim$ (iii).

(i) $Sing(\omega) = \bigcup_{i \neq j} \{z_i = 0, z_j = 0\}$.

(ii) $S^5(1) \setminus (Sing(\omega) \cap S^5(1))$ is transverse to $\mathcal{F}(\omega)$.

(iii) Let $P = \{\alpha z_1 + \beta z_2 + \gamma z_3 = 0\}$, $\alpha, \beta, \gamma \in \mathbf{C}^*$ be a hyperplane through the origin. The restriction $\mathcal{F}_1$ of $\mathcal{F}(\omega)$ to P has only the origin as singularity. P is transverse to $\mathcal{F}(\omega)$ outside $Sing(\omega)$. $\mathcal{F}_1$ is not transverse to $P \cap S^5(1)$ though $Sing(\mathcal{F}_1) \cap (P \cap S^5(1)) = \phi$.

Example 1 (T.Ito and M. Yoshino) Take complex numbers $\lambda_1, \dots, \lambda_n, \mu_1, \dots, \mu_n \in \mathbf{C}^*$ and assume that the origin 0 belongs to $\mathcal{H}(\lambda_1, \dots, \lambda_n)$ and $\mathcal{H}(\mu_1, \dots, \mu_n)$. We make the following assumption: There exist real numbers c_1 and c_2 such that $\mathcal{H}(c_1 \lambda_1 + c_2 \mu_1, \dots, c_1 \lambda_n + c_2 \mu_n) \neq \emptyset$.

Consider linear vector fields $X = \sum_{j=1}^n \lambda_j z_j \partial / \partial z_j$ and $Y = \sum_{j=1}^n \mu_j z_j \partial / \partial z_j$. Then it is clear that

$[X, Y] = 0$ so that X and Y span a foliation $\mathcal{F}$ of complex dimension two on $\mathbf{C}^n$. Also $\mathcal{F}$ has as singular set $Sing(\mathcal{F})$ the union of the coordinate axis. Denote by $\Sigma(X)$ the set of tangency points of X with the spheres $S^{2n-1}(r) \subset \mathbf{C}^n$, $r \geq 0$, then we have $\Sigma(X)$ given by the equation

$\sum_{j=1}^n \lambda_j |z_j|^2 = 0$. This is a real cone. Analogously we define $\Sigma(Y)$ and describe it by the equation

$\sum_{j=1}^n \mu_j |z_j|^2 = 0$. Under the assumption, we have $\Sigma(X) \cap \Sigma(Y) = \{0\}$. $\mathcal{F}$ is transverse to

$S^{2n-1}(r) \setminus (Sing(\mathcal{F}) \cap S^{2n-1}(r))$, $r > 0$. Moreover each leaf of $\mathcal{F}$ accumulates the origin.

Theorem 5 ([5]) Let $\omega = \sum_{j=1}^{2n+1} f_j(z) dz_j$ be a holomorphic one form on $\mathbf{C}^{2n+1}$. Then the sphere

$S^{4n+1}(r)$, $r > 0$ is not transverse to $\mathcal{F}_\omega$.

By this theorem 5 or Proposition 2, the sphere $S^5(1)$ of dimension 5 is not transverse to $\mathcal{F}(\omega_\lambda)$

where $\omega_\lambda = z_1 z_2 z_3 (\sum_{j=1}^3 \lambda_j \frac{dz_j}{z_j})$, $\lambda_j \neq 0$ is a linear logarithmic one form on $\mathbf{C}^3$.

Example 2 ([5]) (1) If λ_i/λ_j , $i \neq j$, are not positive real, then we can construct a smooth embedding $\phi: S^1 \times S^3 \times S^1 \rightarrow \mathbf{C}^3$ such that $\phi(S^1 \times S^3 \times S^1)$ is transverse to $\mathcal{F}(\omega_\lambda)$.

(2) If λ_i/λ_j , $i \neq j$, are not real, then there exists a smooth embeddin $\Phi: T^5 = S^1 \times T^3 \times S^1 \rightarrow \mathbf{C}^3$ such that $\Phi(T^5)$ is transverse to $\mathcal{F}(\omega_\lambda)$.

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Toshikazu Ito
 Department of Natural Science, Ryukoku University
 Fushimi, Kyoto, 612-8577 Japan

B. Azevedo Scárdua
 Instituto de Matemática
 Universidade Federal do Rio de Janeiro
 Caixa Postal 68530
 21945-970 Rio de Janeiro-RJ
 Brazil
 scardua@im.ufrj.br