

A Vista of Mean Zeta-Values. II

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This is a continuation of our former article [5]. We shall use the same set of notations as there. In the last section of [5] we proposed a few central problems on mean values of automorphic L -functions. One of them is to establish an explicit spectral decomposition for the mean square of the L -function attached to a given irreducible Γ -automorphic representation of G , with $\Gamma = \mathrm{PSL}(2, \mathbb{Z})$ and $G = \mathrm{PSL}(2, \mathbb{R})$. The aim of the present article is to indicate briefly a method with which one may settle this problem in the most difficult situation; that is, the case of unitary principal series representations or Maass wave forms. As a consequence, it is now strongly suggested that the inner-product procedure, which was initiated by A. Good and greatly enhanced by M. Jutila, should be the right way to pursue further, if we wish to establish anything like a unified theory of mean values of automorphic L -functions. The Kirillov map $\mathcal{K}$ defined by [5, (5.3)] has turned out to be a key implement, indeed as we envisaged.

Thus, let V be an irreducible subspace of $L^2(\Gamma \backslash G)$ whose spectral data is ν . As it is to be in the unitary principal series, ν is pure imaginary. Let $\phi(\cdot, \alpha) \in U_\nu$ be such that

$$(1) \quad \mathcal{K}\phi(u) = \begin{cases} u^\alpha \exp(-2\pi u) & \text{for } u \geq 0, \\ 0 & \text{for } u < 0, \end{cases}$$

with $\alpha > 0$. This is possible, for $\mathcal{K}$ is surjective and the member on the right side is obviously in $\mathcal{H} = L^2(\mathbb{R}^\times, \pi^{-1}d^\times)$. Let

$$(2) \quad \phi(g, \alpha) = \sum_p c_p \phi_p(g), \quad \phi_p(g) = \phi_p(g; \nu),$$

where $c_p = c_p(\nu, \alpha)$. We may choose an orthonormal base $\{\varphi_p\}$ of V such that

$$(3) \quad \varphi_p(g) = \sum_{n \neq 0} \frac{\varrho_V(n)}{\sqrt{|n|}} \mathcal{A}^{\mathrm{sgn}(n)} \phi_p(a[|n|]g).$$

We put

$$(4) \quad \varphi(g, \alpha) = \sum_p c_p \varphi_p(g),$$

We shall later prove that

$$(5) \quad \varphi(g, \alpha) = \sum_{n \neq 0} \frac{\varrho_V(n)}{\sqrt{|n|}} \mathcal{A}^{\mathrm{sgn}(n)} \phi(a[|n|]g, \alpha),$$

provided α is sufficiently large. Given this, we note that

$$(6) \quad \mathcal{A}^{\text{sgn}(n)} \phi(a[|n|]n[x]a[y], \alpha) = \exp(2\pi i n x) \mathcal{K} \phi(ny, \alpha).$$

Thus

$$(7) \quad \varphi(n[x]a[y], \alpha) = y^\alpha \sum_{n=1}^{\infty} \varrho_V(n) n^{\alpha-\frac{1}{2}} \exp(2\pi i n(x+iy)).$$

This brings us to a situation very much similar to the one with holomorphic cusp forms (see [3]).

Now, we shall prove (5). To this end, we compute explicitly the coefficients c_p : The unitarity of $\mathcal{K}$ gives

$$(8) \quad \begin{aligned} c_p &= \langle \phi, \phi_p \rangle_{U_\nu} = \langle \mathcal{K} \phi, \mathcal{K} \phi_p \rangle_{\mathcal{H}} \\ &= \frac{1}{\pi} \int_0^\infty u^{\alpha-1} \exp(-2\pi u) \overline{\mathcal{A}^+ \phi_p(a[u])} du. \end{aligned}$$

On noting that the Jacquet transform is essentially equal to the Whittaker function (or the confluent hypergeometric function) save for a simple factor, the formula 7.621(3) of [2] becomes relevant here. It implies that

$$(9) \quad c_p = (-1)^p 2^{-2\alpha} \pi^{-\nu-\alpha-\frac{1}{2}} \frac{\Gamma(\alpha+\nu+\frac{1}{2}) \Gamma(\alpha-\nu+\frac{1}{2})}{\Gamma(\frac{1}{2}-\nu+p) \Gamma(\alpha+1-p)}.$$

Or one may argue as follows: The bounds (4.3) and (4.5) of [1] imply that the integral is a regular function of ν in a neighbourhood of the imaginary axis. Let us suppose temporarily that $\text{Re } \nu$ is negative but small. Then we see, by the first equation of (2.16) in [1], that

$$(10) \quad \begin{aligned} c_p &= \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{1}{(\xi^2+1)^{\frac{1}{2}-\nu}} \left(\frac{\xi+i}{\xi-i} \right)^{-p} \int_0^\infty u^{\alpha+\nu-\frac{1}{2}} \exp(-2\pi u(1+i\xi)) du d\xi \\ &= \frac{1}{\pi} (2\pi)^{-\nu-\alpha-\frac{1}{2}} \Gamma\left(\alpha+\nu+\frac{1}{2}\right) \int_{-\infty}^{\infty} \frac{(1+i\xi)^{-\nu-\alpha-\frac{1}{2}}}{(\xi^2+1)^{\frac{1}{2}-\nu}} \left(\frac{\xi+i}{\xi-i} \right)^{-p} d\xi, \end{aligned}$$

where $\arg(1+i\xi)$ varies from $-\frac{1}{2}\pi$ to $\frac{1}{2}\pi$ as ξ runs over $\mathbb{R}$ from $-\infty$ to ∞ . This integral can be computed by the argument given on p. 47 of [4], whence we obtain (9).

In particular, we find that

$$(11) \quad c_p \ll (|p|+1)^{-\alpha-\frac{1}{2}},$$

as $|p|$ tends to infinity, and $\nu \in i\mathbb{R}$ is bounded. Thus, indeed $\phi \in U_\nu$ if $\alpha > 0$, and ϕ becomes smoother if we take α larger. Invoking the uniform bound

$$(12) \quad \mathcal{A}^\delta \phi_p(a[y]) \ll (|p|+|\nu|+1)y^{-\frac{1}{2}} \exp\left(-\frac{y}{|\nu|+|p|+1}\right),$$

we see that (11) confirms (5). This bound is proved in [1, Section 4].

Now, we shall move to an inner-product argument: Let $\tau(\theta)$ be a smooth function supported on a small neighbourhood of $\theta = 0$, and

$$(13) \quad \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} \tau(\theta) d\theta = 1.$$

Let m be a positive integer, and $\operatorname{Re} s > 1$. Put

$$(14) \quad f(g) = y^s \exp(2\pi m i(x + iy)) \tau(\theta).$$

Further, put

$$(15) \quad \mathcal{P}f(g) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} f(\gamma g), \quad \Gamma_\infty = \Gamma \cap N,$$

which is in $L^2(\Gamma \backslash G)$.

With this, consider the inner-product

$$(16) \quad \langle \mathcal{P}f, |\varphi|^2 \rangle_{\Gamma \backslash G}.$$

Let us assume that α is sufficiently large. The unfolding argument gives

$$(17) \quad \begin{aligned} \langle \mathcal{P}f, |\varphi|^2 \rangle_{\Gamma \backslash G} &= \frac{1}{\pi} \int_0^\infty \int_0^1 y^{s-2} \exp(2\pi m i(x + iy)) \\ &\quad \times \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} \tau(\theta) |\varphi(n[x]a[y]k[\theta], \alpha)|^2 d\theta dx dy. \end{aligned}$$

Thus

$$(18) \quad \lim_{\tau} \langle \mathcal{P}f, |\varphi|^2 \rangle_{\Gamma \backslash G} = \frac{1}{\pi} \int_0^\infty \int_0^1 y^{s-2} \exp(2\pi m i(x + iy)) |\varphi(n[x]a[y], \alpha)|^2 dx dy,$$

where the support of τ tends to 0. The expression (7) implies readily that

$$(19) \quad \sum_{n=1}^{\infty} \frac{\varrho_V(n) \overline{\varrho_V(n+m)}}{(n+m)^s (1+m/n)^{\alpha-\frac{1}{2}}} = \frac{\pi(4\pi)^{s+2\alpha-1}}{\Gamma(s+2\alpha-1)} \lim_{\tau} \langle \mathcal{P}f, |\varphi|^2 \rangle_{\Gamma \backslash G}.$$

With this, we may use the argument of [3, Section 1] and attain the inner sum of the expression

$$(20) \quad \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{\varrho_V(n) \overline{\varrho_V(n+m)}}{n^u (n+m)^v} \hat{g}\left(\log\left(1 + \frac{m}{n}\right)\right),$$

where $\hat{g}$ is the Fourier transform of g in the context same as in [1]. In fact, it suffices for us to multiply both sides by the factor

$$(21) \quad m^{-u-v+\xi} \Gamma(u+v-\xi) \cdot \frac{1}{2\pi i} \int_{\text{Im } t = -c} \frac{\Gamma(\frac{3}{2} - u - \alpha + it)}{\Gamma(v + \frac{3}{2} - \alpha - \xi + it)} g(t) dt,$$

with $c > 0$ sufficiently large, and integrate with respect to ξ along an appropriate vertical line. Provided α is sufficiently large and $\text{Re}(u+v) > \text{Re } \xi > 1$, the necessary absolute convergence holds throughout our procedure. Inserting the thus obtained expression into (20), we find that (20) admits an expression in terms of $\langle \mathcal{P}f, |\varphi|^2 \rangle_{\Gamma \backslash G}$, provided $\text{Re}(u+v) > 2$.

The expression (20) is of course related to the non-diagonal part of

$$(22) \quad \int_{-\infty}^{\infty} \left| L_V \left(\frac{1}{2} + it \right) \right|^2 g(t) dt,$$

where

$$(23) \quad L_V(s) = \sum_{n=1}^{\infty} \frac{\varrho_V(n)}{n^s}, \quad \text{Re } s > 1.$$

The inner-product (16) is spectrally decomposed according to the spectral structure of $L^2(\Gamma \backslash G)$. The limit in τ of the decomposition should converge termwise, so do we believe. Then, the left side of (19) admits a spectral decomposition, from which a complete spectral decomposition of (22) ought to transpire.

The above argument appears to extend to bigger groups, at least formally. In our mind is the situation with $G = \text{SL}(3, \mathbb{R})$ and $\Gamma = \text{SL}(3, \mathbb{Z})$. There the minimal parabolic Eisenstein series generates a product of 6 values of the Riemann zeta-function, apart from an unimportant factor. Thus it could be surmised that a mean value of the product of 12 values of the zeta-function is related to the Γ -automorphic structure of G , with this particular combination of G and Γ .

References

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