

Topics from Competitive Game Theory

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Some of the recent works by the present author are given in the two parts **A** and **B**. A lot of interesting open problems are mentioned.

A 2- AND 3- PLAYER GAMES OF SCORE SHOWDOWN

1. 3-Player Games of Score Showdown. Let $X_{ij} (i=1,2,3; j=1,2)$ be the j -th r.v. observed by Player i . Assume that $\{X_{ij}\}$ are i.i.d. with $U[0,1]$ distribution. Each player i first observes $X_{i1} = x_{i1}$ and chooses A/R. If A(R) is chosen, the x_{i1} is accepted (rejected and the second r.v X_{i2} is observed). Player i 's score is

$$S_i(X_{i1}, X_{i2}) = \begin{cases} X_{i1} \\ \varphi(X_{i1}, X_{i2}) \end{cases}, \text{ if } X_{i1} = x_{i1} \text{ is } \begin{cases} \text{accepted.} \\ \text{rejected \& } X_{i2} \text{ is observed.} \end{cases}$$

We consider the cases

$$\varphi(X_{i1}, X_{i2}) = X_{i2}, X_{i2} I(X_{i2} \geq X_{i1}), (X_{i1} + X_{i2}) I(X_{i1} + X_{i2} \geq 1), \frac{1}{2}(X_{i1} + X_{i2})$$

(Keep-or-Exchange, Risky-Exchange, Showcase-Showdown, Competing Average, resp)

Player who gets the highest score is the winner. Each player wants $\Pr(\text{he wins}) \rightarrow \max$

Also we consider the two versions of the games;

Simultaneous-move version — Players' choices are made indep. and are not known to his rivals.

Sequential-move version — Players move sequentially, i.e., I at first, II at second, and III at third. After I's move, II and III are informed of $X_{i1} = x_{i1}$ and I's choice of A/R. After II's move, III is informed of $X_{21} = x_{21}$ and II's choice of A/R. All players are intelligent, and each player should prepare for that any subsequent player must employ their optimal strategies.

In the present article, we use Γ = game, C=common, EQ=equilibrium, S=strategy, V= value, D= draw. Players 1,2,3 are sometimes written by I,II, III, resp.

2. Solutions to the Games, It is shown that games of $\Gamma^{(2)}_{SS}$ and $\Gamma^{(2)}_{RE}$ have the same solution (REF[3]). Some other results are given below (Ref[1,2,3])

• Solution to $\Gamma^{(2)}$ KE (simult.-move): CEQS is "Choose A(R) if his first observ. is $>(<)$
 $a^* = \frac{1}{2}(\sqrt{5}-1) \doteq 0.618$, i.e., a unique root in $(0,1)$ of the equation $a^2+a=1$."
 CEQV = $\frac{1}{2}$. In $\Gamma^{(3)}$ KE, the a^* , the equation and CEQV change to $a^* \doteq$
 0.691, $2a^4 = 1-a+a^2-a^3$ and $\frac{1}{3}$ resp.

• Solution to $\Gamma^{(2)}$ RE (simult.-move): CEQS is "Choose A(R), if his first observ. is $>(<)$
 $a^* \doteq 0.544$, i.e., a unique root in $(0,1)$ of the equation $a^3+a^2+a=1$." $P(D) = \frac{1}{4} a^{*4} \doteq$
 0.022, $P(W_1) = P(W_2) = \frac{1}{2}(1-P(D)) \doteq 0.489$. In $\Gamma^{(3)}$ RE, a^* , the equation and the
 other result change to $a^* \doteq 0.656$, $2a^4+a^5=1-a+a^2-a^3$, $P(D) = \frac{1}{8} a^{*6} \doteq 0.010$, $P(W_1)$
 $=P(W_2) = P(W_3) = \frac{1}{3}(1-P(D)) \doteq 0.330$.

• Solution to $\Gamma^{(2)}$ RE (seq.-move): I's opt.str. is "Choose A(R) $X_{11} = x_1$, if $x_1 >(<) x_0$
 where $x_0 (\doteq 0.570)$ is a unique root in $(\sqrt{2}-1, 1)$ of $\bar{x}^4 = 2x(1-3x^2)$ " II's opt.str. is

"Choose A(R) $X_{21} = x_2$ if $x_2 >(<) \begin{cases} x_1 \\ y_0(x_1) \end{cases}$, in state $\begin{cases} (x_2 | x_1, A) \\ (x_2 | x_1, R) \end{cases}$,
 where $y_0(x_1) \equiv (\sqrt{2}-1)I(x_1 \leq \sqrt{2}-1) + (\bar{x}_1^2/2x_1)I(x_1 > \sqrt{2}-1)$." We obtain $P(D), P(W_1),$
 $P(W_2) \doteq 0.011, 0.477, 0.512$, resp.

• Solution to $\Gamma^{(2)}$ CA (seq.-move): I's opt. str. is "Choose A(R) $X_{11} = x_1$, if $x_1 >(<) x_0$, where
 $x_0 (\doteq 0.549)$ is a unique root in $(\frac{1}{2}, 1)$ of $\eta/2x - 4\sqrt{2}x = 6 - (5/2)x + x^2$." II's opt.
 str. is "Choose A(R) $X_{21} = x_2$ if $x_2 >(<) \begin{cases} x_1 \\ y_0(x_1) \end{cases}$, in state $\begin{cases} (x_2 | x_1, A) \\ (x_2 | x_1, R) \end{cases}$,

where $y_0(x_1) \equiv (\sqrt{2x_1} - \bar{x}_1)I(x_1 < \frac{1}{2}) + (\sqrt{2x_1} - \bar{x}_1)I(x_1 \geq \frac{1}{2})$. We obtain $P(W_1) = 1 - P(W_2) \doteq$
 0.490.

• Solution to $\Gamma^{(3)}$ KE (seq.-move): It is too complicate to write here. We found
 that $P(W_1), P(W_2), P(W_3) \doteq 0.32309, 0.33270, 0.34421$, resp. The sum is 1.

3. Information Types in 2-Player Games. We consider the two information types,
 under which players decide their choices of A/R.

I^{11-11} means that I chooses $X_{11} = x_1$, II chooses $X_{21} = x_2$ and each player informs
 of his observed value to his opponent.

I^{10-11} means that I observes $X_{11} = x_1$, II observes $X_{21} = x_2$ and I informs his x_1
 to II, and II doesn't inform his x_2 to I.

It is clear that I^{10-01} is the information type discussed in Section 2.

Some results are given as follows (Ref[4]).

• Solution to $\Gamma^{(2)}$ RE under I^{11-11} ; In state x_1, x_2 there exists a unique
 saddle point at R-A, R-R, A-R, if $x_1 \vee x_2 \vee (\sqrt{2}-1) = x_2, \sqrt{2}-1, x_1$, resp.

We obtain $P(D) = \frac{1}{4}(\sqrt{2}-1)^4 \doteq 0.007$, $P(W_1) = P(W_2) = \frac{1}{2}(1-P(D)) \doteq 0.496$.

• Solution to $\Gamma^{(2)}$ KE under $I^{(0-1)}$: I's opt. str. is "Choose A (R) $X_{i1} = x_1$, if $x_1 > (<) a^* = \sqrt{3/8} \doteq 0.6124$." II's opt. str. in state $x_1 - x_2$ is given by

Condition	II's opt. Choice
$x_1 < a^*$	A(R), if $x_2 > (<) \frac{1}{2}$
$a^* < x_1 < x_2$	A
$x_1 > a^* \vee x_2$	R

We obtain $P(W_1) = 1 - P(W_2) = \frac{1}{3} + \frac{1}{4}a^* \doteq 0.4864$.

4. 2-Player game of Continue-or-Stop. Player i observes $(X_{i1}, X_{i2}, \dots, X_{in})$, sequentially one-by-one, and facing each X_{ij} chooses Cont/Stop. If Stop is chosen X_{ij} is accepted by player i and his play ends. If Cont. is chosen, X_{ij} is rejected and the next $X_{i,j+1}$ is observed and the game continues. Assume that $\{X_{ij}\}$ are i.i.d. with

$U_{[0,1]}$ distribution. Score is

$$S_i(X_{i1}, \dots, X_{in}) = \begin{cases} X_{i1} \\ X_{i,j+1} \end{cases}, \text{ if } \begin{cases} \text{stop at } X_{i1} \\ X_{i1}, X_{i2}, \dots, X_{ij} \text{ are rejected and stop at } X_{i,j+1} \end{cases}$$

Consider the situation where player i has $n-1$ decision thresholds $(1 >) a_{i1} > a_{i2} > \dots > a_{i,n-1} (> 0)$, so that i chooses Stop (Cont.) if $X_{ij} > (<) a_{ij}$. Player who gets the higher score than his opponent is the winner. Each player aims $\Pr(\text{he wins}) \rightarrow \max$.

• Solution when $n=3$ (Ref[5]): The common optimal thresholds are $(b_1^0, b_2^0) \doteq (0.743, 0.657)$, where (b_1^0, b_2^0) is a unique root in $(0,1)^2$ of

$$\begin{cases} b_2^2 + b_2 - (b_1^{-1} - 1 + b_1) = 0 \\ b_1^2 + (1+b_2)^{-1} b_1 - 1 = 0 \end{cases}$$

$P(W_1) = P(W_2) = \frac{1}{2}$.

5. Remark. There are many open problems of interest around the topic — games of score showdown. Three among them are given below.

(1). Let (X_{ij}, Y_{ij}) , $i=1,2, j=1,2$, are i.i.d. with joint p.d.f.

$f(x,y) = 1 + \gamma(1-2x)(1-2y)$, $\forall (x,y) \in [0,1]^2$,
with $|\gamma| \leq 1$. In the game $\Gamma^{(2)}$ KE (simult.-move), the score is

$$S_i(X_{i1}, Y_{i1}, X_{i2}, Y_{i2}) = \begin{cases} (X_{i1}, Y_{i1}) \\ (X_{i2}, Y_{i2}) \end{cases}, \text{ if } (X_{ij}, Y_{ij}) \text{ is } \begin{cases} \text{accepted} \\ \text{rejected} \end{cases}.$$

Define that (x, y) is higher than [lower than, intermediate to] (x', y') , if $(x, y) \geq [<, \text{otherwise}] (x', y')$. Player with score (x, y) gets 1 $[-1, 0]$. Solve this zero-sum game.

(2). Sequential-move games $\Gamma^{(3)}RE$ and $\Gamma^{(3)}CA$ remain unsolved.

(3). 3-player games under various information types are of interest. $\Gamma^{(3)}KE$, $\Gamma^{(3)}RE$ and $\Gamma^{(3)}CA$, under $I^{III-III-III}$, $I^{IIO-III-III}$,

$I^{III-III-III}$, etc. remain unsolved. The last one $I^{IIO-III-III}$, for example, means that each player observes his own $X_{i1} = x_{i1}$ and, in addition, I knows x_{21} , II knows x_{31} and III knows x_{11} .

REFERENCES

- [1,2,3] M.Sakaguchi, SCMJ, 60(2004), 411-421; 61(2005), 335-344.; 61(2005), 145-151, resp.
[4,5] ———, Game Th. Appl. 10(2005), 148-160; To appear in 11(2006), resp.



MULTISTAGE OPTIMAL STOPPING GAMES

1. Each Player has his Priority. Player I, II, and III observe $X_1, X_2, \dots, X_n$ with i.i.d. $U_{[0,1]}$ distribution sequentially one-by-one. They have their previously given priorities $\langle \beta_1, \beta_2, \beta_3 \rangle$. Facing X_j each player chooses R/A, independently of his rivals.

If only one player chooses A, he gets X_j with his priority, dropping out from the game thereafter, and the remaining two players continue their two-player game with the "revised" priorities. If two players choose A, one player selected according to the "revised" priorities gets X_j , drops out from the game thereafter, and the remaining two players continue their 2-player game with the "revised" priorities. If the choices are A-A-A player i gets X_j with prob. p_i dropping out from the game, and the remaining two players continue their 2-player game with the revised" priorities. If the choices are R-R-R then X_j is rejected, the next X_{j+1} is observed and the subsequent 3-player game continues. Each player aims to maximize the ENV he can get (N in ENV means net, i.e. no-observation-cost and no-discounting).

Let W_n (V_n) be CEQV, for 3-player (2-player) equal-priority game. Then the Opt. Eq. is

$$(W_n, W_n, W_n) = E [\text{eq. val. } M_n(X)] \quad (n \geq 1, W_0 = V_0 = 0)$$

$$M_n(x) = \begin{cases} \text{R by III} \\ \text{A by III} \end{cases} M_{n,A}(x) = \begin{cases} \text{R by I} \\ \text{A by I} \end{cases} M_{n,R}(x) = \begin{cases} \text{R by II} \\ \text{A by II} \end{cases}$$

W, W, W	V, x, V
x, V, V	$\frac{1}{2}(x+V), \frac{1}{2}(x+V), V$

V, V, x	$V, \frac{1}{2}(x+V), \frac{1}{2}(x+V),$
$\frac{1}{2}(x+V), V, \frac{1}{2}(x+V),$	$\frac{1}{3}(x+2V), \frac{1}{3}(x+2V), \frac{1}{3}(x+2V),$

$[W(V) \text{ is } W_{n-1}(V_{n-1})]$

$$(V_n, V_n) = E \left\{ \text{val. } \begin{matrix} \text{R} & \text{A} \\ \text{A} & \text{R} \end{matrix} \begin{matrix} V_{n-1}, & V_{n-1} \\ x, & U_{n-1} \end{matrix} \mid \begin{matrix} U_{n-1}, & x \\ \frac{1}{2}(x+U_{n-1}), & \frac{1}{2}(x+U_{n-1}) \end{matrix} \right\}$$

where $U_n = \frac{1}{2}(1+U_{n-1}^2)$, ($n \geq 1, U_0 = 0$).

Common opt. str. for each player is derived, and CEQ V is computed as

$$U_n = \frac{1}{2}, 0.7417, 0.8364, 0.8791, \text{ for } n=1, 4, 8, 12, \text{ resp.}$$

$$V_n = \frac{1}{4}, 0.6468, 0.7182, 0.8361, \quad \text{ibid.}$$

$$W_n = \frac{1}{6}, 0.5506, 0.7196, 0.7936, \quad \text{ibid.}$$

The cases $\langle p_1, p_2 \rangle = \langle 1, 0 \rangle$ and $\langle p_1, p_2, p_3 \rangle = \langle \frac{1}{2}, \frac{1}{2}, 0 \rangle$, are also solved (Ref [1, 4])

Player with low priority stands at disadvantage, since he remains late in the game, and faces less offers.

The cases where player's aim is Pr. (he gets a r.v. better than opponent) $\rightarrow$ max. are solved in Ref [1, 2]. In the 2-player No-Information case, for example, let us

Define state (i, y) to mean that (1) both players remain in the game, and (2) players currently face the r.v. $Y_i = y$. Let $V(i, y)$ be the value of the game in state (i, y) , for the n -problem. Note that n is fixed throughout, players should choose A-A in state (n, y) and hence draw of the game cannot occur.

Then the Opt. Eq. in state (i, y) is

$$V(i, y) = \text{val.} \begin{cases} \text{R} \\ \text{A} \end{cases} \begin{bmatrix} \mu_{i+1} & 1-g(i, y) \\ g(i, y) & (p-F)g(i, y) + F \end{bmatrix}$$

where $\mu_{i+1} = (i+1)^{-1} \sum_{y=1}^{i+1} V(i+1, y)$, and $1 \leq i \leq n-1, V(n, y) = p \in [\frac{1}{2}, 1], \forall y \in \{1, 2, \dots, n\}$.

$$g(i, y) = \text{Pr.} \{ i\text{-th has absolute rank } y, \text{ in state } (i, y) \}$$

$$= \prod_{j=i}^n \left(1 - \frac{y}{i+j} \right) = \binom{i}{y} \binom{n}{y} = \binom{i}{y} \binom{n}{y}$$

The solution is: Opt. str. pair in state (x, y) is

R-R, R-A, A-A, if $0 < g(x, y) < \bar{\mu}_{x+1}$, $\bar{\mu}_{x+1} < g(x, y) < \frac{1}{2}$, $\frac{1}{2} < g(x, y) < 1$, resp.

The winning prob. for $n=10$ are computed downward in i until reaching $\mu_i = V(i, 1)$. We obtain

$P_i(I \text{ wins}) \doteq 0.578, 0.619$, for $p=3/4, 1$, resp.

3. Committee's Selection

I and II observe (X_j, Y_j) , $j=1, \dots, n$, i.i.d. with $U_{[0,1]}$ distribution sequentially one-by-one, and each player chooses R/A. X_j (Y_j) is I's (II's) evaluation of j -th applicant's ability (Ref.[9]). The Opt. Eq. is

$$(u_n, v_n) = E[pq \text{ val. } M_n(X, Y)], \quad M_n(x, y) = \begin{matrix} & R & A \\ \begin{matrix} R \\ A \end{matrix} & \begin{pmatrix} u_{n-1}, v_{n-1} \\ px + \bar{p}u_{n-1}, py + \bar{p}v_{n-1} \end{pmatrix} & \begin{pmatrix} pu_{n-1} + \bar{p}x, pv_{n-1} + \bar{p}y \\ x, y \end{pmatrix} \end{matrix}$$

The solution is: In state $(X_j = x, Y_j = y)$,

I chooses A (R), if $x \geq (<) u_{n-1}$, indep. of y ,

II chooses A (R), if $y \geq (<) v_{n-1}$, indep. of x ,

where

$$u_n = \frac{1}{2} \{ pu_n^2 + \bar{p}(2u_{n-1} - 1)v_{n-1} + 1 \}, \quad v_n = \text{same with } p \rightarrow \bar{p} \text{ and } u \leftrightarrow v.$$

It is shown that $u_n \uparrow u_\infty$, $v_n \uparrow v_\infty$, and (u_∞, v_∞) is a unique root in $(0, 1)^2$ of

$$u = \frac{\sqrt{1-\bar{p}v}}{\sqrt{1-\bar{p}v} + \sqrt{\bar{p}v}}, \quad v = \frac{\sqrt{1-pu}}{\sqrt{1-pu} + \sqrt{pu}}$$

Convergence is quick:

$$u_{10} = 0.6592, 0.6867, 0.7530, 0.8611, \text{ for } p=0.5, 0.6, 0.8, 1.0, \text{ resp.}$$

$$v_{10} = 0.6592, 0.6331, 0.5788, 1/2, \quad \text{ibid}$$

$$u_\infty = 2/3, 0.6946, 0.7663, 1, \quad \text{ibid.}$$

$$v_\infty = 2/3, 0.6408, 0.5899, 1/2, \quad \text{ibid}$$

Various interesting open problems arise. ① If X_j (Y_j) is the ability of management (foreign language), then X_j and Y_j are not independent. ② The case where players aim ENV of $(X_j | Y_j \geq a) \rightarrow \max$, (Ref.[11]). ③ 3-player game where (X_j, Y_j, Z_j) is observed.

3. High-Hand-Wins Poker.

We first consider a simple n -round poker. Each of two players I and II receives a hand x and y , respectively, in $[0, 1]$, according to a uniform distribution, and chooses one of two alternatives Reject or Accept. If choice-pair is R-R, the game proceeds to the next round and both players are dealt new hands x and y . If the choice-pair is A-A showdown occurs and the game ends with I's reward $\text{sgn}(x-y)$. If players choose different choices, then arbitration comes in, and forces them to take the same choices as I's (II's) with probability p ($\bar{p}$). This zero-sum game is played in n -rounds, and player I(II) aims to maximize(minimize) the expected reward to I.

Let $\phi_n(x)(\psi_n(y))$ be the probability that player I (II) chooses A on the hand x (y). Also let v_n be the value (for I) of the n -round game. Then we have

$$v_n = \max_{\phi_n(\cdot)} \min_{\psi_n(\cdot)} E_{x,y}[(\bar{\phi}_n(x), \phi_n(x)) M_n(x,y) (\bar{\psi}_n(y), \psi_n(y))^T]$$

where

$$M_n(x,y) = \begin{matrix} & \begin{matrix} R & A \end{matrix} \\ \begin{matrix} R \\ A \end{matrix} & \begin{pmatrix} v_{n-1} & \bar{p}\text{sgn}(x-y) + p v_{n-1} \\ p\text{sgn}(x-y) + \bar{p} v_{n-1} & \text{sgn}(x-y) \end{pmatrix} \end{matrix}$$

The solution is

$$\phi_n^*(x) = I(x > a_n), \quad \psi_n^*(y) = I(y > \bar{a}_n), \quad v_n = 2a_{n+1} - 1$$

where $\{a_n\}$ is determined by $a_{n+1} = a_n + \frac{1}{2}(p\bar{a}_n^2 - \bar{p}a_n^2)$ ($n \geq 1, a_1 = \frac{1}{2}$).

We obtain $a_n \uparrow a_\infty = \sqrt{p}/(\sqrt{p} + \sqrt{\bar{p}})$, $v_n \uparrow 2a_\infty - 1 = (\sqrt{p} - \sqrt{\bar{p}})/(\sqrt{p} + \sqrt{\bar{p}})$.

The bilateral-move version, when $p = 1/2$ has an interesting solution. Let w_n be the value of the n -stage game. $w_n < 0$, since I must move first and inevitably gives some information about his true hand. We find that $w_n \downarrow w_\infty = -(1-\sqrt{g})/(1+\sqrt{g}) \doteq -0.1197$ where $g = \frac{1}{2}(\sqrt{5}-1) \doteq 0.618$, the golden bisection number. Also we find that disadvantage disappears when $p = 0.6$, i.e.

$w_n = 0, \forall n \geq 1$. (Ref. [10]).

3-player High-Hand-Wins poker, under simple-majority rule and with bet $B \geq 0$, is also an interesting open problem.

4. Odd-Man-Wins and Odd-Man-Out.

In the three-player two-choice games there often appear Odd-Man and Even-Men. What is the reasonable partition among players of each X_j ?

Let v_n (w_n) be CEQV of n -stage Odd-Man-Wins (Odd-Man-Out). Then the payoff matrices $M_n(X)$ are

			R by III	A by III	
	R by I	R by II	$v_{n-1}, v_{n-1}, v_{n-1}$	0, 0, x	
		A by II	0, x , 0	x , 0, 0	
			R by III	A by III	
	A by I	R by II	x , 0, 0	0, x , 0	
		A by II	0, 0, x	$x/3$, $x/3$, $x/3$	

$n \geq 1, v_i = \frac{1}{6},$

			R by III	A by III	
	R by I	R by II	$w_{n-1}, w_{n-1}, w_{n-1}$	$x/2$, $x/2$, 0	
		A by II	$x/2$, 0, $x/2$	0, $x/2$, $x/2$	
			R by III	A by III	
	A by I	R by II	0, $x/2$, $x/2$	$x/2$, 0, $x/2$	
		A by II	$x/2$, $x/2$, 0	$x/3$, $x/3$, $x/3$	

$n \geq 1, w_i = \frac{1}{6},$

for Odd-Man-Wins, Odd-Man-Out, resp.

Each player must think about : (1) He wants to become the odd-man (an even-man), when the game is Odd-Man-Wins (Odd-Man-Out), especially when he faces a very large X_j , and (2) Since X_j is a random variable, he can expect a larger one may come up in the future.

It is shown that (Ref.[6]), $v_n \uparrow v_\infty \doteq 0.205$ and $w_n \downarrow w_\infty \doteq 0.160$. So, multi-stage

play yields each player a merit of size $v_\infty - \frac{1}{6} \doteq 0.039$ and a demerit of size $\frac{1}{6} - w_\infty \doteq 0.0664$.

The extension to the many-player two-choice simple-majority games is an interesting open problem.

The case where the odd-man has priority p , and the even-men has priority $\bar{p}/2$ is solved in Ref.[7]. When $p = \frac{1}{3}$, $CEQV = \frac{1}{3} \mu_n$, where $\{\mu_n\}$ is the Moser's sequence. When $p = 0$ (1), the game reduces to Odd-Man-Out (Odd-Man-Wins).

REFERENCES

- [1, 2, ..., 7] SCMJ 53(2001) 257-268, 125-134, 56 (2002) 457-473, 317-325, 58(2003) 183-189, 60(2004), 315-325, 403-410..
- [8, 9, 10] GTA 7(2001) 156-169, 8(2002) 95-106, 9(2003) 141-149.
- [11] Bruss and Lecam (edt), Lecture Notes 35 (2000), Inst. Math. Stat., 111-125.

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