

Subrepresentation Theorem for p -adic Symmetric Spaces

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Abstract

The notion of relative cuspidality for distinguished representations attached to p -adic symmetric spaces is introduced. A generalization of Jacquet's subrepresentation theorem to the relative case (symmetric space case) is given, under a reasonable assumption on the relative Cartan decomposition.

1 Introduction

Let $\underline{G}$ be a connected reductive group over a non-archimedean local field F and $\underline{Z}$ be the F -split component (i.e., the maximal F -split central torus) of $\underline{G}$. The group $\underline{G}(F)$ of F -points of $\underline{G}$ is denoted by G , and similarly $\underline{Z}(F)$ by Z .

First we shall recall some of fundamental theory (due to Jacquet and Harish-Chandra) on admissible representations of reductive p -adic groups.

(a) Definition. An admissible representation (π, V) of G is said to be *cuspidal* if the support of every matrix coefficient of π is compact modulo Z .

For a parabolic F -subgroup $\underline{P}$ of $\underline{G}$, let (π_P, V_P) denote the (normalized) Jacquet module of (π, V) along $P = \underline{P}(F)$. The following criterion for cuspidality in terms of Jacquet modules is well-known.

(b) **Theorem.** (Jacquet, Harish-Chandra) *An admissible representation (π, V) of G is cuspidal if and only if $V_P = 0$ for any proper parabolic F -subgroup P of G .*

Let $P = \underline{M} \ltimes \underline{U}$ be a Levi decomposition of P . For an admissible representation ρ of $M = \underline{M}(F)$ let $\text{Ind}_P^G(\rho)$ be the normalized induction. The Frobenius reciprocity asserts that the (functorial) isomorphism

$$\text{Hom}_G(\pi, \text{Ind}_P^G(\rho)) \simeq \text{Hom}_M(\pi_P, \rho)$$

holds. After (b) Jacquet's *subrepresentation theorem* directly follows from the above isomorphism (and the induction on the semi-simple rank etc).

(c) **Theorem.** (Jacquet) *For any irreducible admissible representation (π, V) of G , there exists a parabolic F -subgroup $P = \underline{M} \ltimes \underline{U}$ of G and an irreducible cuspidal representation ρ of $M = \underline{M}(F)$ such that π is embedded in $\text{Ind}_P^G(\rho)$.*

In this work we shall give the *relative version* (denoted by (A), (B), (C) below) of the above (a), (b), (c) respectively, assuming a version of the *relative Cartan decomposition* (#) which is a description of orbits in G/H under a maximal compact subgroup of G . Details of (#) will be explained in section 4. There are many concrete examples for which (#) is valid.

From now on we assume that the residual characteristic of F is not equal to 2. Let σ be an F -involution on G and let H be the σ -fixator, that is, the subgroup consisting of σ -fixed points in G . An admissible representation (π, V) of G is said to be *H-distinguished* if the space $(V^*)^H$ of H -invariant linear forms on V is non-zero. For each $\lambda \in (V^*)^H$ and $v \in V$ let $\phi_{\lambda, v}$ be the corresponding *generalized matrix coefficient* given by

$$\phi_{\lambda, v}(g) = \langle \lambda, \pi(g^{-1})v \rangle$$

for $g \in G$. These are right H -invariant smooth functions on G . We call such functions *H-matrix coefficients* of π . We put the following definition.

(A) **Definition.** An H -distinguished representation (π, V) of G is said to be *H-relatively cuspidal* if the support of every H -matrix coefficient of π is compact modulo ZH .

A parabolic F -subgroup $\underline{P}$ is said to be σ -split if $\underline{P}$ and $\sigma(\underline{P})$ are opposite. For such a $\underline{P}$ we shall always take $\underline{M} = \underline{P} \cap \sigma(\underline{P})$ as a (σ -stable) Levi subgroup of $\underline{P}$. In section 3 we shall construct a linear mapping

$$r_P : (V^*)^H \rightarrow ((V_P)^*)^{M \cap H}$$

between the spaces of invariant linear forms by using Casselman's *canonical lifts*. We shall give an asymptotic relation between H -matrix coefficients of π defined by $\lambda \in (V^*)^H$ and $(M \cap H)$ -matrix coefficients of the Jacquet module π_P defined by $r_P(\lambda)$. Using this relation, our criterion for relative cuspidality is given, in terms of Jacquet modules along σ -split parabolics, as follows.

(B) Theorem. Assume (#). An H -distinguished representation (π, V) of G is H -relatively cuspidal if and only if $r_P((V^*)^H) = 0$ for any proper σ -split parabolic F -subgroup $\underline{P}$ of $\underline{G}$.

After this characterization of relative cuspidality, our *relative subrepresentation theorem* is given (by the Frobenius reciprocity etc) as follows.

(C) Theorem. Assume (#). For any irreducible H -distinguished representation (π, V) of G , there exists a σ -split parabolic F -subgroup $\underline{P} = \underline{M} \ltimes \underline{U}$ of $\underline{G}$ and an irreducible $(M \cap H)$ -relatively cuspidal representation ρ of $M = \underline{M}(F)$ such that π is embedded in $\text{Ind}_P^G(\rho)$.

We hope that this theorem (C) provides a new foundation for the classification of distinguished representations attached to symmetric spaces, as (c) did for the classification of admissible representations.

Our statements (A), (B), (C) are generalizations of (a), (b), (c) respectively in the following sense: take a connected reductive F -group $\underline{G}_0$ and let $\underline{G}$ be the direct product $\underline{G} = \underline{G}_0 \times \underline{G}_0$. Consider the involution σ on $\underline{G}$ which permutes the factors. Then the corresponding symmetric space $\underline{G}/\underline{H}$ is isomorphic to the underlying space of $\underline{G}_0$. Such a situation is referred to as *the group case*. The assumption (#) is true for the group case by the ordinary Cartan decomposition for G_0 . The statements (A), (B), (C) applied to the group case will recover (a), (b), (c) for the group G_0 respectively. See section 6 for details.

Complete proofs of the statements in this article will be given in our forthcoming paper.

2 Notation for subgroups associated to σ

Let $\underline{G}$, σ and H be as in the introduction. For any F -subgroup $\underline{R}$ of $\underline{G}$, the group $\underline{R}(F)$ of F -points of $\underline{R}$ is denoted by R (by deleting the underbar).

An F -split subtorus $\underline{S}$ of $\underline{G}$ is said to be (σ, F) -split if $\sigma(s) = s^{-1}$ for all $s \in \underline{S}$. Fix a maximal (σ, F) -split torus $\underline{S}_0$ of $\underline{G}$. Take a maximal F -split torus $\underline{A}_0$ of $\underline{G}$ containing $\underline{S}_0$ and let Φ be the root system of $(\underline{G}, \underline{A}_0)$. Since $\underline{A}_0$ turns out to be σ -stable ([HW]), σ naturally acts on Φ . As in [HH] choose a σ -basis Δ of Φ satisfying

$$\alpha > 0, \sigma(\alpha) \neq \alpha \implies \sigma(\alpha) < 0$$

under the corresponding order.

Let $\underline{P}_0$ be the minimal parabolic F -subgroup of $\underline{G}$ containing $\underline{A}_0$, corresponding to the choice of Δ as above. Parabolic F -subgroups containing $\underline{P}_0$ are called standard parabolics. They correspond to subsets of Δ . For a subset $I \subset \Delta$ let $\underline{P}_I$ be the corresponding standard parabolic subgroup. Let $\underline{A}_I$ be the identity component of the intersection of all $\ker(\alpha)$, $\alpha \in I$, and set $\underline{M}_I = Z_{\underline{G}}(\underline{A}_I)$, the centralizer of $\underline{A}_I$ in $\underline{G}$. Then $\underline{A}_I$ is the F -split component of $\underline{M}_I$. One has a Levi decomposition $\underline{P}_I = \underline{M}_I \ltimes \underline{U}_I$ where $\underline{U}_I$ denotes the unipotent radical of $\underline{P}_I$. Let $\underline{P}_I^-$ be the unique parabolic subgroup such that $\underline{P}_I \cap \underline{P}_I^- = \underline{M}_I$ and $\underline{U}_I^-$ be its unipotent radical.

Recall that a parabolic F -subgroup $\underline{P}$ of $\underline{G}$ is said to be σ -split if $\underline{P}$ and $\sigma(\underline{P})$ are opposite. Let Δ_σ be the set of all σ -fixed roots in Δ . The condition for a standard parabolic subgroup $\underline{P}_I$ to be σ -split is given as follows ([HH]).

$\underline{P}_I$ is σ -split if and only if $\Delta_\sigma \subset I$ and the subsystem Φ_I generated by I is σ -stable.

Note that every σ -split parabolic F -subgroup of $\underline{G}$ arises as $\underline{P}_I$ in this way, for a suitable choice of $\underline{S}_0$, $\underline{A}_0$ and Δ .

For a standard σ -split parabolic F -subgroup $\underline{P}_I$ of $\underline{G}$, let $\underline{S}_I$ be the identity component of $\underline{A}_I \cap \underline{S}_0$. We call $\underline{S}_I$ the (σ, F) -split component of $\underline{P}_I$. Given a positive real number $\varepsilon > 0$, set

$$S_I^-(\varepsilon) = \{s \in S_I \mid |s^\alpha|_F \leq \varepsilon \ (\forall \alpha \in \Delta \setminus I)\}.$$

We shall often drop the subscript I if there is no fear of confusion. We shall say briefly that P is a σ -split parabolic subgroup of G if it is the group of F -points of a σ -split parabolic F -subgroup $\underline{P}$ of $\underline{G}$, and also say that S is the

(σ, F) -split component of P if it is the group of F -points of the (σ, F) -split component $\underline{S} = \underline{S}_I$ of $\underline{P} = \underline{P}_I$, and so on.

Lemma 2.1. *Let $P = M \ltimes U$ be a σ -split parabolic subgroup with the (σ, F) -split component S . For any two open compact subgroups U_1, U_2 of U , there exists a positive real number $\varepsilon \leq 1$ such that*

$$sU_1s^{-1} \subset U_2$$

for all $s \in S^-(\varepsilon)$.

For an open compact subgroup K of G and a parabolic subgroup $P = M \ltimes U$, set $U_K = U \cap K$, $M_K = M \cap K$ and $U_K^- = U^- \cap K$. If K is σ -stable and P is σ -split, it is obvious that $\sigma(U_K) = U_K^-$ and $\sigma(U_K^-) = U_K$. We say that K has the Iwahori factorization with respect to P if the product map

$$U_K^- \times M_K \times U_K \rightarrow K$$

is bijective.

To study Jacquet modules along σ -split parabolics we use a particular fundamental system $\{K_n\}$ of open neighborhoods of the identity in G : it consists of σ -stable open compact subgroups of G , having the Iwahori factorization with respect to all standard σ -split parabolic subgroups. (We just replace each K_n in [C, 1.4.4] by $K_n \cap \sigma(K_n)$.) We say that such a family $\{K_n\}$ is adapted to $(\underline{S}_0, \underline{A}_0, \Delta)$.

The following lemma is important for the investigation of invariant linear forms on Jacquet modules.

Lemma 2.2. *Let $K = K_n$ be an open compact subgroup in the family adapted to $(\underline{S}_0, \underline{A}_0, \Delta)$. Then for any corresponding standard σ -split parabolic subgroup P of G one has*

$$U_K \subset HM_K U_K^-.$$

3 Invariant linear forms on Jacquet modules

In this section we shall explain how to construct the mapping

$$r_P : (V^*)^H \rightarrow ((V_P)^*)^H$$

between the spaces of invariant linear forms mentioned in the introduction, and give the result on the asymptotic behaviour of H -matrix coefficients.

Let (π, V) be an admissible representation of G and $P = M \ltimes U$ be a σ -split parabolic subgroup of G , with the (σ, F) -split component S . The Jacquet module (π_P, V_P) of (π, V) along P is defined as follows: the space V_P is the quotient $V/V(U)$, where $V(U)$ denotes the subspace of V generated by all the elements of the form $\pi(u)v - v$, $u \in U$, $v \in V$. Let $j_P : V \rightarrow V_P$ be the canonical projection. The action π_P of M is normalized so that

$$\pi_P(m)j_P(v) = \delta_P^{-1/2}(m)j_P(\pi(m)v)$$

for $m \in M$.

Now we recall Casselman's *canonical lifting* ([C, §4]). For a compact subgroup K of G let V^K be the subspace of V of all K -fixed vectors and let $\mathcal{P}_K : V \rightarrow V^K$ be the projection operator given by

$$\mathcal{P}_K(v) = \frac{1}{\text{vol}(K)} \int_K \pi(k)v dk.$$

For a compact subgroup U_1 of U set

$$V(U_1) = \{v \in V \mid \int_{U_1} \pi(u)v du = 0\}.$$

It is known ([C, 3.2.1]) that $V(U)$ is the union of all $V(U_1)$ where U_1 ranges over all compact subgroups of U . Now, given $\bar{v} \in V_P$, take an open compact subgroup $K = K_n$ from the family $\{K_n\}$ adapted to $(\underline{S}_0, \underline{A}_0, \Delta)$ so that $\bar{v} \in (V_P)^{M_K}$. Next let us choose an open compact subgroup U_1 of U so that $V^K \cap V(U) \subset V(U_1)$. Finally, by 2.1 we can choose a positive real number $\varepsilon \leq 1$ so that for all $s \in S^-(\varepsilon)$, we have $sU_1s^{-1} \subset U_K$. Then, for all $s \in S^-(\varepsilon)$ the spaces $\mathcal{P}_K(\pi(s)V^K)$ are identical ([C, 4.1.6]) and by the restriction of $j_P : V \rightarrow V_P$ we have an isomorphism

$$\mathcal{P}_K(\pi(s)V^K) \xrightarrow{\cong} (V_P)^{M_K}$$

for any $s \in S^-(\varepsilon)$ ([C, 4.1.4]). The element $v \in \mathcal{P}_K(\pi(s)V^K)$ such that $j_P(v) = \bar{v}$ is called the canonical lift of $\bar{v} \in V_P$ with respect to K . It depends on the choice of K , but not on U_1 and ε . If v' is another canonical lift of $\bar{v}$, say, with respect to K' , then assuming that K' is contained in K we have ([C, 4.1.8])

$$v' \in V^{M_K U_K^-}, \quad v = \mathcal{P}_K(v') = \mathcal{P}_{U_K}(v').$$

By this relation and the lemma 2.2 we have:

Lemma. Let λ be an H -invariant linear form on an H -distinguished representation (π, V) of G . Let $P = M \ltimes U$ be a σ -split parabolic subgroup of G and $v, v' \in V$ be canonical lifts of the same element $\bar{v} \in V_P$. Then

$$\langle \lambda, v \rangle = \langle \lambda, v' \rangle.$$

After this lemma we may define a linear form $r_P(\lambda)$ on the Jacquet module V_P along a σ -split parabolic subgroup P as follows:

Definition. Let $\lambda \in (V^*)^H$ be an H -invariant linear form on an H -distinguished representation (π, V) of G and P be a σ -split parabolic subgroup of G . The linear form $r_P(\lambda)$ on the Jacquet module V_P is defined by

$$\langle r_P(\lambda), \bar{v} \rangle = \langle \lambda, v \rangle$$

for each $\bar{v} \in V_P$ if $v \in V$ is a canonical lift of $\bar{v}$.

This construction of $r_P(\lambda)$ is a relative version of Casselman's *canonical pairing* of Jacquet modules ([C, 4.2.2]). See section 6.

Next we give the following proposition which describes the asymptotic behaviour of H -matrix coefficients.

Proposition. Let (π, V) be an H -distinguished representation of G and λ be an H -invariant linear form on V . Let $P = M \ltimes U$ be a σ -split parabolic subgroup of G with the (σ, F') -split component S .

- (i) For each $v \in V$, there exists a positive real number $\varepsilon \leq 1$ such that for any $s \in S^-(\varepsilon)$ one has

$$\langle \lambda, \pi(s)v \rangle = \delta_P^{1/2}(s) \langle r_P(\lambda), \pi_P(s)j_P(v) \rangle.$$

- (ii) Assume that $\bar{\lambda}$ is a linear form on V_P having the following property: for each $v \in V$, there exists a positive real number $\varepsilon \leq 1$ such that for any $s \in S^-(\varepsilon)$ one has

$$\langle \lambda, \pi(s)v \rangle = \delta_P(s)^{1/2} \langle \bar{\lambda}, \pi_P(s)j_P(v) \rangle.$$

Then $\bar{\lambda}$ coincides with $r_P(\lambda)$.

This is a relative version of [C, 4.2.3].

The $(M \cap H)$ -invariance of the linear form $r_P(\lambda)$ is shown after (ii) of the above proposition.

Corollary(1). *The linear form $r_P(\lambda)$ on V_P is $M \cap H$ -invariant and the mapping $r_P : (V^*)^H \rightarrow ((V_P)^*)^{M \cap H}$ is linear.*

This is seen as follows: for any $m \in M \cap H$ put $\bar{\lambda} = r_P(\lambda) \circ \pi_P(m)$. Then $\bar{\lambda}$ has the property that $r_P(\lambda)$ must have in (ii). As a consequence we have $r_P(\lambda) = r_P(\lambda) \circ \pi_P(m)$.

(ii) of the above proposition has one more important corollary on the transitivity with respect to the inclusion of σ -split parabolics: let P, Q be σ -split parabolic subgroups of G with $P \supset Q$. Let M, L be the σ -stable Levi subgroup of P, Q respectively. In such a case, $M \cap Q$ is a σ -split parabolic subgroup of M . As is well-known, $(V_P)_{M \cap Q}$ is naturally isomorphic to V_Q as an L -module. There are induced mappings

$$r_P : (V^*)^H \rightarrow ((V_P)^*)^{M \cap H}, \quad r_{M \cap Q} : ((V_P)^*)^{M \cap H} \rightarrow (((V_P)_{M \cap Q})^*)^{L \cap H}$$

and

$$r_Q : (V^*)^H \rightarrow ((V_Q)^*)^{L \cap H} \left(\simeq (((V_P)_{M \cap Q})^*)^{L \cap H} \right)$$

of invariant linear forms.

Corollary(2). *For P, Q as above, one has*

$$r_{M \cap Q} \circ r_P = r_Q.$$

That is, the diagram

$$\begin{array}{ccc} (V^*)^H & \xrightarrow{r_P} & (V_P^*)^{M \cap H} \\ r_Q \downarrow & & \downarrow r_{M \cap Q} \\ (V_Q^*)^{L \cap H} & \xrightarrow{\simeq} & (((V_P)_{M \cap Q})^*)^{L \cap H} \end{array}$$

is commutative.

Indeed, $\bar{\lambda} := r_{M \cap Q} \circ r_P(\lambda)$ has the property that $r_Q(\lambda)$ must have in (ii) of the proposition.

4 Characterization of relative cuspidality

In this section we shall explain our assumption on the orbit decomposition of G/H , which we call *relative Cartan decomposition*. Then we shall give a rough sketch of the way to obtain our theorem (B) from the assumption (#).

Choose a maximal (σ, F) -split torus $\underline{S}_0$, a maximal F -split torus $\underline{A}_0$ containing $\underline{S}_0$, and a σ -basis Δ . Let $\underline{M}_0$ be the centralizer of $\underline{S}_0$ in $\underline{G}$ (which coincides with $\underline{M}_{\Delta_\sigma}$) and set

$$S_0^+ = \{s \in S_0 \mid |s^\alpha|_F \geq 1 (\forall \alpha \in \Delta)\} = \{s^{-1} \mid s \in S_0^-(1)\}.$$

Assumption (#). For a suitable choice of maximal compact subgroup $K_{\max}$ of G , there exists a finite subset Γ of $(\underline{M}_0 \cdot \underline{H})(F)$ such that

$$G = K_{\max} \cdot S_0^+ \cdot \Gamma \cdot H.$$

There are many examples of symmetric pairs (G, H) satisfying this assumption, such as

$$(G, H) = (\mathrm{GL}_n(F), \mathrm{O}_n(F)), (\mathrm{GL}_n(E), \mathrm{U}_n(E/F)), (\mathrm{GL}_{2n}(F), \mathrm{Sp}_n(F)), \\ (\mathrm{GL}_n(F), \mathrm{GL}_r(F) \times \mathrm{GL}_{n-r}(F)), (\mathrm{GL}_n(E), \mathrm{GL}_n(F)), \dots$$

where E/F is a quadratic extension. See [H] for the first four and [T] for the last one. See also [U] for related matters.

We shall briefly explain how to derive theorem (B) under the assumption (#). Let (π, V) be an H -distinguished representation of G and $\lambda \in (V^*)^H$ be an H -invariant linear form on V . For each $v \in V$ consider the H -matrix coefficient $\phi_{\lambda, v}$ defined by

$$\phi_{\lambda, v}(g) = \langle \lambda, \pi(g^{-1})v \rangle.$$

Let P be a σ -split parabolic subgroup of G and S be the (σ, F) -split component of P . Since $v \in V$ is $K_{\max}$ -finite, we may choose a positive real number $\varepsilon \leq 1$ in (i) of the proposition of section 4 so that the relation

$$\langle \lambda, \pi(s)\pi(k)v \rangle = \delta_P^{1/2}(s) \langle r_P(\lambda), \pi_P(s)j_P(\pi(k)v) \rangle$$

holds for all $s \in S^-(\varepsilon)$ and all $k \in K_{\max}$. Now assume that $r_P(\lambda) = 0$. Then for all $h \in H$, $s \in S^-(\varepsilon)$ and $k \in K_{\max}$ we must have

$$\phi_{\lambda, v}(k^{-1}s^{-1}h) = \langle \lambda, \pi(s)\pi(k)v \rangle = 0.$$

That is, $\phi_{\lambda,v}$ is zero on the double coset $K_{\max} \cdot s^{-1} \cdot H$ for all $s \in S^-(\varepsilon)$. The compactness of the support of $\phi_{\lambda,v}$ in the union

$$\bigcup_{s \in S_0^-(1)} K_{\max} \cdot s^{-1} \cdot H$$

modulo ZH readily follows by varying P in the proper standard σ -split parabolics. Unfortunately $\bigcup K_{\max} \cdot s^{-1} \cdot H$ does not cover all of G in any example we examined. We need a complementary finite set Γ to cover all of G as in (#). Roughly speaking, it is possible to show the compactness of the support in

$$\bigcup_{s \in S_0^-(1)} K_{\max} \cdot s^{-1} \gamma^{-1} \cdot H$$

by a similar discussion at least if $\gamma \in (\underline{M}_0 \cdot \underline{H})(F)$. Thus, assuming that the complementary elements γ can be chosen from $(\underline{M}_0 \cdot \underline{H})(F)$, we have one direction of the theorem.

(B) Theorem. (Characterization of Relative Cuspidality)
Assume (#) for (G, σ) . An H -distinguished representation (π, V) of G is H -relatively cuspidal if and only if $r_P((V^)^H) = 0$ for any proper σ -split parabolic F -subgroup $\underline{P}$ of $\underline{G}$.*

5 Relative subrepresentation theorem

We need the following lemma. It is non-trivial but the proof is elementary.

Lemma. *A finitely generated H -relatively cuspidal representation has an irreducible H -distinguished quotient.*

Now we shall give a rough sketch of the proof of our main theorem (C). If (π, V) is H -relatively cuspidal there is nothing to prove. If not, then there is a proper σ -split parabolic subgroup P of G such that $r_P((V^*)^H) \neq 0$. Let $P = M \ltimes U$ be minimal one. If we assume (#) for G , then by the corollary (2) of section 3 it is seen that the Jacquet module π_P is $(M \cap H)$ -relatively cuspidal. Apply the above lemma to take an irreducible $M \cap H$ -distinguished quotient ρ of π_P . By the Frobenius reciprocity

$$\mathrm{Hom}_G(\pi, \mathrm{Ind}_P^G(\rho)) \simeq \mathrm{Hom}_M(\pi_P, \rho) (\neq 0),$$

there is an embedding of π into $\text{Ind}_P^G(\rho)$. If ρ is not relatively cuspidal apply the same procedure for ρ . Here we need to assume $(\#)$ also for M . In this way we have

(C) Theorem. (Relative Subrepresentation Theorem)

Assume $(\#)$ for all σ -stable Levi subgroups of σ -split parabolic subgroups of G . For any irreducible H -distinguished representation (π, V) of G , there exists a σ -split parabolic F -subgroup $P = M \ltimes U$ of G and an irreducible $(M \cap H)$ -relatively cuspidal representation ρ of M such that π is embedded in $\text{Ind}_P^G(\rho)$.

6 The group case

Take a connected reductive F -group $\underline{G}_0$ and let $\underline{G}$ be the direct product $\underline{G} = \underline{G}_0 \times \underline{G}_0$. Let σ be the involution on $\underline{G}$ which permutes the factors. Then the σ -fixator H in G is the diagonal subgroup

$$H = \Delta(G_0) = \{(g, g) \in G_0 \times G_0 \mid g \in G_0\}.$$

The map $(g_1, g_2) \mapsto g_1 g_2^{-1}$ induces an identification $G/H = (G_0 \times G_0) / \Delta(G_0) \simeq G_0$. We shall apply our theory to this situation.

- Distinguishedness.

Any irreducible admissible representation π of $G = G_0 \times G_0$ is of the form $\pi_0 \otimes \pi'_0$ where π_0, π'_0 are irreducible admissible representations of G_0 . It is $H = \Delta(G_0)$ -distinguished if and only if $\pi'_0 \simeq \tilde{\pi}_0$, that is, π is of the form $\pi_0 \otimes \tilde{\pi}_0$ for an irreducible admissible representation π_0 of G_0 .

- (A) for the group case means (a).

The natural pairing between π_0 and $\tilde{\pi}_0$ gives a non-zero $\Delta(G_0)$ -invariant linear form $\lambda \in ((\pi_0 \otimes \tilde{\pi}_0)^*)^{\Delta(G_0)}$ (which is unique up to constant) by

$$\langle \lambda, v_0 \otimes \tilde{v}_0 \rangle = \langle \tilde{v}_0, v_0 \rangle_{\tilde{\pi}_0 \times \pi_0}.$$

The H -matrix coefficients defined by λ are identified with the usual matrix coefficients of π_0 through the map $(g_1, g_2) \mapsto g_1 g_2^{-1}$ as follows:

$$\langle \lambda, (\pi_0 \otimes \tilde{\pi}_0)(g_1, g_2)^{-1}(v_0 \otimes \tilde{v}_0) \rangle = \langle \lambda, \pi_0(g_1^{-1})v_0 \otimes \tilde{\pi}_0(g_2^{-1})\tilde{v}_0 \rangle$$

$$= \langle \tilde{\pi}_0(g_2^{-1})\tilde{v}_0, \pi_0(g_1^{-1})v_0 \rangle_{\tilde{\pi}_0 \times \pi_0} = \langle \tilde{v}_0, \pi_0((g_1 g_2^{-1})^{-1})v_0 \rangle_{\tilde{\pi}_0 \times \pi_0}.$$

Thus it is obvious that $\pi = \pi_0 \otimes \tilde{\pi}_0$ is H -relatively cuspidal if and only if π_0 is cuspidal as a representation of G_0 .

- (#) is true for the group case.

In the group case the decomposition in (#) follows from the *ordinary Cartan decomposition* for the group G_0 : take a maximal F -split torus $\underline{A}_0$ of $\underline{G}_0$ and let K_0 be an $\underline{A}_0$ -good maximal compact subgroup of G_0 . The ordinary Cartan decomposition asserts that

$$G_0 = K_0 \cdot A_0^+ \cdot \Gamma_0 \cdot K_0$$

for a suitable finite subset Γ_0 of $M_0 = Z_{G_0}(A_0)$ [S, §0.6]. Now the map $(g_1, g_2) \mapsto g_1 g_2^{-1}$ induces an identification

$$(K_0 \times K_0) \backslash (G_0 \times G_0) / \Delta(G_0) \simeq K_0 \backslash G_0 / K_0,$$

which implies that (#) is true by taking $K_{\max} = K_0 \times K_0$.

- The mapping r_P for the group case.

The σ -split parabolic F -subgroups of $G = G_0 \times G_0$ are those of the form $P_0 \times P_0^-$ where P_0 and P_0^- are opposite parabolic F -subgroups of G_0 . Set $M_0 = P_0 \cap P_0^-$. For an *irreducible* $\Delta(G_0)$ -distinguished representation $\pi_0 \otimes \tilde{\pi}_0$ of $G_0 \times G_0$, let $\lambda \in ((\pi_0 \otimes \tilde{\pi}_0)^*)^{\Delta(G_0)}$ be as above. Then $r_P(\lambda) = r_{P_0 \times P_0^-}(\lambda)$ is a linear form on the Jacquet module

$$(\pi_0 \otimes \tilde{\pi}_0)_{P_0 \times P_0^-} \simeq (\pi_0)_{P_0} \otimes (\tilde{\pi}_0)_{P_0^-}$$

which is invariant under

$$(M_0 \times M_0) \cap \Delta(G_0) = \Delta(M_0).$$

So $r_P(\lambda)$ gives an M_0 -invariant bilinear form on $(\pi_0)_{P_0} \times (\tilde{\pi}_0)_{P_0^-}$. It coincides with the one constructed by Casselman in [C, §4].

- (B) for the group case implies (b).

Now the linear form $r_{P_0 \times P_0^-}(\lambda)$ vanishes if and only if the Jacquet module $(\pi_0)_{P_0}$ vanishes, since Casselman's pairing was shown to be non-degenerate ([C, 4.2.4]). Thus our theorem (B) applied to the group case actually recovers (b).

- (C) for the group case implies (c).

Finally, apply our theorem (C) to the group case. We then assert that for any irreducible admissible ΔG_0 -distinguished representation $\pi_0 \otimes \tilde{\pi}_0$ of $G_0 \times G_0$, there exists a σ -split parabolic subgroup $P_0 \times P_0^-$, and an irreducible ΔM_0 -relatively cuspidal representation $\rho_0 \otimes \tilde{\rho}_0$ of $M_0 \times M_0$, such that $\pi_0 \otimes \tilde{\pi}_0$ can be embedded in

$$\mathrm{Ind}_{P_0 \times P_0^-}^{G_0 \times G_0}(\rho_0 \otimes \tilde{\rho}_0) \simeq \mathrm{Ind}_{P_0}^{G_0}(\rho_0) \otimes \mathrm{Ind}_{P_0^-}^{G_0}(\tilde{\rho}_0).$$

Now (c) is recovered at the first factor. At the second factor we also have an embedding $\tilde{\pi}_0 \hookrightarrow \mathrm{Ind}_{P_0^-}^{G_0}(\tilde{\rho}_0)$ as in [S, 3.3.1].

7 Concluding remarks

- By theorem (B) it turns out that *cuspidal distinguished representations are relatively cuspidal in our sense*. Examples of such representations were constructed by Hakim and Mao, for symmetric pairs $(G, H) = (\mathrm{GL}_n(F), \mathrm{O}_n(F))$, $(\mathrm{GL}_n(E), \mathrm{U}_n(E/F))$. See [HM1], [HM2].
- We have studied several examples of *non-cuspidal but relatively cuspidal* representations, for symmetric pairs $(G, H) = (\mathrm{GL}_{2n}(F), \mathrm{Sp}_n(F))$ and $(\mathrm{GL}_n(F), \mathrm{GL}_{n-1}(F) \times \mathrm{GL}_1(F))$. For these pairs it is known that there is no cuspidal distinguished representation (see [HR] and [P]). Details will be included in our forthcoming paper.
- In recent preprint ([BD]) Blanc and Delorme studied the distinguishedness of a class of induced representations. They used only σ -split parabolic subgroups as the inducing subgroups, and distinguished representations of σ -stable Levi subgroups as the inducing representations. Their work seems to include the *adjoint* operation (in some sense) to our construction of the mapping r_P .

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