

Perfect Isometries for Blocks with Abelian Defect

Groups and Klein Four Inertial Quotients

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1. Alperin's weight conjecture for the case of abelian defect groups

Let p be a prime number, k an algebraically closed field of characteristic p , O a complete discrete valuation ring with residue field k and quotient field K of characteristic zero, G a finite group, b a p -block of G (i.e. a primitive idempotent of $Z(kG)$), P a defect group of b , e a root of b in $C_G(P)$ and E the inertial quotient $N_G(P, e)/PC_G(P)$. We assume that K is large enough.

Alperin's weight conjecture states that the number $l(b)$ of isomorphism classes of simple kGb -modules can be calculated by the function of its local structure. When P is abelian, this is equivalent to the following one. ([1])

Conjecture 1. If P is abelian, then $l(b)$ is the number of isomorphism classes of simple $kN_G(P, e)e$ -modules.

This is known to be true if $|E| \leq 3$ by the results of Brauer (cf. [3], Proposition (6G)) and Usami [13] (except the case $|E| = 3$ and $p = 2$). Here we introduce the result which proves it in the case E is a Klein four group (and in the case $|E| = 3$ and $p = 2$).

2. Reformed conjecture

First we want to reform Conjecture 1 in terms of a suitable k^* -central extension of E . Setting $\bar{N}_G(P, e) = N_G(P, e)/P$, $\bar{C}_G(P) = C_G(P)/P$ and denoting by $\bar{e}$ the image of e in $k\bar{C}_G(P)$, it is well known from Brauer that $k\bar{C}_G(P)\bar{e}$ is a simple k -algebra (i.e. a full matrix algebra over k) and , in particular, we have $Z(k\bar{C}_G(P)\bar{e}) \cong k$; hence, by Skolem-Noether's theorem , we have an exact sequence

$$1 \longrightarrow k^* \longrightarrow (k\bar{C}_G(P)\bar{e})^* \xrightarrow{\pi} \text{Aut}(k\bar{C}_G(P)\bar{e}) \longrightarrow 1$$

so that $(k\bar{C}_G(P)\bar{e})^*$ can be seen as a k^* -central extension. Since $N_G(P, e)$ acts on $k\bar{C}_G(P)\bar{e}$, we have a group homomorphism $\mathcal{P}: \bar{N}_G(P, e) \longrightarrow \text{Aut}(k\bar{C}_G(P)\bar{e})$ and then $\widehat{N}_G(P, e)$ is the k^* -central extension of $\bar{N}_G(P, e)$ induced by $(k\bar{C}_G(P)\bar{e})^*$: that is to say, $\widehat{N}_G(P, e)$ is the subgroup of

$$(\bar{a} , \bar{n}) \in (k\bar{C}_G(P)\bar{e})^* \times \bar{N}_G(P, e)$$

such that $\pi(\bar{a}) = \mathcal{P}(\bar{n})$ and we get a commutative and exact diagram

$$\begin{array}{ccccccc} 1 & \longrightarrow & k^* & \longrightarrow & (k\bar{C}_G(P)\bar{e})^* & \xrightarrow{\pi} & \text{Aut}(k\bar{C}_G(P)\bar{e}) \longrightarrow 1 \\ & & \uparrow \text{id} & & \uparrow \widehat{\mathcal{P}} & & \uparrow \mathcal{P} \\ 1 & \longrightarrow & k^* & \longrightarrow & \widehat{N}_G(P, e) & \longrightarrow & \bar{N}_G(P, e) \longrightarrow 1 \end{array}$$

Now , the twisted algebra $k_*\widehat{N}_G(P, e)$ is the quotient of the full group algebra by the ideal generated by the elements $\lambda(\bar{a} , \bar{n}) - (\lambda\bar{a} , \bar{n})$ where λ runs over k^* and $(\bar{a} , \bar{n})$ over $\widehat{N}_G(P, e)$. (We can

define $0_* \hat{N}_G(P, e)$, since there is a unique section $k^* \longrightarrow 0^*$ of the canonical homomorphism $0^* \longrightarrow k^*$.) Moreover, we have an injective group homomorphism

$$\bar{C}_G(P) \longrightarrow \hat{N}_G(P, e)$$

mapping $\bar{z} \in \bar{C}_G(P)$ on $(\bar{z}\bar{e}, \bar{z}) \in \hat{N}_G(P, e)$ and its image is a normal subgroup of $\hat{N}_G(P, e)$ intersecting trivially the image of k^* , so the corresponding quotient is a k^* -central extension of E . We denote by $\hat{E}$ the opposite one; that is to say, denoting by $\hat{N}_G(P, e)^\circ$ the set $\hat{N}_G(P, e)$ endowed with opposite product, we have the exact sequence

$$1 \longrightarrow \bar{C}_G(P) \longrightarrow \hat{N}_G(P, e)^\circ \xrightarrow{\hat{\sigma}} \hat{E} \longrightarrow 1$$

where $\bar{z} \in \bar{C}_G(P)$ maps on $(\bar{z}\bar{e}, \bar{z})^{-1}$. The following more or less known lemma explains the role of $\hat{E}$ (see also [9], Proposition 14.6 in [11], Proposition 2.1 in [10] and Lemma 2.5 in [12]).

Lemma 1. With the notation above, there is an algebra isomorphism

$$k\bar{N}_G(P, e)\bar{e} \cong k\bar{C}_G(P)\bar{e} \otimes_k k_* \hat{E}$$

mapping $\bar{n}\bar{e}$ on $\hat{f}(\hat{n}) \otimes \hat{\sigma}(\hat{n})^{-1}$, where $\bar{n} \in \bar{N}_G(P, e)$ and $\hat{n}$ is an element of $\hat{N}_G(P, e)$ lifting $\bar{n}$.

Let $\hat{L}$ be the semidirect product of $\hat{E}$ and P . Since the number of isomorphism classes of simple $k_* \hat{L}$ -modules is equal to the number of isomorphism classes of simple $k_* \hat{E}$ -modules, we can reform Conjecture 1 by Lemma 1 as follows.

Conjecture 2. If P is abelian, then $l(b)$ is the number of

isomorphism classes of simple $k_*\hat{L}$ -modules.

Hence we must study the relation between $OG\hat{b}$ and $0_*\hat{L}$, where $\hat{b}$ denotes the unique primitive idempotent lifting b to $Z(OG)$. We denote respectively by $L_K(\hat{L})$ and $L_K(G, b)$ the Grothendieck groups of the categories of $K_*\hat{L}$ -modules and ordinary K -representations of G in b . We expect that there exists a special kind of bijective isometry between $L_K(\hat{L})$ and $L_K(G, b)$.

3. Preliminaries and the main theorem

Following [2] and [6], we consider Brauer morphism Br_Q for a p -subgroup Q of G and (b, G) -Brauer pairs. Note that (P, e) is a maximal (b, G) -Brauer pair and for a p -subgroup Q of P , $(Q, e^{C_G(Q)})$ is a (b, G) -Brauer pair contained in (P, e) . One of the typical properties of blocks with abelian defect groups is the following one.

Lemma 2. (Proposition 4.21 in [2]) Assume that P is abelian. If (Q, f) is a (b, G) -Brauer pair such that $(Q, f) \subset (P, e)$ and x an element of G such that $(Q, f)^x \subset (P, e)$, then there are $z \in C_G(Q)$ and $n \in N_G(P, e)$ such that $x = zn$. In particular, if U is a set of representatives for the orbits of E in P , then $\{(u, e^{C_G(u)})\}_{u \in U}$ is a set of representatives for the conjugacy classes of (b, G) -Brauer elements.

It is not difficult to handle $0_*\hat{L}$, since there are a finite

subgroup L' of $\widehat{L}$ and a p -block b' of L' such that the inclusion $L' \subset L$ induces a bijective isometry $L_K(\widehat{L}) = L_K(L', b')$ and an algebra isomorphism $O_{*}\widehat{L} \cong OL'b'$ (see Remark 5 in section 1 in [8], Lemma 5.5 and Proposition 5.15 in [11]). Furthermore P is also a defect group of b' and E is also the inertial quotient of b' , since P is the normal Sylow p -subgroup of L' and $(P, \text{Br}_P(b'))$ is the unique maximal (b', L') -Brauer pair. We remark that

(3.1) $(Q, \text{Br}_Q(b'))$ is the unique (b', L') -Brauer pair for a fixed p -subgroup Q of P .

From now on we introduce some general notation and results without any hypothesis until we state Theorem 1 (i.e. E is arbitrary and we do not assume that P is abelian for the moment). We denote by $CF_K(G)$ and $CF_0(G)$ the sets of K - and O -valued central functions over G , so that $CF_0(G) \subset CF_K(G)$, and we identify $CF_K(G)$ with $K \otimes_O CF_0(G)$ and with the set of central K -linear forms over KG (or OG). We denote respectively by $L_K(G)$ and $L_k(G)$ the Grothendieck groups of the categories of KG - and kG -modules (of finite dimension) and we identify $L_K(G)$ with its image in $CF_0(G)$. We also identify any element of $L_k(G)$ with its Brauer character; that is to say, denoting respectively by $BCF_K(G)$ and $BCF_0(G)$ (BCF for "Brauer central function") the sets of K - and O -valued G -central functions over the set G_p , of elements of G of order prime to p , we also identify $L_k(G)$ with its image in $BCF_0(G)$ and $BCF_K(G)$ with $K \otimes_O BCF_0(G)$. Recall that the inclusion $L_k(G) \subset BCF_0(G)$ induces an isomorphism

$$(3.2) \quad O \otimes_Z L_k(G) \cong BCF_0(G).$$

Following Brauer, we denote by

$$d_G : CF_K(G) \longrightarrow BCF_K(G)$$

the restriction map, which fulfills

$$(3.3) \quad d_G(L_K(G)) = L_K(G).$$

Moreover we denote by $CF_K^\circ(G)$ the kernel of d_G and set $L_K^\circ(G) = CF_K^\circ(G) \cap L_K(G)$. It is clear that d_G induces a bijection between the orthogonal subspace of $CF_K^\circ(G)$ and $BCF_K(G)$, and then the inverse map determines a section of d_G

$$e_G : BCF_K(G) \longrightarrow CF_K(G)$$

and induces an scalar product on $BCF_K(G)$; thus, d_G and e_G become adjoint maps.

More generally, following Broué [4], for any p -element u of G we consider the "twisted" restriction

$$d_G^u : CF_K(G) \longrightarrow BCF_K(C_G(u))$$

mapping $\chi \in CF_K(G)$ on the $C_G(u)$ -central function over $C_G(u)_p$, which maps $s \in C_G(u)_p$, on $\chi(us)$, and denote by

$$e_G^u : BCF_K(C_G(u)) \longrightarrow CF_K(G)$$

the adjoint K -linear map, which is a section of d_G^u .

It is well-known that any idempotent of $Z(kG)$ determines a selfadjoint projector over $CF_K(G)$ which stabilizes $CF_0(G)$ and $L_K(G)$, and commutes with $e_G \circ d_G$, so that it determines a self-adjoint projector over $BCF_K(G)$ stabilizing $BCF_0(G)$ and $L_K(G)$. In particular, for any element χ of $CF_K(G)$ or $BCF_K(G)$, we denote by $b.\chi$ the image of χ by the projector determined by b and set

$$b.CF_K(G) = CF_K(G, b) \text{ and } b.BCF_K(G) = BCF_K(G, b).$$

Moreover, for any p -element u of G , we have (cf. [4] Appendixes)

$$(3.4) \quad d_G^u(b.\chi) = Br_u(b).d_G^u(\chi) \text{ and } e_G^u(Br_u(b).\varphi) = b.e_G^u(\varphi)$$

for any $\chi \in CF_K(G)$ and any $\varphi \in BCF_K(C_G(u))$ (where $Br_u = Br_{\langle u \rangle}$).

Consequently, for any $\chi \in CF_K(G, b)$ and any (b, G) -Brauer element (u, g) we consider the central function

$$\chi^{(u, g)} = e_G^u(g \cdot d_G^u(\chi))$$

which still belongs to $CF_K(G, b)$. Notice that we have

$$\chi^{(u, g)}(u) = \chi(u\hat{g}),$$

where $\hat{g}$ is the unique primitive idempotent of $Z(OC_G(u))$ lifting g . We remark that

$$(3.5) \quad \chi = \sum_{(u, g)} \chi^{(u, g)}$$

and for any $\chi, \chi' \in CF_K(G, b)$ we get

$$(3.6) \quad (\chi, \chi')_G = \sum_{(u, g)} (\chi^{(u, g)}, \chi'^{(u, g)})_G$$

where (u, g) runs over a set of representatives for the conjugacy classes of (b, G) -Brauer elements.

Following [6], a central function λ over P is called (G, e) -stable if, for any (b, G) -Brauer element (u, g) such that $(\langle u \rangle, g) \subset (P, e)$ and any $x \in G$ such that $(\langle u^x \rangle, g^x) \subset (P, e)$, we have $\lambda(u^x) = \lambda(u)$. In that case, for any $\chi \in CF_K(G, b)$, we consider the new central function

$$\lambda * \chi = \sum_{(u, g)} \lambda(u) \chi^{(u, g)}$$

where (u, g) runs over a set of representatives such that $(\langle u \rangle, g) \subset (P, e)$ for the conjugacy classes of (b, G) -Brauer elements, which still belongs to $CF_K(G, b)$ and does not depend on the choice of the set of representatives. We remark that

$$g \cdot d_G^u(\lambda * \chi) = \lambda(u)(g \cdot d_G^u(\chi)).$$

Then, by the main result in [6], if λ and χ are generalized characters, so is $\lambda * \chi$. Notice that, by Lemma 2, if P is

abelian, a central function over P is (G, e) -stable if and only if it is E -stable. We denote by $CF_0(P)^E$ the O -module of E -stable O -valued central functions over P .

We are ready to state our main theorem (Theorem 1.5 in [12]).

Theorem 1. With the notation above, assume that P is abelian and E is a Klein four group. Then there is a bijective isometry

$$\Delta : CF_0(\widehat{L}) \longrightarrow CF_0(G, b)$$

such that

$$\Delta(L_K(\widehat{L})) = L_K(G, b)$$

and

$$(3.7) \quad \Delta(\lambda * \eta) = \lambda * \Delta(\eta)$$

for any $\lambda \in CF_0(P)^E$ and any $\eta \in CF_0(\widehat{L})$.

(3.7) implies that Δ fulfills Definition 4.3 in [5] (i.e. (3.7) guarantees the existence of a local system in Broué's terms) and therefore, by Lemma 4.5 in [5], Δ is a perfect isometry in Broué's terms. (We discuss perfect isometries in section 5.) By Proposition 1.3 and Theorem 1.5 in [5] we have a following corollary.

Corollary 1. If P is abelian and E is a Klein four group, then the following hold with the notation of Theorem 1.

- (i) Δ is a perfect isometry from $L_K(\widehat{L})$ onto $L_K(G, b)$.
- (ii) Δ induces a bijective isometry from $L_K(\widehat{L})$ onto $L_K(G, b)$ and hence Alperin's weight conjecture (Conjecture 2) holds in this case.

- (iii) The algebra isomorphism Δ^* from $Z(K_*\hat{L})$ onto $Z(KG\hat{b})$ determined by the isometry Δ maps $Z(O_*\hat{L})$ onto $Z(OG\hat{b})$.
- (iv) Δ preserves the height of irreducible ordinary characters. In particular, all the irreducible ordinary characters of G in b have height zero and Alperin-McKay conjecture holds in this case.
- (v) Δ preserves the elementary divisors of the Cartan matrices.

4. Local systems for blocks with abelian defect groups

Before we introduce Theorem 2 in this section, we assume only that P is abelian (and E is arbitrary).

By Lemma 2 E controls the fusion of (b, G) -Brauer pairs (resp. (b', L') -Brauer pairs). Then in the summation of (3.5) we have only to make u run over U .

Applying inductive method, we hope to construct a bijective isometry Δ_Q from $CF_0(C_L^\wedge(Q))$ onto $CF_0(C_G(Q), f)$ for each p -subgroup Q of P where $f = e^{C_G(Q)}$. (That is to say, first we construct it for $Q=P$ and we hope to construct it for $Q=1$ finally.) We note that (P, e) is also a maximal $(f, C_G(Q))$ -Brauer pair and (u, g) is a $(f, C_G(Q))$ -Brauer element contained in it for any element u of P , where $g = e^{C_G(Q\langle u \rangle)}$. By Lemma 2 $C_E(Q)$ controls the fusion of $(f, C_G(Q))$ -Brauer pairs, and by (3.5) for any $\chi \in CF_K(C_G(Q), f)$

$$(4.1) \quad \chi = \sum_{u \in U_Q} e_{C_G(Q)}^u (g \cdot d_{C_G(Q)}^u (\chi))$$

where U_Q is a set of representatives for the orbits of $C_E(Q)$ in P . Similarly we note that $(P, Br_P(b'))$ is the maximal $(Br_Q(b'), C_L(Q))$ -Brauer pair and $(u, Br_{Q\langle u \rangle}(b'))$ is a $(Br_Q(b'), C_L(Q))$ -Brauer element contained in it for any element u of P by (3.1). By Lemma 2 and (3.5) for any $\eta \in CF_K(C_L^\wedge(Q))$

$$(4.2) \quad \eta = \sum_{u \in U_Q} e_{C_L^\wedge(Q)}^u (d_{C_L^\wedge(Q)}^u(\eta)),$$

since we can omit $Br_{Q\langle u \rangle}(b')$ by (3.1) and (3.4).

Let X be an E -stable non-empty set of subgroups of P and assume that X contains any subgroup of P containing an element of X . We call any map Γ (G, b) -local system over X , if Γ is defined over X and sends $Q \in X$ to a bijective isometry

$$\Gamma_Q : BCF_K(C_L^\wedge(Q)) \cong BCF_K(C_G(Q), f)$$

where $f = e_{C_G(Q)}^{C_G(Q)}$, and fulfills the following two conditions:

(4.3) For any $Q \in X$, any $\eta \in BCF_K(C_L^\wedge(Q))$ and any $s \in E$, we have

$$\Gamma_Q(\eta)^s = \Gamma_{Q^s}(\eta^s).$$

(4.4) For any $Q \in X$ and any $\eta \in L_K(C_L^\wedge(Q))$, the sum

$$\sum_{u \in U_Q} e_{C_G(Q)}^u (\Gamma_{Q\langle u \rangle}(d_{C_L^\wedge(Q)}^u(\eta)))$$

is a generalized character of $C_G(Q)$.

We examine these conditions to give more explicit expression.

For any $Q \in X$ and any $\eta \in CF_K(C_L^\wedge(Q))$, the sum

$$(4.5) \quad \Delta_Q(\eta) = \sum_{u \in U_Q} e_{C_G(Q)}^u (\Gamma_{Q\langle u \rangle}(d_{C_L^\wedge(Q)}^u(\eta)))$$

is certainly an element of $CF_K(C_G(Q), f)$ (cf. (3.4), (4.1) and (4.2)), and we have, setting $g = e_{C_G(Q\langle u \rangle)}^{C_G(Q\langle u \rangle)}$,

$$(4.6) \quad \Delta_Q(\eta)^{(u,g)} = e_{C_G(Q)}^u(\Gamma_{Q\langle u \rangle}(d_{C_L(Q)}^u(\eta)))$$

and therefore, for any $\eta' \in CF_K(C_L(Q))$ we get (cf.(3.6))

$$\begin{aligned} & (\Delta_Q(\eta), \Delta_Q(\eta'))_{C_G(Q)} \\ &= \sum_{u \in U_Q} (d_{C_L(Q)}^u(\eta), d_{C_L(Q)}^u(\eta'))_{C_L(Q\langle u \rangle)} \\ &= (\eta, \eta')_{C_L(Q)} \end{aligned}$$

(recall that $e_{C_G(Q)}^u$ and $e_{C_L(Q)}^u$ are isometries !). Hence for

any $Q \in X$ we get a bijective isometry

$$(4.7) \quad \Delta_Q = \sum_{u \in U_Q} e_{C_G(Q)}^u \circ \Gamma_{Q\langle u \rangle} \circ d_{C_L(Q)}^u$$

from $CF_K(C_L(Q))$ onto $CF_K(C_G(Q), f)$ and condition (4.3) insures that Δ_Q does not depend on the choice of U_Q whereas condition (4.4) demands that $L_K(C_G(Q), f)$ contains $\Delta_Q(L_K(C_L(Q)))$ which actually implies the equality

$$(4.8) \quad \Delta_Q(L_K(C_L(Q))) = L_K(C_G(Q), f)$$

since both members have orthonormal basis of the same cardinal (cf.(4.7)). Moreover, notice that $d_{C_G(Q)} \circ \Delta_Q = \Gamma_Q \circ d_{C_L(Q)}$ (cf.

(4.5)) and therefore we get (cf.(3.3) and (4.8))

$$\Gamma_Q(L_K(C_L(Q))) = L_K(C_G(Q), f)$$

which then implies (cf.(3.2))

$$(4.9) \quad \Gamma_Q(BCF_0(C_L(Q))) = BCF_0(C_G(Q), f).$$

Consequently, since (4.9) is true for any $R \in X$ and the maps $d_{C_L(R)}^u$ and $e_{C_G(R)}^u$ send 0-valued functions to 0-valued functions, we have for any $Q \in X$

$$(4.10) \quad \Delta_Q(CF_0(C_L(Q))) = CF_0(C_G(Q), f).$$

An immediate consequence of the definition of Δ_Q , which does not depend on conditions (4.3) and (4.4), is that for any $Q \in X$,

any $\lambda \in CF_K(P)^{C_E(Q)}$ and any $\eta \in CF_K(C_L^\wedge(Q))$ we have

$$(4.11) \quad \Delta_Q(\lambda * \eta) = \lambda * \Delta_Q(\eta).$$

These (4.8), (4.10) and (4.11) show that for any $Q \in X$, Δ_Q (for $C_L^\wedge(Q)$ and $(C_G(Q), f)$) fulfills the similar conditions to Δ (for $\hat{L}$ and (G, b)) in Theorem 1. (Hence, if $1 \in X$, then $\Delta = \Delta_1$ is a required one in Theorem 1.) Since $C_L^\wedge(P) \cong k^* \times P$, we can easily prove that there are exactly two (G, b) -local systems defined over $\{P\}$ (cf. (4.11)). (Notice that, up to sign, there is just one possibility for the isometry Γ_P .)

We want to extend X and Γ step by step. Assume that X does not contain all the subgroups of P and let Q be a subgroup of P which is maximal such that $Q \notin X$. We will discuss now a necessary and sufficient condition to extend Γ to a (G, b) -local system Γ' over the union X' of X and the E -orbit of Q . Since any subgroup R of P properly containing Q belongs to X , for any $u \in P - Q$ we still have the map (as in (4.6))

$$e_{C_G(Q)}^u \circ \Gamma_{Q\langle u \rangle} \circ d_{C_L^\wedge(Q)}^u : CF_K(C_L^\wedge(Q)) \longrightarrow CF_K(C_G(Q), f)$$

where $f = e_{C_G(Q)}^{C_E(Q)}$. We consider the sum

$$\Delta_Q^\circ = \sum_{u \in U_Q - Q} e_{C_G(Q)}^u \circ \Gamma_{Q\langle u \rangle} \circ d_{C_L^\wedge(Q)}^u$$

where, as above, U_Q is a set of representatives for the orbits of $C_E(Q)$ in P and by condition (4.3) again, Δ_Q° does not depend on the choice of U_Q .

Denote by $\bar{f}$ the image of f in $k\bar{C}_G(Q)$, where we set $\bar{C}_G(Q) =$

$C_G(Q)/Q$. We also set $C_L^\wedge(Q) = C_L^\wedge(Q)/Q$. In [12] we proved following propositions (Proposition 3.7 and Proposition 3.11).

Proposition 1. With the notation and the hypothesis above , Δ_Q° induces a bijective isometry

$$\bar{\Delta}_Q^\circ : CF_K^\circ(\bar{C}_L^\wedge(Q)) \cong CF_K^\circ(\bar{C}_G(Q)) , \bar{f})$$

such that $\bar{\Delta}_Q^\circ(L_K(\bar{C}_L^\wedge(Q))) \subset L_K^\circ(\bar{C}_G(Q)) , \bar{f})$.

Proposition 2. With the notation and the hypothesis above, the (G,b) -local system Γ over X can be extended to a (G,b) -local system Γ' over X' if and only if the bijective isometry $\bar{\Delta}_Q^\circ$ can be extended to a $N_E(Q)$ -stable bijective isometry

$$\bar{\Delta}_Q : CF_K(\bar{C}_L^\wedge(Q)) \cong CF_K(\bar{C}_G(Q)) , \bar{f})$$

such that $\bar{\Delta}_Q(L_K(\bar{C}_L^\wedge(Q))) = L_K(\bar{C}_G(Q)) , \bar{f})$.

Now we try to extend $\bar{\Delta}_Q^\circ$ to a $N_E(Q)$ -stable bijective isometry $\bar{\Delta}_Q$. When E is a Klein four group , we obtain the following slightly stronger theorem (Theorem 4.2 in [12]) than Theorem 1. Unfortunately we do not succeed when E is cyclic of order 4 .

Theorem 2. If P is abelian and E is a Klein four group , then there is a (G,b) -local system over the set of all the subgroups of P .

5. Perfect isometry

In this section we introduce some Broué's terms. Let (H, f) (resp. (H', f')) be a pair of a finite group H and its block f (resp. a finite group H' and its block f').

Definition 1 (Definition 1.4 and Proposition 4.1 in [5]). A bijective isometry I from $L_K(H, f)$ onto $L_K(H', f')$ is called a perfect isometry if it induces a bijection from $CF_0(H, f)$ onto $CF_0(H', f')$ and a bijection from $BCF_K(H, f)$ onto $BCF_K(H', f')$. (We can extend I K -linearly.)

Such special kind of bijective isometry has various properties as follows.

Proposition 3 (Proposition 1.3 and Theorem 1.5 in [5]). If I is a perfect isometry from $L_K(H, f)$ onto $L_K(H', f')$, then the following hold.

- (i) I induces a bijective isometry from $L_k(H, f)$ onto $L_k(H', f')$ and then $l(f) = l(f')$.
- (ii) I induces a bijective isometry from the $\mathbb{Z}$ -module generated by the characters of projective $\widehat{OHf}$ -modules onto the $\mathbb{Z}$ -module generated by the characters of projective $\widehat{OH'f'}$ -modules.
- (iii) The bijection between primitive idempotents of $ZKH\widehat{f}$ and $ZKH'\widehat{f'}$ defined by I induces an algebra isomorphism between $ZOH\widehat{f}$ and $ZOH'\widehat{f'}$.
- (iv) I preserves the height of irreducible ordinary characters and the elementary divisors of the Cartan matrices.

Let (P, f_P) be a maximal (f, H) -Brauer pair and for any p -subgroup Q of P , let (Q, f_Q) be a (f, H) -Brauer pair contained in it.

Definition 2. Let $\text{Br}_f(H)$ be the category whose objects are (f, H) -Brauer pairs and whose morphisms from (Q, f_Q) to (R, f_R) are the homomorphisms from Q to R induced by the inner automorphisms of G which send (Q, f_Q) onto a pair contained in (R, f_R) . This is called the Brauer category of H for f .

Hypothesis for pairs (H, f) and (H', f') (Hypothesis 4.2 in [5]). We suppose that P is a defect group of f and f' . We also suppose that the inclusions of P in H and H' induce an equivalence of Brauer categories $\text{Br}_f(H)$ and $\text{Br}_{f'}(H')$.

Definition 3 (Definition 4.3 in [5]). With the above Hypothesis, a linear map I from $\text{CF}_K(H, f)$ to $\text{CF}_K(H', f')$ is called compatible with the fusion, if for every cyclic subgroup $\langle u \rangle$ of P , there exists a linear map $I_{P'}^{\langle u \rangle}$ from $\text{BCF}_K(C_H(u), f_{\langle u \rangle})$ onto $\text{BCF}_K(C_{H'}(u), f'_{\langle u \rangle})$ such that

$$(f'_{\langle u \rangle} \cdot d_{H'}^u) \circ I = I_{P'}^{\langle u \rangle} \circ (f_{\langle u \rangle} \cdot d_H^u).$$

Here the family $\{ I_{P'}^{\langle u \rangle} \mid \langle u \rangle \subseteq P \}$ is called the local system of I .

Definition 4 (Definition 4.6 and "good definition" in its

Remark in [5]). We say that the pair (H, f) and (H', f') are the same type (in "good definition") , if the following conditions are satisfied.

- (i) The Brauer categories $\text{Br}_f(H)$ and $\text{Br}_{f'}(H')$ are equivalent.
- (ii) There exists a family of perfect isometries

$$\{I^Q : L_K(C_H(Q), f_Q) \longrightarrow L_K(C_{H'}(Q), f'_Q) \mid Q \subseteq P\}$$

such that if for any Q we denote by

$$I_{P'}^Q : \text{BCF}_K(C_H(Q), f_Q) \longrightarrow \text{BCF}_K(C_{H'}(Q), f'_Q)$$

the map induced by I^Q , then I^Q is compatible with the fusion and its local system is

$$\{I_{P'}^{Q \langle u \rangle} \mid \langle u \rangle \subseteq C_P(Q)\} .$$

Broué conjectured that if b has an abelian defect group P , and (P, e) is a maximal (b, G) -Brauer pair , then (G, b) and $(N_G(P, e), e)$ are the same block type (Conjecture 6.1 in [5]). By Lemma 2 $\text{Br}_b(G)$ and $\text{Br}_e(N_G(P, e))$ are equivalent. Notice that by (4.5) for any p -subgroup Q of P and any $u \in U_Q$ we have

$$g \cdot d_{C_G(Q)}^u \circ \Delta_Q = \Gamma_{Q \langle u \rangle} \circ d_{C_L^\wedge(Q)}^u$$

and in particular, Γ_Q is the restriction of Δ_Q to $\text{BCF}_K(C_L^\wedge(Q))$.

Then by (4.8), (4.10) and Theorem 2, this conjecture holds when E is a Klein four group (and it also holds when $|E| \leq 3$).

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