

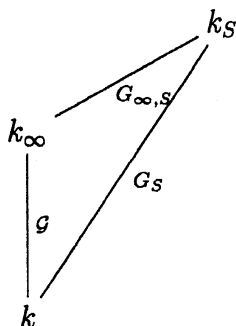
A spectral sequence for Iwasawa adjoints

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Abstract

This article is, apart from some small corrections, an unchanged version of a note written in 1994. It was circulated as a manuscript and quoted in several articles. Its aim was to provide a purely algebraic tool for the Iwasawa theory of arbitrary p -adic Lie groups, by constructing a spectral sequence which relates the generalized 'Iwasawa adjoints' $E^i(M)$ of certain Iwasawa modules M to projective limits often encountered in the applications. A somewhat extended version will be published elsewhere.

Let k be a number field, fix a prime p , and let k_∞ be some Galois extension of k such that $\mathcal{G} = \text{Gal}(k_\infty/k)$ is a p -adic Lie-group (e.g., $\mathcal{G} \cong \mathbb{Z}_p^r$ for some $r \geq 1$). Let S be a finite set of primes containing all primes above p and ∞ , and all primes ramified in k_∞/k , and let k_S be the maximal S -ramified extension of k ; by assumption, $k_\infty \subseteq k_S$. Let $G_S = \text{Gal}(k_S/k)$ and $G_{\infty,S} = \text{Gal}(k_S/k_\infty)$.



Let A be a discrete (left) G_S -module which is isomorphic to $(\mathbb{Q}_p/\mathbb{Z}_p)^r$ for some $r \geq 1$ as an abelian group (e.g., $A = \mathbb{Q}_p/\mathbb{Z}_p$ with trivial action, or $A = E[p^\infty]$, the group of p -power torsion points of an elliptic curve E/k with good reduction outside S). We are not assuming that $G_{\infty,S}$ acts trivially.

Let $\Lambda = \mathbb{Z}_p[[\mathcal{G}]]$ be the completed group ring. For a finitely generated Λ -module M we put

$$E^i(M) = \text{Ext}_\Lambda^i(M, \Lambda).$$

Hence $E^0(M) = \text{Hom}_\Lambda(M, \Lambda) =: M^+$ is just the Λ -dual of M . This has a natural structure of a Λ -module, by letting $\sigma \in \mathcal{G}$ act via

$$\sigma f(m) = f(m) \cdot \sigma^{-1}$$

for $f \in M^+$, $m \in M$. It is known that Λ is a noetherian ring (here we use that $\mathcal{G}$ is a p -adic Lie group), by results of Lazard [La]. Hence M^+ is a finitely generated Λ -module again (choose a projection $\Lambda^r \twoheadrightarrow M$; then we have an injection $M^+ \hookrightarrow (\Lambda^r)^+ = \Lambda^r$). By standard homological algebra, the $E^i(M)$ are finitely generated Λ -modules for all $i \geq 0$.

Examples a) If $\mathcal{G} = \mathbb{Z}_p$, then $\Lambda = \mathbb{Z}_p[[\mathcal{G}]] \cong \mathbb{Z}_p[[X]]$ is the classical Iwasawa algebra, and, for a Λ -torsion module M , $E^1(M)$ is isomorphic to the Iwasawa adjoint, which can be defined as

$$\text{ad}(M) = \varprojlim_n (M/\alpha_n M)^\vee$$

where $(\alpha_n)_{n \in \mathbb{N}}$ is a sequence of elements in Λ such that $\lim_{n \rightarrow \infty} \alpha_n = 0$ and (α_n) is prime to the support of M for every $n \geq 1$, and where

$$N^\vee = \text{Hom}(N, \mathbb{Q}_p/\mathbb{Z}_p)$$

is the Pontrjagin dual of a compact $\mathbb{Z}_p$ -module N . For any finitely generated Λ -module M , $E^1(M)$ is quasi-isomorphic to $\text{Tor}_\Lambda(M)^\sim$, where $\text{Tor}_\Lambda(M)$ is the Λ -torsion submodule of M , and $M^\sim$ is the "Iwasawa twist" of a Λ -module M : the action of $\gamma \in \mathcal{G}$ is changed to the action of γ^{-1} .

b) If $\mathcal{G} = \mathbb{Z}_p^r$, $r \geq 1$, then the $E^i(M)$ are the standard groups considered in local duality. By duality for the ring $\mathbb{Z}_p[[\mathcal{G}]] = \mathbb{Z}_p[[x_1, \dots, x_r]]$, they can be computed in terms of local cohomology groups (with support) or by a suitable Koszul complex.

The topic of this note is the following observation.

Theorem 1 *There is a spectral sequence*

$$E_2^{p,q} = E^p(H^q(G_{\infty,S}, A)^\vee) \Rightarrow \varprojlim_{k',m} H^{p+q}(G_S(k'), A[p^m]).$$

Here the limit runs through the natural numbers m and the finite extensions k'/k contained in k_∞ , via the natural maps

$$H^n(G_S(k'), A[p^{m+1}]) \rightarrow H^n(G_S(k'), A[p^m]) \quad (m \geq 1)$$

and the corestrictions.

Before we give the proof, we discuss what this spectral sequence gives in more down-to-earth terms. First of all, we always have the 5-low-terms exact sequence

$$\begin{aligned} 0 &\rightarrow E^1(H^0(G_{\infty,S}, A)^\vee) \xrightarrow{\text{inf}^1} \varprojlim_{k'} H^1(G_S(k'), T_p A) \\ &\rightarrow (H^1(G_{\infty,S}, A)^\vee)^+ \rightarrow E^2(H^0(G_{\infty,S}, A)^\vee) \xrightarrow{\text{inf}^2} \varprojlim_{k'} H^2(G_S(k'), T_p A). \end{aligned}$$

To say more, we consider some assumptions.

A.1 Assume that $p > 2$ or that k_∞ is totally imaginary. Then

$$H^r(G_{\infty,S}, A) = 0 = \varprojlim_{k'} H^r(G_S(k'), T_p A)$$

for all $r > 2$.

Corollary 2 *Assume that $H^2(G_{\infty,S}, A) = 0$ (This is the so-called "weak Leopoldt conjecture" for A . It is stated classically for $A = \mathbb{Q}_p/\mathbb{Z}_p$, and there are precise conjectures when this is expected for representations coming from algebraic geometry, cf. [Ja2]).*

Then the cokernel of $\inf^2$ is

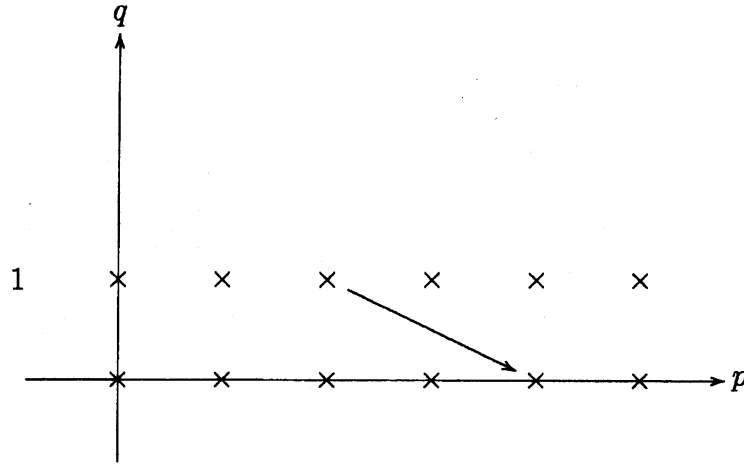
$$\ker(E^1(H^1(G_{\infty, S}, A)^\vee) \rightarrow E^3(H^0(G_{\infty, S}, A)^\vee)),$$

and there are isomorphisms

$$E^i(H^1(G_{\infty, S}, A)^\vee) \xrightarrow{\sim} E^{i+2}(H^0(G_{\infty, S}, A)^\vee)$$

for $i \geq 2$.

Proof This comes from A.1 and the following picture of the spectral sequence



Corollary 3 Assume that $H^0(G_{\infty, S}, A) = 0$. Then

(a)

$$\varprojlim_{k'} H^1(G_S(k'), T_p A) \xrightarrow{\sim} H^1(G_{\infty, S}, A)^\vee{}^+$$

(b) There is an exact sequence

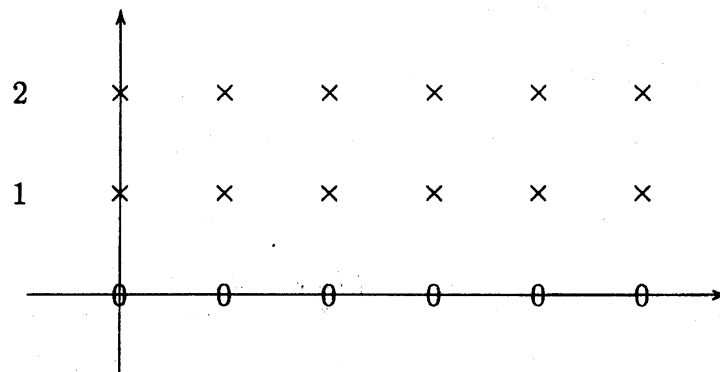
$$\begin{aligned} 0 \rightarrow E^1(H^1(G_{\infty, S}, A)^\vee) &\rightarrow \varprojlim_{k'} H^2(G_S(k'), T_p A) \\ &\rightarrow (H^2(G_{\infty, S}, A)^\vee)^+ \rightarrow E^2(H^1(G_{\infty, S}, A)^\vee) \rightarrow 0 \end{aligned}$$

(c) There are isomorphisms

$$E^i(H^2(G_{\infty, S}, A)^\vee) \xrightarrow{\sim} E^{i+2}(H^1(G_{\infty, S}, A)^\vee)$$

for $i \geq 1$.

Proof In this case, the spectral sequence looks like



Corollary 4 Assume that $\mathcal{G}$ is a p -adic Lie group of dimension 1 (equivalently: an open subgroup is $\cong \mathbb{Z}_p$). Then $E^i(-) = 0$ for $i \geq 3$. Let

$$B = \text{im} (\inf^2 : E^2(H^0(G_{\infty, S}, A)^\vee) \rightarrow \varprojlim_{k'} H^2(G_S(k'), T_p A))$$

Then B is finite, and there is an exact sequence

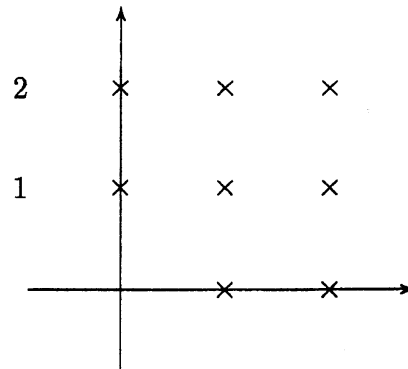
$$\begin{aligned} 0 \rightarrow E^1(H^1(G_{\infty, S}, A)^\vee) &\rightarrow \varprojlim_{k'} H^2(G_S(k'), T_p A)/B \rightarrow (H^2(G_{\infty, S}, A)^\vee)^+ \\ &\rightarrow E^2(H^1(G_{\infty, S}, A)^\vee) \rightarrow 0, \end{aligned}$$

and

$$E^1(H^2(G_{\infty, S}, A)^\vee) = 0 = E^2(H^2(G_{\infty, S}, A)^\vee),$$

i.e., $(H^2(G_{\infty, S}, A)^\vee)$ is a projective Λ -module.

Proof Quite generally, for a p -adic Lie group $\mathcal{G}$ of dimension n one has $\text{vcd}_p(\mathcal{G}) = n$ for the virtual cohomological p -dimension of $\mathcal{G}$, and hence $E^i(-) = 0$ for $i > n + 1$, cf. [Ja3]. The finiteness of $E^2(M)$ (for a finitely generated Λ -module M) in our case is well-known, cf. [Ja3]. The remaining claims follow from the following shape of the spectral sequence:



Lemma 5 Assume that $\mathcal{G}$ is a p -adic Lie group of dimension n (e.g., $\mathcal{G}$ contains an open subgroup $\cong \mathbb{Z}_p^n$). Then $E^i(H^0(G_{\infty, S}, A)^\vee) = 0$ for $i \neq n, n + 1$.

(a) If $H^0(G_{\infty, S}, A)$ is divisible (e.g., if $G_{\infty, S}$ acts trivially on A), then

$$E^i(H^0(G_{\infty, S}, A)^\vee) = \begin{cases} 0 & \text{for } i \neq n \\ \text{Hom}(D, H^0(G_{\infty, S}, A)), & \text{for } i = n, \end{cases}$$

where D is the dualising module for $\mathcal{G}$ ($D = \mathbb{Q}_p/\mathbb{Z}_p$ if $\mathcal{G} = \mathbb{Z}_p^n$).

(b) If $H^0(G_{\infty, S}, A)$ is finite, then

$$E^i(H^0(G_{\infty, S}, A)^\vee) = \begin{cases} 0 & \text{for } i \neq n + 1 \\ \text{Hom}(H^0(G_{\infty, S}, A), D)^\vee & \text{for } i = n + 1 \end{cases}$$

Proof This is well-known, see [Ja3].

Corollary 6 Let $\mathcal{G}$ is a p -adic Lie group of dimension 2 (e.g., $\mathcal{G}$ contains an open subgroup $\cong \mathbb{Z}_p^2$). If $G_{\infty, S}$ acts trivially on A , then there are exact sequences

$$\begin{aligned} 0 \rightarrow \varprojlim_{k'} H^1(G_S(k'), T_p A) &\rightarrow (H^1(G_{\infty, S}, A)^\vee)^+ \\ &\rightarrow T_p A \xrightarrow{\inf^2} \varprojlim_{k'} H^2(G_S(k'), T_p A) \end{aligned}$$

and

$$\begin{aligned} 0 \rightarrow E^1(H^1(G_{\infty, S}, A)^\vee) &\rightarrow \varprojlim_{k'} H^2(G_S(k'), T_p A) / \text{im } \inf^2 \\ &\rightarrow (H^1(G_{\infty, S}, A)^\vee)^+ \rightarrow E^2(H^1(G_{\infty, S}, A)^\vee) \rightarrow 0, \end{aligned}$$

an isomorphism

$$E^1(H^2(G_{\infty, S}, A)^\vee) \xrightarrow{\sim} E^3(H^1(G_{\infty, S}, A)^\vee),$$

and one has

$$E^2(H^2(G_{\infty, S}, A)^\vee) = 0 = E^3(H^2(G_{\infty, S}, A)^\vee).$$

Proof The spectral sequence looks like

Corollary 7 Let $\mathcal{G}$ be a p -adic Lie group of dimension 2 (So $E^i(-) = 0$ for $i \geq 4$). If $H^0(G_{\infty, S}, A)$ is finite, then

$$\varprojlim_{k'} H^1(G_S(k'), T_p A) \cong (H^1(G_{\infty, S}, A)^\vee)^+.$$

If

$$d_2^{1,1} : E^1(H^1(G_{\infty, S}, A)^\vee) \rightarrow E^3(H^0(G_{\infty, S}, A)^\vee)$$

is the differential of the spectral sequence in the theorem, then one has an exact sequence

$$\begin{aligned} 0 \rightarrow \ker d_2^{1,1} &\rightarrow \varprojlim_{k'} H^2(G_S(k'), T_p A) \rightarrow \\ &\rightarrow \ker(d_2^{0,2} : (H^2(G_{\infty, S}, A)^\vee)^+ \rightarrow E^2(H^1(G_{\infty, S}, A)^\vee)) \rightarrow \text{coker } d_2^{1,1} \rightarrow 0, \end{aligned}$$

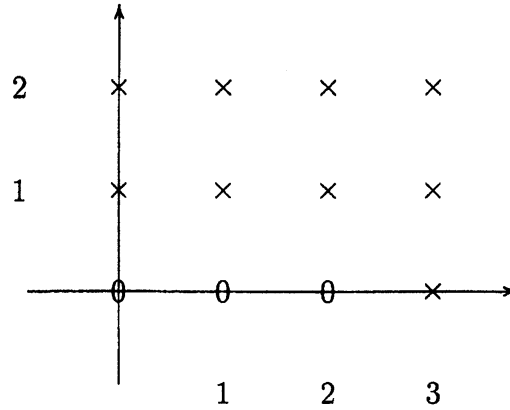
an isomorphism

$$E^1(H^2(G_{\infty, S}, A)^\vee) \xrightarrow{\sim} E^3(H^1(G_{\infty, S}, A)^\vee),$$

and the vanishing

$$E^2(H^2(G_{\infty, S}, A)^\vee) = 0 = E^3(H^2(G_{\infty, S}, A)^\vee).$$

Proof The spectral sequence looks like



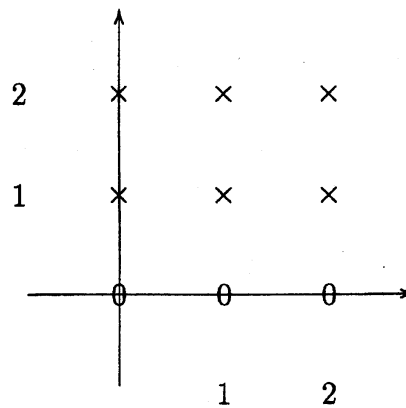
Remark In the situation of Corollary 5, one has an exact sequence up to *finite modules*:

$$\begin{aligned} 0 \rightarrow E^1(H^1(G_{\infty, S}, A)^\vee) &\rightarrow \varprojlim_{k'} H^2(G_S(k'), T_p A) \\ &\rightarrow (H^2(G_{\infty, S}, A)^\vee)^+ \rightarrow E^2(H^1(G_{\infty, S}, A)^\vee) \rightarrow 0. \end{aligned}$$

Corollary 8 Let $\mathcal{G}$ be a p -adic Lie group of dimension > 2 . Then

$$(H^1(G_{\infty, S}, A)^\vee)^+ \cong \varprojlim_{k'} H^1(G_S(k'), T_p A)$$

Proof The first three columns of the spectral sequence look like



We now turn to the **proof of Theorem 1**:

Let G be any profinite group, let $H \leq G$ be a closed normal subgroup, and denote $\mathcal{G} = G/H$. Let M_G be the category of discrete G -modules which are torsion as abelian groups, and let $M_G^{\mathbb{N}}$ be the category of inverse systems (A_n) in M_G as in [Ja1]. For any A in M_G , one gets an inverse system $T_p A := (A[p^n])$, where the transition maps $A[p^{n+1}] \rightarrow A[p^n]$ are induced by multiplication with p in A . For reasons explained later, denote by $H_{\text{cont}}^m(G, H; RT_p A)$ the value at A of the m -th derived functor of the left exact functor

$$F : A \mapsto \varprojlim_n \varprojlim_U H^0(U, A[p^n])$$

where U runs through all open subgroups $U \subset G$ containing H , and the transition maps are the corestriction maps and these coming from $A[p^{n+1}] \rightarrow A[p^n]$, respectively. We can write F as the composition of the functors

$$\begin{aligned} T_p : M_G &\rightarrow M_G^{\mathbb{N}} \\ A &\mapsto (A[p^n]) \end{aligned}$$

and

$$\begin{aligned} H_{\text{cont}}^0(G, H; -) : M_G^{\mathbb{N}} &\rightarrow \text{Ab} \\ (A_n) &\mapsto \varprojlim_n \varprojlim_U H^0(U, A_n) \end{aligned}$$

where Ab is the category of abelian groups. Because T_p maps injectives to $H_{\text{cont}}^0(G, H; -)$ -acyclics we get a spectral sequence

$$E_2^{p,q} = H_{\text{cont}}^p(G, H; R^q T_p A) \Rightarrow H_{\text{cont}}^{p+q}(G, H; RT_p A).$$

One has

$$R^q T_p A = \begin{cases} (A/p^m A) & q = 1 \\ 0 & q > 1 \end{cases}$$

and hence short exact sequences

$$0 \rightarrow H_{\text{cont}}^n(G, H; T_p A) \rightarrow H_{\text{cont}}^n(G, H; RT_p A) \rightarrow H_{\text{cont}}^{n-1}(G, H; R^1 T_p A) \rightarrow 0$$

explaining the notation for $R^n F$. In fact, $H_{\text{cont}}^n(G, H; RT_p A)$ is the hypercohomology with respect to $H_{\text{cont}}^0(G, H; -)$ of a complex $RT_p A$ in $M_G^{\mathbb{N}}$ computing the $R^i T_p A$. If A is p -divisible, then $R^q T_p A = 0$ for all $q > 0$, the spectral sequence degenerates and gives isomorphisms $H_{\text{cont}}^n(G, H; RT_p A) \cong H_{\text{cont}}^n(G, H; T_p A)$. There is another spectral sequence from deriving the inverse limit, viz.

$$E_2^{p,q} = R^p \varprojlim_{n,U} H^q(U, A_n) \Rightarrow H_{\text{cont}}^{p+q}(G, H; (A_n))$$

If G/H has a countable basis of neighbourhoods of identity, i.e., if there is a countable family U_ν of open subgroup, $H \leq U_\nu \leq G$, with $\bigcap_\nu U_\nu = H$, then $R^p \varprojlim_{n,U} = 0$ for $p > 1$, and $R^1 \varprojlim_{n,U}$ has the usual description ([Ja1]). If, in addition, all $H^q(U, A_n)$ are finite, then $R^1 \varprojlim_{n,U} H^q(U, A_n) = 0$, and we get isomorphisms

$$H_{\text{cont}}^n(G, H; (A_n)) = \varprojlim_n \varprojlim_U H^n(U, A_n).$$

All this applies in the situation of the theorem, so that

$$H_{\text{cont}}^n(G_S, G_{\infty, S}; RT_p A) = \varprojlim_n \varprojlim_U H^n(G_S(k'), A[p^m])$$

where k' runs through the finite extensions k'/k contained in k_{∞} .

On the other hand, we can write F as the composition of the left exact functors

$$\begin{array}{ccc} H^0(H, -) : M_G & \rightarrow & M_G \\ A & \mapsto & A^H \end{array}$$

and

$$\begin{array}{ccc} M_G & \rightarrow & Ab \\ B & \mapsto & \text{Hom}_{\Lambda}(B^{\vee}, \Lambda) \end{array}$$

where $\Lambda = \mathbb{Z}_p[[\mathcal{G}]]$ is the completed $\mathbb{Z}_p$ -group ring of $\mathcal{G}$ and $B^{\vee}$ is the Pontrjagin dual of B , which is a compact Λ -module. In fact, one has (cf. [Ja3] p. 179).

$$\begin{aligned} \text{Hom}_{\Lambda}(B^{\vee}, \Lambda) &= \varprojlim_U \text{Hom}_{\mathcal{G}}(B^{\vee}, \mathbb{Z}_p[G/U]) \\ &= \varprojlim_n \varprojlim_U \text{Hom}(H^0(U/H, B)^{\vee}, \mathbb{Z}/p^n \mathbb{Z}) \\ &= \varprojlim_n \varprojlim_U \text{Hom}(H^0(U/H, B[p^n])^{\vee}, \mathbb{Z}/p^n \mathbb{Z}) \\ &= \varprojlim_n \varprojlim_U H^0(U/H, B[p^n]) \end{aligned}$$

and hence

$$\text{Hom}_{\Lambda}(H^0(H, A)^{\vee}, \Lambda) = \varprojlim_n \varprojlim_U (H^0(U, A[p^n]) = F(A).$$

Since taking Pontrjagin duals is an exact functor

$$M_G \rightarrow (\text{compact } \Lambda\text{-modules})$$

taking injectives to projectives, the derived functors of the functor $B \mapsto \text{Hom}_{\Lambda}(B^{\vee}, \Lambda)$ are the functors $B \mapsto \text{Ext}_{\Lambda}^i(B^{\vee}, \Lambda) = E^i(B^{\vee})$, and we get a spectral sequence

$$E_2^{p,q} = E^p(H^q(H, A)^{\vee}) \Rightarrow R^{p+q}F(A) = H_{\text{cont}}^{p+q}(G, H; RT_p A).$$

The theorem follows.

References

- [Ja1] U. Jannsen: *Continuous étale cohomology*, Math. Ann. **280** (1988), 207–245
- [Ja2] U. Jannsen: *On the l -adic cohomology of varieties over number fields and its Galois cohomology*, Galois groups over Q (Berkeley, CA, 1987), 315–360, Math. Sci. Res. Inst. Publ., 16, Springer, New York, 1989
- [Ja3] U. Jannsen: *Iwasawa modules up to isomorphism*, Algebraic Number theory, 171–207, Adv. Stud. Pure Math. **17**, Academic Press Boston 1989
- [La] M. Lazard: *Groupes analytiques p -adiques*, Publ. Math. IHES **26** (1965), 389–603.