

Quasi limit theorems for open systems

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Survey and motivations

- I will present a few preliminary results of an ongoing paper with J Atnip, C Gonzalez-Tokman, G Froyland and Y Nakano.
- The following two contributions share our kind of investigation.
- M. M. Castro, J. S. W. Lamb, G. Olicon Mendez, and M. Rasmussen. *Existence and uniqueness of quasi-stationary and quasi-ergodic measures for absorbing Markov processes: a Banach lattice approach*, 2021. arXiv:2111.13791v6. To appear in Stochastic Processes and their Applications. **FOR MARKOV CHAINS**
- P. Collet, S. Martínez and J. San Martín, *Quasi-Stationary Distributions, Markov Chains, Diffusions and Dynamical Systems*, Springer, 2013. **FOR GIBBS STATE**
- Our results apply to **deterministic dynamical systems and dynamical systems perturbed in a quenched way**.
- Basic reference for quenched (fibred) systems: L Arnold: **Random Dynamical Systems**.

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Random Dynamical Systems: quenched

- Suppose we have a complete metric space X and a probability space $(\Omega, \mathcal{F}, m)$ and an ergodic invertible map σ . Define I_ω (**the FIBER**) a closed subset of X and
- For each $\omega \in \Omega$ we consider the map

$$T_\omega : I_\omega \rightarrow I_{\sigma\omega}$$

- and the random composition

$$T_\omega^n = T_{\sigma^{n-1}\omega} \circ \dots \circ T_{\sigma\omega} \circ T_\omega.$$

- Put $I_c := \bigcup_{\omega \in \Omega} \{\omega\} \times I_\omega$; writing $B \in I_c$ means $\forall \omega, B_\omega \in I_\omega, \{\omega\} \times B_\omega = B \cap \{\omega\} \times I_\omega$.
- Ex. Take σ an irrational rotation of the circle $\sigma\omega = \overline{\omega + \alpha}$, $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, $\Omega = \bigcup_{j=1}^M \Omega_j$, a finite partition of the circle.
- On each Ω_j the map $\omega \in \Omega_j \rightarrow T_\omega$ is constant.

$$T_\omega^3(x) = T_{\overline{\omega+2\alpha}} T_{\overline{\omega+\alpha}} T_\omega(x).$$

- In this context we will consider a probability measure satisfying the equivariant condition:

$$\int_{I_\omega} f \circ T_\omega d\mu_\omega = \int_{I_{\sigma\omega}} f d\mu_{\sigma\omega}, \quad f \in L^1(I, \mu_{\sigma\omega}). \quad (1)$$

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Another example: the i.i.d. case

- Take a probability space $(\Omega, \mathcal{F}, m)$ and associate to each $\omega \in \Omega$ a map T_ω .
- Then construct $(\Omega^{\mathbb{Z}}, \mathcal{F}^{\mathbb{Z}}, m^{\mathbb{Z}})$ and take an element $\underline{\omega} \in \Omega^{\mathbb{Z}}$ as $\underline{\omega} = (\dots, \omega_{-1}, \omega_0, \omega_1, \dots)$, where $\omega_j \in \Omega$.
- Define the shift

$$(\sigma \underline{\omega})_i = \underline{\omega}_{i+1}$$

and put

$$\underline{\omega} \rightarrow T_{\underline{\omega}} = T_{\omega_0},$$

where ω_0 is the 0-th coordinate of $\underline{\omega}$.

- Then

$$T_{\underline{\omega}}^n = T_{\sigma^{n-1}\underline{\omega}} \circ \dots \circ T_{\sigma\underline{\omega}} \circ T_{\underline{\omega}} = T_{\omega_{n-1}} \circ \dots \circ T_{\omega_0}.$$

- Suppose we define the shift σ over $\Omega^{\mathbb{N}}$. Then we could define the **stationary** measure μ_s satisfying

$$\int f(x) d\mu_s(x) = \int \int f(T_\omega(x)) d\mu_s(x) dm(\omega).$$

It is the stationary measure of the associated Markov chain Z_n with transition probability

$$\mathbb{P}(Z_{n+1} \in B, Z_n = z) = m(\omega, T_\omega(z) \in B).$$

This is the **ANNEALED scenario**.

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What makes randomness so different?

- In the following we could forget the presence of ω : the results will be still **NEW**.
- **Invariance** $\mu(T^{-1}A) = \mu(A)$ is replaced by **equivariance** $\mu_\omega(T_\omega^{-1}(A)) = \mu_{\sigma\omega}(A)$.
- The rate of decay could be affected by norms of functions along fibers $f_{\sigma^k\omega}(x), x \in I_{\sigma^k\omega}$. Use **ergodicity** of σ an especially **temperedness**. The function $a(\omega)$ is tempered if for ω a.e. we have $\lim_{|n| \rightarrow \infty} \frac{1}{n} \log |a(\sigma^n\omega)| = 0$, which is equivalent to say that $\forall \varepsilon, \exists A_\omega > 0, \forall n : a(\sigma^n\omega) \leq A_\omega e^{\varepsilon|n|}$.
- More important **powers** are replaced by **compositions or cocycles**; **quasicompactness of operators** are replaced by **quasicompactness of cocycles**.

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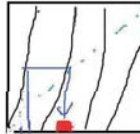
Open systems

- Given a dynamical system (X, μ, T) with μ which is T -invariant, Birkhoff ergodic theorem states that for any $f \in L^1(\mu)$:

$$\lim_{n \rightarrow \infty} \frac{1}{n} S_n(x) := \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=0}^{n-1} f(T^j x) = \tilde{f}(x), \quad \mu - a.s.$$

with $\int f d\mu = \int \tilde{f} d\mu$; $\tilde{f}(Tx) = \tilde{f}(x)$. If μ is ergodic then $\tilde{f}(x) = \int f d\mu$, a.s.

- How could we formulate it for **open systems**?
- Namely for systems which admit a set (the **hole**), which absorbs the trajectories?



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QUESTIONS and Holes

- When the orbit enters the hole it disappears. Do we have points whose orbit never disappears? **The surviving set**.
- How does mass escape into the hole? **Escape rate**.
- How could we describe the **metastable** statistics of convergence to the **equilibrium** state on the surviving set?
- The **hole**. Let H be a measurable set in I_c . For each $\omega \in \Omega$ the fiber sets $H_\omega \subseteq I_\omega$ are uniquely determined by the condition that $\{\omega\} \times H_\omega = H \cap (\{\omega\} \times I_\omega)$. For each $\omega \in \Omega$ and $n \geq 0$ we define

$$X_{\omega,n} := \left\{ x \in I_\omega : T_\omega^j(x) \notin H_{\sigma^j\omega} \text{ for all } 0 \leq j \leq n \right\} = \bigcap_{j=0}^n T_\omega^{-j}(X_{\sigma^j\omega,0}) \quad (2)$$

to be the set of points in $X_{\omega,0}$ whose trajectories avoid the hole for n iterates. Note that $X_{\omega,0} = I_\omega \setminus H_\omega$ and $X_{\omega,n} = \{\tau(\omega, \cdot) > n\}$, where τ is the first hitting time to H (i.e. $\tau(\omega, x) = \inf\{j \geq 1 : T_\omega^j(x) \in H_{\sigma^j\omega}\}$). We call $X_{\omega,n}$ the **nth level surviving set**. We then naturally define

$$X_{\omega,\infty} := \bigcap_{n=0}^{\infty} X_{\omega,n} = \bigcap_{n=0}^{\infty} T_\omega^{-n}(X_{\sigma^n\omega,0}) \quad (3)$$

the ω -**surviving set**.

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Measures

- A probability measure ζ on $(\Omega \times X, F \otimes B)$ is said to be a **RANDOM PROBABILITY MEASURE** if it satisfies the following properties:
 - For every $B \in B$, the map $\Omega \ni \omega \mapsto \zeta_\omega(B) \in [0, 1]$ is measurable,
 - For μ -a.e. $\omega \in \Omega$, the map $B \ni B \mapsto \zeta_\omega(B) \in [0, 1]$ is a Borel probability measure.
- **CONDITIONALLY INVARIANT**
Det $\forall n : \eta(B) = \frac{\eta(T_\omega^{-n} B \cap X_\omega)}{\eta(X_\omega)}$
Ran $\forall n : \rho_{\sigma^n\omega}(B) = \frac{\rho_\omega(T_\omega^{-n} B \cap X_{\omega,n})}{\rho_\omega(X_{\omega,n})}$
- **YAGLOM LIMIT**
Det $\frac{\zeta(T_\omega^{-n} B \cap X_\omega)}{\zeta(X_\omega)} \rightarrow \eta(B)$
Ran $\frac{\zeta_\omega(T_\omega^{-n} B \cap X_{\omega,n})}{\zeta_\omega(X_{\omega,n})} - \rho_{\sigma^n\omega}(B) \rightarrow 0$
- **CONDITIONING LIMIT**
Det $\frac{\zeta(B \cap X_\omega)}{\zeta(X_\omega)} \rightarrow \text{some measure}$
Ran $\frac{\zeta_\omega(B \cap X_{\omega,n})}{\zeta_\omega(X_{\omega,n})} \rightarrow \text{some random measure}$

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- From the abstract point of view, we will assume the existence of a measure (**CONFORMAL**) and of operators (**transfer or Perron-Fröbenius**) verifying:
- Det:** $\int L_c(\phi)\psi d\nu_c = \lambda \int \phi \psi \circ T d\nu_c$, $\lambda = \int L_c(1) d\nu_c$
- Ran:**
 $\int L_{\omega,c}(\phi_\omega)\psi d\nu_{\sigma\omega,c} = \lambda_\omega \int \phi_\omega \psi \circ \sigma_\omega d\nu_{\omega,c}$, $\lambda_\omega = \int L_{\omega,c}(1_\omega) d\nu_{\sigma\omega,c}$
- Warning: the choice of the functional spaces is essential.**
- (Ex: for billiard systems and hyperbolic diffeos with singularities,

$$L_c : \mathcal{B} \rightarrow \mathcal{B}, L_c(h)(\phi) = h(\phi \circ T),$$

where $h \in \mathcal{B}$, and ϕ is a test function.)

- We will consider iterates of these operators

$$L_{\omega,c}^n(f) = L_{\sigma^{n-1}\omega,c} \cdots L_{\omega,c}(f)$$

which become powers in the deterministic setting

$$L_c^n(f).$$

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The functional space

- We will consider a weight function $g : X \rightarrow (0, \infty)$.
- We assume there exists a Banach space $(\mathcal{B}, \|\cdot\|)$ of complex-valued functions on X such that
- there is $C_0 > 0$ for which

$$\|f \cdot v\| \leq C_0 \|f\| \|v\|, \quad \|f\|_\infty \leq C_0 \|f\| \quad (4)$$

for any $f, v \in \mathcal{B}$,

- $\mathcal{B}$ is dense in $L^1(\nu_{\omega,c})$ for any $\omega \in \Omega$.
- $\hat{X}_n, g \in \mathcal{B}$ a.s. for any $n \in \mathbb{N}$ (recall that $\hat{A}$ is the indicator function of A),
- the fiberwise transfer operator $\mathcal{L}_{\omega,c} : \mathcal{B} \rightarrow \mathcal{B}$ (associated with g) given by

$$\mathcal{L}_{\omega,c}(f)(x) := \sum_{T_\omega(y)=x} f(y) g_\omega(y), \quad f \in \mathcal{B}, x \in X_{\sigma\omega} \quad (5)$$

is well-defined
for each $\omega \in \Omega$,

- the map

$$(\omega, x) \mapsto (\mathcal{L}_c f)_\omega(x) \quad (6)$$

is measurable for any $\mathcal{B}$ -valued random variable f , where

$$(\mathcal{L}_c f)_\omega := \mathcal{L}_{\sigma^{-1}\omega,c} f_{\sigma^{-1}\omega} \quad \text{for } \omega \in \Omega.$$

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Operators on the open system I

- We now introduce the operators for the open systems

$$L_\omega f = L_{\omega,c}(\mathbf{1}_{X_\omega,0})$$

and its iterates

$$L_\omega^n f = L_{\omega,c}^n(\mathbf{1}_{X_\omega,n-1})$$

- Suppose one could prove the existence of a random measure $\nu_{\omega,\infty}$ supported on $X_{\omega,\infty}$ and such that (Assumption **A**):
 - $\lambda_\omega \int f_\omega d\nu_{\omega,\infty} = \int L_\omega f_\omega d\nu_{\sigma\omega,\infty}$, $\lambda_\omega = \int L_\omega 1 d\nu_{\sigma\omega,\infty}$
 - $L_\omega \phi_\omega = \lambda_\omega \phi_{\sigma\omega}$, $\int \phi_\omega d\nu_{\omega,\infty} = 1$,
 - $\|L_\omega^n f_\omega - \nu_{\omega,\infty}(f_\omega) \phi_{\sigma^n \omega}\|_\infty \leq D \|f\| \kappa^n$
- Then a few interesting results follow, notably
- The measure $\mu_{\omega,\infty} = \phi_\omega \nu_{\omega,\infty}$ is **equivariant**

$$\int f_{\sigma\omega} \circ T_\omega d\mu_{\omega,\infty} = \int f_{\sigma\omega} d\mu_{\sigma\omega,\infty}$$

- Assumption A** is basically the quasi-compactness in the deterministic case and the quasi-compactness for random cocycles using MET.

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Operators on the open system II

- The measure

$$\eta_\omega(\mathcal{B}) = \frac{\nu_{\omega,c}(\mathbf{1}_B \phi_\omega \mathbf{1}_{X_{\omega,0}})}{\nu_{\omega,c}(\phi_\omega \mathbf{1}_{X_{\omega,0}})}$$

is conditionally invariant.

- η_ω is the Yaglom limit of any other measure absolutely continuous wrt $\nu_{\omega,c}$ and with positive Radon-Nykodim derivative, class $P_{(\nu_{\omega,c},+)}$
- Each element of $P_{(\nu_{\omega,c},+)}$ has a (conditioning) limit and in particular $\nu_{\omega,\infty}$ is the conditioning limit of $\nu_{\omega,c}$ and $\mu_{\omega,\infty}$ of η_ω
- The previous results were known in a few cases for deterministic systems (see Demers, Young, Liverani, Maume, Todd, etc, they are new for random systems. We now begin to state results which are new in both framework).
- In the case of billiards quoted above, Demers proved that the conditionally invariant measure ν is simply the eigenmeasure of $L = \mathbf{1}_{X_0} L_c(\mathbf{1}_{X_0})$

$$L\nu = \lambda_e \nu,$$

where $\nu(X_n) = \lambda_e^n$, λ_e being the escape rate.

- In that case ν is singular wrt to the Lebesgue measure, but it is its Yaglom limit, so the escape rate could be practically computed.

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New results I

- We remind that $\zeta_{\omega,n}(\mathcal{B}) = \frac{\zeta_{\omega}(B \cap X_{\omega,n})}{\zeta_{\omega}(X_{\omega,n})}$ and η_{ω} is the conditionally invariant. Choose $\zeta \in P_{(\nu_{\omega,c},+)}$.
- We already said that $\zeta_{\omega,n}$ has a (conditioning) limit $\zeta_{\omega,\infty}$.

Theorem: Quasi Exponential Mixing

Under **Assumption A**, there exist are $\kappa \in (0, 1)$ and a measurable function $C: \Omega \rightarrow (0, \infty)$ such that

$$\left| \zeta_{\omega,n} \left(\left(f_{\sigma^k \omega} \circ T_{\omega}^k \right) v_{\omega} \right) - \eta_{\sigma^k \omega, n-k} (f_{\sigma^k \omega}) \zeta_{\omega,\infty} (v_{\omega}) \right| \leq \frac{C(\omega) \|\psi_{\omega}\| \|\phi_{\omega}\|}{\inf \phi_{\sigma^k \omega} \inf \psi_{\omega}} \|f_{\sigma^k \omega}\|_{\infty} \|v\|_{\mathcal{B}}^k$$

for every $\zeta \in P_{(\nu_{\omega,c},+)}$ with density ψ , measurable $f: \mathcal{I}_c \rightarrow \mathbb{R}$, $v \in \mathcal{B}$, integers $n \geq 1$, $1 \leq k \leq n$ and m -a.e. $\omega \in \Omega$. Moreover, if $\omega \mapsto \log \|f_{\omega}\|_{\infty}$ is in $L^1(m)$ then the right-hand side goes to zero exponentially fast as $k, n \rightarrow \infty$.

- We should add $\inf \phi_{\omega}$ is **tempered**; moreover $\log \|f_{\omega}\|_{\infty}$ is also tempered.

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New results II

- We could now state the ergodic theorem for our open systems

Theorem: Quasi Strong Law of Large Numbers

Under **Assumption A** Then, $\mu_{\omega,\infty}$ is the unique quasi-ergodic probability measure with respect to $P_{(\nu_{\omega,c},+)}$. Furthermore, for each $\zeta_{\omega} \in P_{(\nu_{\omega,c},+)}$ and $\mathcal{B}$ -valued random variable $f_{\omega}: \mathcal{I}_{\omega} \rightarrow \mathbb{R}$ with $\|f\|_{\infty} \in L^1(m)$, there exist sets $\Gamma_{\omega,n} \subseteq X_{\omega,n}$ with $\zeta_{\omega,n}(\Gamma_{\omega,n}) = 1$ a.s. for each $n \geq 0$ such that for m -a.e. $\omega \in \Omega$ and any $\{x_n\}_{n \geq 0} \in \prod_{n=0}^{\infty} \Gamma_{\omega,n}$, we have

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=0}^{n-1} f_{\sigma^j \omega} \circ T_{\omega}^j(x_n) = \int f_{\omega} d\mu_{\omega,\infty}.$$

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New results III

- We now establish a few **limit theorems**
- We first define the **dynamical variance**.
- Let $(S_n f)_{\omega}(x) := \sum_{j=0}^{n-1} f_{\sigma^j \omega} \circ T_{\omega}^j(x)$. Assume that

$$\mu_{\omega,\infty}(f_{\omega}) = 0 \quad m\text{-almost surely.}$$

Then,

$$(s^2(f))_{\omega} := \mu_{\omega,\infty} \left(\frac{f_{\omega}^2}{\phi_{\omega}} \right) + 2 \sum_{n=1}^{\infty} \mu_{\sigma^{-n} \omega, \infty} (f_{\omega} \circ T_{\sigma^{-n} \omega} \cdot f_{\sigma^{-n} \omega})$$

exists m -almost surely. Set

$$\Sigma^2(f) := \int s^2(f) dm.$$

Furthermore,

$\Sigma^2(f) > 0$ if and only if f is not cohomologous to a constant;

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New results IV

Theorem: Quasi Central Limit Theorem

Under **Assumption A**. Let $f: X \rightarrow \mathbb{R}$ be a $\mathcal{B}$ -valued random variable. Assume that $\mu_{\omega,\infty}(f) = 0$ m -a.s. and $\Sigma^2(f) > 0$. Then, for any $\zeta \in P_{(\nu_{\omega,c},+)}$ and $a \in \mathbb{R}$,

$$\lim_{n \rightarrow \infty} \zeta_{\omega,n} \left(\frac{1}{\sqrt{n}} \sum_{j=0}^{n-1} f_{\sigma^j \omega} \circ T_{\omega}^j \leq a \right) = \frac{1}{\sqrt{2\pi \Sigma^2(f)}} \int_{-\infty}^a \exp \left(-\frac{z^2}{2\Sigma^2(f)} \right) dz \quad (7)$$

We emphasise that the right-hand-side of (7) is a (non-random) constant.

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Theorem: Quasi Large Deviation Principle

Under **Assumption A**. Let $f: X \rightarrow \mathbb{R}$ be a $\mathcal{B}$ -valued random variable. Assume that $\mu_{\omega, \infty}(f) = 0$ m -a.s. and $\Sigma^2(f) > 0$. Then, for any $\zeta \in P_{(\nu_{\omega, c}, +)}$, there is a nonnegative, non-random, continuous, strictly convex function $c: (-\delta_0, \delta_0) \rightarrow \mathbb{R}$ with some $\delta_0 > 0$ such that c vanishes at 0 and that for any $\zeta \in P_{\nu_{\omega, c}, +}(\nu_c)$ and $\delta \geq 0$,

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log \zeta_{\omega, n} \left(\frac{1}{n} \sum_{j=0}^{n-1} f_{\sigma^j \omega} \circ T_{\omega}^j > \delta \right) = -c(\delta) \quad , \quad m - a.s.$$

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Theorem: Quasi Berry-Essen Theorem

Under **Assumption A**. Let $f: X \rightarrow \mathbb{R}$ be a $\mathcal{B}$ -valued random variable. Assume that $\mu_{\omega, \infty}(f) = 0$ m -a.s. and $\Sigma^2(f) > 0$. Then for any $\zeta \in P_{(\nu_{\omega, c}, +)}$, there is $C = C_{f, \zeta}$ such that

$$\sup_{a \in \mathbb{R}} \left| \zeta_{\omega, n} \left(\frac{1}{\sqrt{n}} \sum_{j=0}^{n-1} f_{\sigma^j \omega} \circ T_{\omega}^j(\omega, x) \leq a \right) - \frac{1}{\sqrt{2\pi\Sigma^2(f)}} \int_{-\infty}^a \exp\left(-\frac{z^2}{2\Sigma^2(f)}\right) dz \right| \leq \frac{C}{\sqrt{n}}$$

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- The limit theorem are proved by adapting the **Nagaev-Guivarc'h method** in terms of the twisted operator to the cocycle of transfer operators.
- This theory, which used an implicit function result of Hennion, was proposed in our two papers:
 - 1 D. DRAGICEVIC, G. FROYLAND, C. GONZALEZ-TOKMAN, S. VAIENTI, [A spectral approach for quenched limit theorems for random expanding dynamical systems](#), Comm.Math. Phys., (2018).
 - 2 D. DRAGICEVIC, G. FROYLAND, C. GONZALEZ-TOKMAN, S. VAIENTI, [A spectral approach for quenched limit theorems for random hyperbolic dynamical systems](#), Trans. Amer Math. Soc, (2020).
- We used the quasi-compactness of the cocycle (and therefore the **Multiplicative Ergodic Theorem**) plus the implicit function theorem .

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- Focusing on the CLT, we will try to prove it by the convergence of the characteristic function of our process. By denoting with S_n the (random) Birkoff sum, we must control a term like

$$\zeta_{\omega, n}(e^{\theta(S_n f)\omega}) = \zeta_{\omega, n}(e^{\theta \sum_{j=0}^{n-1} f_{\sigma^j \omega} \circ T_{\omega}^j}) = \frac{\nu_{\omega, c}(e^{\theta(S_n f)\omega} \hat{X}_{\omega, n} \cdot \psi|_{\omega})}{\nu_{\omega, c}(\hat{X}_{\omega, n} \cdot \psi|_{\omega})},$$

where ψ is the density of ζ_{ω} w.r.t. $\nu_{\omega, c}$ and $\hat{F} = 1_F$.

- By using duality one reduces the previous quantity to

$$\frac{\nu_{\sigma^n \omega, c}(\tilde{L}_{\omega, n}^{\theta, n} \psi|_{\omega})}{\nu_{\sigma^n \omega, c}(\tilde{L}_{\omega, n}^{\theta, n} \psi|_{\omega})},$$

where

$$\tilde{L}_{\omega}^{\theta}(\varphi) := \tilde{L}_{\omega}(e^{\theta f_{\omega}} \cdot \varphi) = \lambda_{\omega}^{-1} L_{\omega, c}(e^{\theta f_{\omega}} \hat{X}_{\omega} \cdot \varphi), \quad \varphi \in \mathcal{B}.$$

is the **twisted operator**. For the CLT $\theta = -\frac{it}{\sqrt{n}}$

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Twisted operator III

- There are an open set $U \subset \mathbb{C}$ containing 0, C^1 maps $\phi : U \rightarrow L^\infty(\Omega, \mathcal{B})$, $\nu : U \rightarrow L^\infty(\Omega, \mathcal{B}^*)$, and a C^3 map $\lambda : U \rightarrow L^\infty(m)$ such that the following hold:

- (1) For every $\theta \in U$ and m -almost every $\omega \in \Omega$,

$$\tilde{\mathcal{L}}_\omega^\theta \phi_\omega^\theta = \lambda_\omega^\theta \phi_{\sigma\omega}^\theta, \quad \text{and}$$

$$\lambda^\theta = \nu_\omega(e^{\theta f} \phi_\omega^\theta)$$

- (2) For every $\theta \in U$, there is an $\mathcal{L}_\omega^\theta$ -equivariant splitting $\mathcal{B} = \mathcal{C}\phi_\omega^\theta \oplus \text{Ker}(\nu_\omega^\theta)$, where $\text{Ker}(\nu_\omega^\theta) = \{u \in \mathcal{B} : \nu_\omega^\theta(u) = 0\}$. Moreover, there is $\kappa_1 \in (0, 1)$ such that for every $\theta \in U$ and m -almost every $\omega \in \Omega$,

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log \|\mathcal{L}_\omega^{\theta, n}\|_{\mathcal{C}\phi_\omega^\theta} = \int \hat{\Lambda}(\theta) dm$$

and

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log \|\mathcal{L}_\omega^{\theta, n}\|_{\text{Ker}(\nu_\omega^\theta)} \leq \kappa_1 \int \hat{\Lambda}(\theta) dm,$$

where $\hat{\Lambda}$ is the function $\hat{\Lambda} : U \rightarrow L^\infty(m)$ given by $\hat{\Lambda}(\theta)_\omega = \log \lambda_\omega^\theta$ for $\theta \in U$ and $\omega \in \Omega$,

- (3) It holds that

$$\hat{\Lambda}(0) = \hat{\Lambda}'(0) = 0 \quad \int \hat{\Lambda}''(0) dm = \Sigma^2(f).$$

Furthermore, by shrinking U if necessary, there are $C > 0$, $r \in (0, 1)$ such that for every $\theta \in U$ and m -almost every $\omega \in \Omega$,

$$\|\mathcal{L}_\omega^{\theta, n}\|_{\text{Ker}(\nu_\omega^\theta)} \leq C r^n.$$

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Proof of the strong law of large numbers

- We show quasi strong law of large numbers in the following abstract form. We introduce the correlation function

$$C_{\zeta, f, \nu}(n, k) := \zeta_n(f \circ T^k \cdot \nu) - (\mu_\infty(f))_k \zeta_n(\nu)$$

for each $\zeta \in P_{\nu_\zeta}$, each $\mathcal{B}$ -valued random variable f and each $\nu \in \mathcal{B}$.

Proposition

Let $\zeta \in P_{\nu_\zeta}$, and f be a $\mathcal{B}$ -valued random variable such that $\|f\|_\infty \in L^1(m)$. Suppose that

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} |C_{\zeta, f, \nu}(n, k)| = 0$$

for every $\nu \in \mathcal{B}$. Then, there exist sets $\Gamma_n \subseteq X_n$ with $\zeta_n(\Gamma_n) = 1$ a.s. for each $n \geq 0$ such that for m -a.e. $\omega \in \Omega$ and any $\{x_n\}_{n \geq 0} \in \prod_{n=0}^\infty \Gamma_{\omega, n}$, we have

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} f \circ T_\omega^k(x_n) = \int f d\mu_\infty.$$

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Proof.

Suppose that

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} |C_{\zeta, f, \nu}(n, k)| = 0$$

for every $\nu \in \mathcal{B}$. Then, since $\mathcal{B}$ is dense in $L^1(\nu_{\omega, \zeta})$ for each $\omega \in \Omega$, we can assume that

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} |C_{\zeta, f, \nu}(n, k)| = 0$$

for every $\nu \in L^1(\zeta_{\omega, n})$. For each $\omega \in \Omega$, let $Z_\omega := \prod_{j=0}^\infty \zeta_{\omega, j}$, which is a probability measure on $\prod_{j=0}^\infty X_{\omega, j}$. To get the conclusion, it is enough to show that

$$\int \left(\frac{1}{n} \sum_{j=0}^{n-1} (f_{\sigma^k \omega} \circ T_\omega^k(x_n) - \mu_{\sigma^k \omega, \infty}(f_{\sigma^k \omega})) \right) V(x_0, x_1, \dots) dZ_\omega(x_0, x_1, \dots) \rightarrow 0$$

as $n \rightarrow \infty$, for and $V \in L^1(Z_\omega)$ and m -a.e. $\omega \in \Omega$ (recall that

$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=0}^{n-1} \mu_{\sigma^j \omega, \infty}(f_{\sigma^j \omega}) = \int f d\mu_\infty$ by the ergodicity of (σ, m)).

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On the other hand, for every $n \in \mathbb{N}$,

$$\left| \int \left(\frac{1}{n} \sum_{j=0}^{n-1} (f_{\sigma^j \omega} \circ T_\omega^j(x_n) - \mu_{\sigma^j \omega, \infty}(f_{\sigma^j \omega})) \right) V dZ_\omega \right| \leq \frac{1}{n} \sum_{j=0}^{n-1} |C_{\zeta, f, \tilde{\nu}, \omega}(n, j)|$$

where

$$\tilde{\nu}(x) = \int V(x_0, x_1, \dots, x_{n-1}, x_{n+1}, \dots) d \left(\prod_{i \neq n} \zeta_{\omega, i} \right).$$

On the other hand, by construction $\tilde{\nu}$ belongs to $L^1(\zeta_{\omega, n})$. Hence, from the claim at the beginning of this proof, we get the desired convergence.

The quasi-ergodicity follows from the above estimate (just Lebesgue dominated convergence theorem).

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QODS

Examples

- J. ATNIP, G. FROYLAND, C. GONZALEZ-TOKMAN, S. VAIENTI, Thermodynamic formalism for random weighted covering systems, *Communication Mathematical Physics*, 386, pp. 819–902 (2021)
- J. ATNIP, G. FROYLAND, C. GONZALEZ-TOKMAN, S. VAIENTI, Equilibrium states for non-transitive random open and closed dynamical systems, *Ergodic Theory and Dynamical systems*, 43, (10), pp. 3193–3215, (2023)
- J. ATNIP, G. FROYLAND, C. GONZALEZ-TOKMAN, S. VAIENTI, Thermodynamic Formalism and Perturbation Formulae for Quenched Random Open Dynamical Systems, submitted, 160p, to appear on *Disser. Math.*

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Checking the general assumptions for a large class of examples I

- Suppose that the spaces $\mathcal{T}_\omega = [0, 1]$ for each $\omega \in \Omega$ and the maps $T_\omega : [0, 1] \rightarrow [0, 1]$ are surjective, finitely-branched, piecewise monotone, nonsingular (with respect to Lebesgue measure), and that there exists $C \geq 1$ such that
$$\text{ess sup}_\omega |\mathcal{T}'_\omega| \leq C \quad \text{and} \quad \text{ess sup}_\omega d(T_\omega) \leq C, \quad (8)$$
- where $d(T_\omega) := \sup_{y \in [0, 1]} \# T_\omega^{-1}(y)$. We let $\mathcal{Z}_{\omega, 0}$ denote the (finite) monotonicity partition of T_ω and for each $k \geq 2$ we let $\mathcal{Z}_{\omega, 0}^{(k)}$ denote the partition of monotonicity of T_ω^k .
- The map $\sigma : \Omega \rightarrow \Omega$ is a homeomorphism, the skew-product map $T : \Omega \times [0, 1] \rightarrow \Omega \times [0, 1]$ is measurable, and $\omega \mapsto T_\omega$ has countable range.
- Recall that the variation of $f : [0, 1] \rightarrow \mathbb{R}_+$ on $Z \subset [0, 1]$ be

$$\text{var}_Z(f) = \sup_{x_0 < \dots < x_k, x_j \in Z} \sum_{j=0}^{k-1} |f(x_{j+1}) - f(x_j)|,$$

and $\text{var}(f) := \text{var}_{[0, 1]}(f)$. We let $\text{BV} = \text{BV}([0, 1])$ denote the set of complex-valued functions on $[0, 1]$ that have bounded variation. We define the Banach space $\text{BV}_1 \subseteq L^\infty()$ to be the set of (equivalence classes of) functions of bounded variation on $[0, 1]$, with norm given by

$$\|f\|_{\text{BV}_1} := \inf_{\tilde{f} = f \text{ a.e.}} \text{var}(\tilde{f}) + \|\tilde{f}\|.$$

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Checking the general assumptions for a large class of examples II

- We set $J_\omega := |\mathcal{T}'_\omega|$ and define the random Perron–Frobenius operator, acting on functions in BV

$$P_\omega(f)(x) := \sum_{y \in T_\omega^{-1}(x)} \frac{f(y)}{J_\omega(y)}.$$

The operator P satisfies the well-known property that

$$\int_{[0, 1]} P_\omega(f) d = \int_{[0, 1]} f d \quad (9)$$

for m -a.e. $\omega \in \Omega$ and all $f \in \text{BV}$. Recall that $g_0 = \{g_{\omega, 0}\}_{\omega \in \Omega}$ and that

$$\mathcal{L}_{\omega, 0}(f)(x) := \sum_{y \in T_\omega^{-1}(x)} g_{\omega, 0}(y) f(y), \quad f \in \text{BV}.$$

- We assume that the weight function $g_{\omega, 0}$ lies in BV for each $\omega \in \Omega$ and satisfies

$$\text{ess sup}_\omega \|g_{\omega, 0}\|_{\infty, 1} < \infty, \quad (F2)$$

and

$$\text{ess inf}_\omega \inf g_{\omega, 0} > 0. \quad (F3)$$

- We also assume a uniform covering condition: For every subinterval $J \subset [0, 1]$ there is a $k = k(J)$ such that for a.e. ω one has $T_\omega^k(J) = [0, 1]$. We will use it with the further requirement that there are only finitely many maps T_ω .

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Checking the general assumptions for a large class of examples III

- There is a uniform-in- n and uniform-in- ω upper bound $h \geq 1$ on the number of connected components of $H_{\omega, n}$
- We also assume that

$$\lim_{n \rightarrow \infty} \text{ess sup}_\omega (H_{\omega, n}) = 0. \quad (F6)$$

- Further we suppose that there exists $N' \geq 1$ such that (contracting potential)

$$(9 + 12hN') \cdot \text{ess sup}_\omega \|g_{\omega, 0}^{(N')}\|_{\infty, 1} < \text{ess inf}_\omega \inf \mathcal{L}_{\omega, 0}^{N'} \mathbb{1}. \quad (F8)$$

- With these assumptions we can prove the quasi-compactness of the unperturbed operator by using two important facts:
 - 1 The construction of the closed measure $\nu_{\omega, 0}$ given in our paper *Thermodynamic Formalism for Random Weighted Covering Systems, CMP (2021)*.
 - 2 The generalisation to general potentials of the quasi-compactness decomposition obtained for geometric potential in *D. Dragicevic, G. Froyland, C. Gonzalez-Tokman, S.V. Almost Sure Invariance Principle for random piecewise expanding maps, Nonlinearity*

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Checking the general assumptions for a large class of examples IV

- We now use quasi-compactness (or hyperbolicity) of the unperturbed transfer operator cocycle to guarantee that we have hyperbolic cocycles for large n for the perturbed operators. This requires

- 1 $\hat{\mathcal{L}}_{\omega,0}$ is a hyperbolic transfer operator cocycle on BV_1 with norm $\|\cdot\|_{BV_1}$ and a one-dimensional leading Oseledec's space and slow and fast growth rates.
- 2 The family of cocycles $\{\hat{\mathcal{L}}_{\omega,n,s}\}_{n \geq n_0}$ satisfy a uniform Lasota–Yorke inequality

$$\|\hat{\mathcal{L}}_{\omega,n,s}^k f\|_{BV_1} \leq A\alpha^k \|f\|_{BV_1} + B^k \|f\|_1$$

for a.e. ω and $n \geq n_0$, where $\alpha \leq \gamma < \Gamma \leq B$.

- 3 $\lim_{n \rightarrow \infty} \text{ess sup}_{\omega} \left\| \hat{\mathcal{L}}_{\omega,0} - \hat{\mathcal{L}}_{\omega,n,s} \right\| = 0$, where $\|\cdot\|$ is the $BV - L^1()$ triple norm.
- The main technical problem is to translate to BV_1 what we got for $\nu_{i,0}$.
 - We may now apply Theorem 4.8 in *H. Crimmins, Stability of hyperbolic Oseledec's splittings for quasi-compact operator cocycles*, to conclude that given $\delta > 0$ there is an $\varepsilon_0 > 0$ such that for all $\varepsilon \leq \varepsilon_0$ the cocycle generated by $\hat{\mathcal{L}}_{\omega,n,s}$ is hyperbolic, with
 - (i) the existence of an equivariant family $\hat{\phi}_{\omega,n,s} \in BV$ with $\text{ess sup}_{\omega} \|\hat{\phi}_{\omega,n,s} - \phi_{\omega,0}\|_{BV_1} < \delta$,
 - (ii) existence of corresponding Lyapunov multipliers $\hat{\lambda}_{\omega,n,s}$ satisfying $|\hat{\lambda}_{\omega,n,s} - 1| < \delta$,
 - (iii) operators $\hat{Q}_{\omega,n,s}$ satisfying $\|(\hat{Q}_{\omega,n,s})^n\|_{BV_1} \leq K'(\gamma + \delta)^n$, where γ is the decay rate for $Q_{\omega,0}$.

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