

ON THE MULTIPLICITY OF LUCAS SEQUENCES

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A Lucas sequence of the first kind is a sequence $\{U_n\}$ of rational integers satisfying a linear recurrence relation

$$(1) \quad U_{n+2} = M U_{n+1} - N U_n, \quad U_0 = 0, U_1 = 1$$

where M and N are relatively prime integer constants. The recurrence $\{U_n\}$ is called non-degenerate if the roots and the ratio of the roots of the companion polynomial $X^2 - MX + N = 0$ are non-zero non-roots of unity. The multiplicity of $\{U_n\}$ is the supremum taken over all integers c of the number $m(c)$ of times the integer c occurs in $\{U_n\}$.

In [4], it was shown that with the single exception of the Lucas sequence of multiplicity 4 corresponding to $M = -1$ and $N = 2$, non-degenerate Lucas sequences of the first kind have multiplicity at most three. This will be sharpened as follows.

Theorem. - A non-degenerate Lucas sequence of the first kind has multiplicity at most two except in the cases $M = 1$, $N = 3$ or 5 and $M = \pm 1$, $N = 2$.

For applications to exponential diophantine equations, a more useful multiplicity is given by $m(c) + m(-c)$. The above theorem can be made more precise in the following way.

Theorem. - If $c \neq \pm 1$, then for every non-degenerate Lucas sequence, one has the inequality

$$(2) \quad m(c) + m(-c) \leq 2.$$

If $M = \pm 1$, the same inequality holds for $c = 1$ except in the cases $N = 2, 3$, and 5 . If $M \neq \pm 1$, then $m(1) + m(-1) \leq 3$, and inequality (2) holds with $c = 1$ provided that $N \not\equiv 2 \pmod{48}$.

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In the cases $M = 1$, $N = 2, 3, 5$, the multiplicity of all integers occurring more than once in $\{U_n\}$ has been determined [1,12]. These results will be generalized for various infinite classes of Lucas sequences. Amongst others, the following results will be shown.

Theorem.- Let $\{U_n\}$ be a non-degenerate Lucas sequence of the first kind satisfying Eq. (1) with $M^2 - 4N < 0$ and $N \neq 2, 3, 5$. If $M = -1$, then the sequence $\{U_n\}$ is of multiplicity one. If $M = 1$, then $U_1 = U_2 = 1$ are the only occurrences of 1 and no other integer occurs more than once in $\{U_n\}$.

Theorem.- Let $\{U_n\}$ be a non-degenerate Lucas sequence of the first kind satisfying Eq. (1) with $M^2 - 4N < 0$. Then $\{U_n\}$ is of multiplicity one in each of the following cases.

- (i) $M \equiv 3$ or $5 \pmod{8}$ and $N \equiv 1 \pmod{8}$
- (ii) $2 \parallel M$ and $N \equiv 1 \pmod{8}$
- (iii) $4 \mid M$ and $N \equiv 3 \pmod{8}$
- $8 \mid M$ and $N \equiv 7 \pmod{16}$

The above results, and especially their more precise forms given below yield by a standard translation [6,1], results on the existence and uniqueness of solutions of certain kinds of exponential diophantine equations. One might mention in particular that assertion (c) above suffices to prove a conjecture of Lewis [6, p. 1068] to the effect that the equation $X^2 + 7 = N^y$ where N is a fixed odd integer, has at most one solution.

1.- Preliminaries.

A number of definitions and formulas essential to the subsequent argument are collected together in this section. Recall that a second order linear recurrence is a sequence $\{a_n\}$ of rational integers satisfying a recurrence relation

$$(3) \quad a_{n+2} = M a_{n+1} - N a_n, \quad |a_0| + |a_1| > 0$$

where M and N are integer constants which except where otherwise noted are assumed relatively prime. A Lucas sequence of the second kind is a second order linear recurrence satisfying

$$(4) \quad V_{n+2} = M V_{n+1} - N V_n, \quad V_0 = 2, \quad V_1 = M.$$

We denote by β_1, β_2 (resp. Δ) the roots (resp. discriminant) of the companion polynomial $X^2 - M X + N = 0$ and say that the recurrence $\{a_n\}$ is non-degenerate if β_1, β_2 and β_1/β_2 are non-zero non-roots of unity. The multiplicity of $\{a_n\}$ and the function $m(c)$ are defined as in the case of Lucas sequences of the first kind.

An easy induction argument shows that

$$(5) \quad a_n = A_1 \beta_1^n + A_2 \beta_2^n$$

for $n \geq 0$ where A_1 and A_2 are determined by the system of equations

$$(6) \quad A_1 + A_2 = a_0, \quad A_1 \beta_1 + A_2 \beta_2 = a_1.$$

In particular, one has

$$(7) \quad U_n = \frac{\beta_1^n - \beta_2^n}{\beta_1 - \beta_2},$$

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$$(8) \quad v_n = \beta_1^n + \beta_2^n$$

for all $n \geq 0$; from these, we derive

$$(9) \quad \beta_1^n - \beta_2^n = U_n \sqrt{\Delta}$$

$$(10) \quad \beta_1^n = U_n \beta_1 - N U_{n-1} \quad \text{for } n > 0$$

$$(11) \quad v_n = M U_n - 2 N U_{n-1}$$

where the square root is chosen so that $\sqrt{\Delta} = \beta_1 - \beta_2$.

An induction argument using the recurrence relation (3) shows

$$(12) \quad a_{n+m} = U_m a_{n+1} - N U_{m-1} a_n$$

for all $n \geq 0$, $m \geq 1$ where $\{U_m\}$ is the Lucas sequence of the first kind satisfying the same linear recurrence relation as does $\{a_n\}$. Some useful special cases of this formula are the following

$$(13) \quad \begin{aligned} U_{nd+i} &= U_{d+1} U_{(n-1)d+i} - N U_d U_{(n-1)d+i-1} \equiv U_{d+1} U_{(n-1)d+i} \\ &\equiv \dots \equiv U_{d+1}^n U_i \pmod{U_d} \end{aligned} ,$$

$$(14) \quad \begin{aligned} U_{nd+1} &= U_{d+1} U_{(n-1)d+1} - N U_d U_{(n-1)d} \equiv U_{d+1} U_{(n-1)d+1} \\ &\equiv \dots \equiv U_{d+1}^n \pmod{U_d^2} \end{aligned} ,$$

and

$$(15) \quad \begin{aligned} U_{nd-1} &= U_d U_{nd} - N U_{d-1} U_{nd-1} \equiv (-N U_{d-1}) U_{nd-1} \\ &\equiv \dots \equiv (-N U_{d-1})^{n-1} U_{d-1} \pmod{U_d^2} \end{aligned}$$

which can be rewritten as

$$(16) \quad 1 + N U_{nd-1} \equiv 1 - (-N U_{d-1})^n \pmod{U_d^2}.$$

The above congruences are consequences of (12) and the following result of Lucas [9].

Lemma 1.— Let $\{U_n\}$ (resp. $\{V_n\}$) be the Lucas sequence of first (resp. second) kind which satisfies Eq. (1) (resp. Eq. (4)).

(i) For all $n > 0$, one has

$$(U_n, N) = (V_n, N) = 1 \quad \text{and} \quad (U_n, V_n) = 1 \text{ or } 2.$$

(ii) For all $n, m > 0$, one has $(U_n, U_m) = |U_{(m,n)}|$.

(iii) If for some prime p , one has $p^t \parallel U_m$, $p^u \parallel k$, $t > 0$, and $k \geq 0$, then $p^{t+u} \parallel U_{km}$. If further one has $p^t > 2$, then $p^{t+u} \parallel U_{km}$.

For all integers $n \geq m$, one has

$$(17) \quad U_n^2 = U_{n+m} U_{n-m} + N^{n-m} U_m^2$$

since by Eq. (9)

$$\begin{aligned} \Delta(U_n^2 - U_{n+m} U_{n-m}) &= (\beta_1^n - \beta_2^n)^2 - (\beta_1^{n+m} - \beta_2^{n+m})(\beta_1^{n-m} - \beta_2^{n-m}) \\ &= -2(\beta_1 \beta_2)^n + \beta_1^{n+m} \beta_2^{n-m} + \beta_1^{n-m} \beta_2^{n+m} = N^{n-m} (\beta_1^m - \beta_2^m)^2 \\ &= N^{n-m} \Delta U_m^2. \end{aligned}$$

Combining Eqs. (15, 17), one obtains

$$\begin{aligned} (18) \quad U_{dn-1}^2 &\equiv (-N)^{2(n-1)} U_{d-1}^{2n} = (-N)^{2(n-1)} (U_d U_{d-2} + N^{d-2})^n \\ &\equiv N^{nd-2} \pmod{U_d}. \end{aligned}$$

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The formula

$$(19) \quad U_n = \sum_{i=0}^{\infty} \binom{n-i-1}{n-2i-1} M^{n-2i-1} (-N)^i$$

where $\binom{m}{j}$ is defined to be zero for $j < 0$ is useful whenever one needs to express some U_n as a polynomial in M and N ; it is easily verified using the Pascal triangle identity and Eq.(1). In particular, one has

$$(20) \quad U_n \equiv M^{n+1} \pmod{N}$$

$$(21) \quad U_{2n+1} \equiv (-N)^n \pmod{M}.$$

If $r > 0$ and $s \geq 0$ are fixed integers, then $b_n = a_{rn+s}$ defines a linear recurrence satisfying

$$(22) \quad b_{n+2} = V_r b_{n+1} - N^r b_n,$$

as is easily verified using Eqs. (5,8) and $N = \beta_1 \beta_2$. In particular, the sequences $\{U_{rn}/U_r\}$ and $\{V_{rn}\}$ are Lucas sequences of the first and second kinds respectively. If $\{a_n\}$ is non-degenerate, then so is $\{a_{rn+s}\}$ since the roots of the characteristic polynomial $X^2 - V_r X + N^r = 0$ are just β_1^r and β_2^r by Eq. (8).

2.- The p-adic argument.

The following application of Strassman's Lemma is a refinement of Theorem 1 of [4]. The proof does not require M and N to be relatively prime.

Theorem 1.- Let $\{a_n\}$ be a non-degenerate rational integer second order linear recurrence satisfying Eq. (3) and $\{U_n\}$ be the Lucas sequence of the first kind satisfying the same recurrence relation. For $q \in \mathbb{N}^+$, $c \in \mathbb{Z}$, and p a rational prime not dividing N , set

$$K = \min (\text{ord}_p U_q, \text{ord}_p (N U_{q-1} + 1))$$

$$e = \delta_{2p} \quad (\text{Kronecker } \delta) \quad .$$

If $K > e$, then for each fixed index i with $0 \leq i < q$, the equation

$$a_{qn+i} = c$$

has at most one non-negative integer solution n unless

$$a_{qm+i} \equiv c \pmod{p^{2K-e}}$$

for all non-negative integers m .

Proof.- With the notation of the last section, one has by the definition of K and Eq. (10) that $\beta_j^q = U_q \beta_j - N U_{q-1} \equiv 1 \pmod{p^K}$ for $j = 1, 2$. Let $\delta_j = \beta_j^q$. Since $A_2 \beta_2^i = a_i - A_1 \beta_1^i$ by Eq. (5), one has also that

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$$\begin{aligned}
(23) \quad a_{qn+i} &= A_1 \beta_1^i \delta_1^n + A_2 \beta_2^i \delta_2^n \\
&= \sum_{j=0}^{\infty} A_1 \beta_1^i \binom{n}{j} (\delta_1 - 1)^j + A_2 \beta_2^i \binom{n}{j} (\delta_2 - 1)^j \\
&= a_i + n (a_{q+i} - a_i) \\
&\quad + \sum_{j=2}^{\infty} \binom{n}{j} \{A_1 \beta_1^i (\delta_1 - 1)^j + (a_i - A_1 \beta_1^i) (\delta_2 - 1)^j\} \\
&= a_i + n (a_{q+i} - a_i) + h(n)
\end{aligned}$$

where

$$\begin{aligned}
h(n) &= \sum_{j=2}^{\infty} \binom{n}{j} (A_1 \beta_1^i \{(\delta_1 - 1)^j - (\delta_2 - 1)^j\} + a_i (\delta_2 - 1)^j) \\
&= \sum_{j=2}^{\infty} \binom{n}{j} c_j .
\end{aligned}$$

Now

$$A_1 \beta_1^i \{(\delta_1 - 1)^j - (\delta_2 - 1)^j\} = \left\{ \sum_{t=0}^{j-1} (\delta_1 - 1)^{j-t-1} (\delta_2 - 1)^t \right\} A_1 (\beta_1 - \beta_2) \beta_1^i u_q$$

since by Eq. (9) one has

$$(\delta_1 - 1) - (\delta_2 - 1) = \beta_1^q - \beta_2^q = (\beta_1 - \beta_2) u_q .$$

By Cramer's rule applied to Eq. (6) ,

$$(\beta_1 - \beta_2) A_j = - \begin{vmatrix} 1 & 1 \\ \beta_1 & \beta_2 \end{vmatrix} A_j \in \mathbb{Z}$$

and so it follows that $p^{Kk} | c_k$ for all $k \geq 2$. Since $j! \binom{n}{j}$ is a polynomial in n with integer coefficients, it is straightforward to verify that the coefficients of $h(n)$ considered as a power series in n are all divisible by p^{2K-e} . The condition $a_{qn+i} = c$ can be written

$$0 = (a_i - c) + n(a_{q+i} - a_i) + h(n) .$$

By Strassman's Lemma [10,11], it follows that the number of solutions of

$a_{qn+i} = c$ is no more than one unless δ

$$a_i - c \equiv a_{q+i} - a_i \equiv 0 \pmod{p^{2K-e}}.$$

But then $a_{qn+i} \equiv c \pmod{p^{2K-e}}$ for all $n \geq 0$ by Eq. (23). This proves Theorem 1.

The next result is a natural analogue of Theorem 2 of [4].

Theorem 2.- Let $\{U_n\}$ be a non-degenerate Lucas sequence of the first kind satisfying Eq. (1) with $\Delta = M^2 - 4N < 0$. Suppose that for some positive integer d , one has $p^t \parallel U_d$ where p is a rational prime and $t > e$, $e = \delta_{2p}$ (Kronecker δ). Let v be the multiplicative order of $-N U_{d-1}$ modulo p^{e+1} , $p^u \parallel (-N U_{d-1})^v - 1$, and c be any integer.

- (i) If $u \neq t$ and $p \nmid c$, then for each integer i with $0 \leq i < d-1$ and $p \nmid U_{i+1}$ there is at most one occurrence of c in the subsequence $\{U_{nd+i}\}$.
- (ii) If $c = 1$ or -1 and $p^{t-2e} \nmid M$, then c occurs at most once in the subsequence $\{U_{nd-1}\}$.
- (iii) The integer c occurs at most once in each subsequence $\{U_{dvn+kd}\}$, $0 \leq k < v$.

Proof.- Let r be the multiplicative order of $-N U_{d-1}$ modulo p^t and $q = dr$. Then $r = p^w v$ where $w = \max(0, t-u)$. Further, by Eq. (16) and Lemma 1, the parameter K of Theorem 1 is at least t . Suppose that $p \nmid c$. If $p \mid U_i$ for some fixed i , then by Eq. (13) we have $p \mid U_{dn+i}$ for all $n \geq 0$, and so c does not occur in the subsequence $\{U_{dn+i}\}$. On the other hand, if $p \nmid U_i$, then by the same equation and the definition of r , there is for fixed i at most one integer s such that $0 \leq s < r$ and $U_{qn+sr+i} \equiv c \pmod{p^t}$ for some and hence all $n \geq 0$. For the other values of s , the integer c cannot occur in $\{U_{qn+sr+i}\}$.

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For the first assertion, one may assume that $p \nmid U_i$. Note that by Eqs.

(12,1), one has

$$\begin{aligned}
 (24) \quad U_{q+j} - U_j &= U_{q+1} U_j - N U_q U_{j-1} - U_j \\
 &= (M U_q - N U_{q-1}) U_j - N U_q U_{j-1} - U_j \\
 &= (M U_j - N U_{j-1}) U_q - U_j (1 + N U_{q-1}) \\
 &= U_{j+1} U_q - U_j (1 + N U_{q-1}) .
 \end{aligned}$$

Since $p \nmid U_i, U_{i+1}$, one knows that $p \nmid U_{ds+i}, U_{ds+i+1}$ by Eq. (13). Further, by Lemma 1 and Eq. (16), one has

$$\text{ord}_p U_q = t + w \neq u + w = \text{ord}_p (1 + N U_{q-1}) .$$

Therefore, since $w < t-e$, we have by Eq. (24) with $j = ds + i$ that

$$\text{ord}_p (U_{q+ds+i} - U_{ds+i}) = \min(t+w, u+w) < 2t-e \leq 2K-e .$$

In particular, U_{q+ds+i} and U_{ds+i} cannot both be congruent to c modulo p^{2K-e} , and so by Theorem 1 the integer c can occur at most once in the subsequence $\{U_{qn+ds+i}\}$. This proves the first assertion.

For the second assertion, recall that with $i = d-1$, the integer s was chosen so that $U_{ds+d-1} \equiv c = \pm 1 \pmod{p^t}$. By Eq. (18) with $n = s+1$, it follows that $N^{d(s+1)-2} \equiv U_{ds+d-1}^2 \equiv 1 \pmod{p^t}$. Using Eq. (17), one has

$$\begin{aligned}
 (-N U_{d-1})^{2(d(s+1)-2)} &= (-N)^{2(d(s+1)-2)} (U_d U_{d-2} + N^{d-2})^{d(s+1)-2} \\
 &\equiv (N^{d(s+1)-2})^d \equiv 1 \pmod{p^t} ,
 \end{aligned}$$

and so $r \mid 2(d(s+1)-2)$. By Theorem 1 applied with $q = rd$, the subsequence

$\{U_{qn+d(s+1)-1}\}$ can contain more than one occurrence of c only if

$$U_{d(s+1)-1} \equiv U_{q+d(s+1)-1} \equiv c = \pm 1 \pmod{p^{2t-e}}.$$

By Eq. (15), this means

$$(25) \quad (-N)^s U_{d-1}^{s+1} \equiv (-N)^{r+s} U_{d-1}^{r+s+1} \equiv c = \pm 1 \pmod{p^{2t-e}},$$

and so $(-N U_{d-1})^r \equiv 1 \pmod{p^{2t-e}}$. Since $r \mid 2(d(s+1)-2)$, it follows that

$$(-N U_{d-1})^{2d(s+1)-4} \equiv 1 \pmod{p^{2t-e}}.$$

Combining with Eq. (25) gives $(-N)^{2d-4} \equiv U_{d-1}^4 \pmod{p^{2t-e}}$, and so by Eq. (17),

$$\begin{aligned} 0 &\equiv U_{d-1}^4 - (-N)^{2d-4} = (U_d U_{d-2} + N^{d-2})^2 - N^{2d-4} \\ &= 2 U_d U_{d-2} N^{d-2} \pmod{p^{2t-e}}. \end{aligned}$$

Since $p \nmid N$ by Lemma 1, it follows that $p^{t-2e} \mid U_{d-2}$ and so

$$p^{t-2e} \mid (U_d, U_{d-2}) = |U_{(d,d-2)}| = \begin{cases} |U_2| = |M| & \text{if } d \text{ is even} \\ |U_1| = 1 & \text{if } d \text{ is odd} \end{cases}$$

which proves the second assertion.

For the third assertion, we need a formula for $W_{dn} = U_{dn}/U_d$. Let $\{V_n\}$ be the Lucas sequence of the second kind satisfying Eq. (4). By solving Eqs. (8, 9) with $n = d$, one obtains

$$\beta_1^d, \beta_2^d = \left(\frac{V_d}{2}\right) (1 \pm U_d \sqrt{\Delta/V_d})$$

and so

$$\beta_1^{dn}, \beta_2^{dn} = \left(\frac{V_d}{2}\right)^n \left(1 \pm \frac{U_d \sqrt{\Delta}}{V_d}\right)^n = \left(\frac{V_d}{2}\right)^n \sum_{j=0}^{\infty} \binom{n}{j} \left(\frac{\pm U_d \sqrt{\Delta}}{V_d}\right)^j.$$

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Therefore by Eq. (7)

$$(26) \quad W_{dn} = U_{dn}/U_d = \frac{\beta_1^{dn} - \beta_2^{dn}}{\beta_1^d - \beta_2^d} = \left(\frac{V_d}{2}\right)^{n-1} \sum_{j=0}^{\infty} \binom{n}{2j+1} \left(\frac{U_r^2 \Delta}{V_r^2}\right)^j$$

By Eq. (8,9), one has

$$V_d = U_d \sqrt{\Delta + 2 \beta_2^d} \equiv 2 \beta_2^d \pmod{p^{e+1}}$$

and $p \nmid N = \beta_1 \beta_2$ by Lemma 1 ; hence $V_d/2$ is a p -adic unit. Let $\gamma = (V_d/2)^s - 1$ where s is the multiplicative order of $V_d/2 \pmod{p^{e+1}}$. For k fixed in the interval $0 \leq k < s$, one has by Eq. (26)

$$\begin{aligned} W_{dsn+dk} &= (1 + \gamma)^n \left(\frac{V_r}{2}\right)^{k-1} \sum_{j=0}^{\infty} \binom{sn+k}{2j+1} \left(\frac{\Delta U_d^2}{V_d^2}\right)^j \\ &= \left(\frac{V_r}{2}\right)^{k-1} (sn+k) + h(n) \end{aligned}$$

where, as it is easy to see, $h(n)$ is a power series in n convergent at all p -adic integers and having coefficients all divisible by p^{e+1} . By Strassman's Lemma [11,10], the quantity c/U_d can occur at most once in each subsequence $\{W_{dsn+kd}\}$, $0 \leq k < s$.

By Eq. (11), $V_d/2 = -N U_{d-1} \pmod{p^{t-e}}$ and so $s = v$ if $p^t \neq 4$. This proves assertion (iii) in the case where $p \geq 3$. If $p = s = 2$, then by Lemma 1, $p^{t+1} \mid U_{dn}$ precisely when n is even. In particular, c/U_d occurs at most once in $\{W_{dsn}\} \cup \{W_{dsn+d}\}$. Since $s = 1$ when $p = 2$ and $s \neq 2$, we have in the $p = 2$ case that c occurs at most once in $\{U_{dn}\}$. This completes the proof of Theorem 2.

For future reference, we restate Theorems 1 and 2 of [4].

Theorem 3. - Let $\{a_n\}$ be a non-degenerate second order linear recurrence satisfying Eq. (3) with $M^2 - 4N < 0$, $\{U_n\}$ (resp. $\{V_n\}$) be the Lucas sequence of first (resp. second) kind satisfying the same linear recurrence relation, and β_1, β_2 be the roots of the characteristic polynomial $\chi^2 - M\chi + N = 0$. Suppose that $c \in \mathbb{Z}$, p is a rational prime not dividing N , and π is a prime element of the completion of the ring of integers of $\mathbb{Q}(\beta_1)$ at a prime ideal $\mathfrak{P}$ lying over p .

(i) Suppose $p = 2$ and let q be the least positive integer with

$$\beta_1^q \equiv \beta_2^q \equiv 1 \pmod{\pi^\kappa}, \quad \kappa = \left\lfloor \frac{e}{p-1} \right\rfloor + 1$$

where e is the absolute ramification index of $\mathfrak{P}$. Then for i fixed, the equation $a_{qn+i} = c$ has at most two solutions with $n \geq 0$. Further, if the equation has two solutions when $i = i_1, i_2$ where $0 \leq i_1 < i_2 < q$, then $q = 2(i_2 - i_1)$.

(ii) Suppose $p \geq 3$, $p \mid U_r$, $r \geq 1$, and s is the multiplicative order of $V_r/2 \pmod{p}$. Set

$$\epsilon = \begin{cases} 1 & \text{if } p = 3 \text{ and } \beta_i^{rs} - 1 \not\equiv 0 \pmod{3\pi} \text{ for } i = 1 \text{ or } 2 \\ 0 & \text{otherwise} \end{cases}.$$

If $p \nmid c$, then with the possible exception of one value of i in the interval $0 \leq i < r$, the equation $a_{rn+i} = c$ has at most one solution; for the exceptional value of i , it has at most $2 + \epsilon$ solutions.

3.- The real case.

If $M^2 - 4N \geq 0$, then the situation is very simple as we see in the next proposition.

Proposition 1.- Let $\{U_n\}$ be a non-degenerate Lucas sequence of the first kind satisfying Eq. (1) with $\Delta = M^2 - 4N \geq 0$. For all integers c , one has $m(c) + m(-c) \leq 1$ except when $c = \pm 1$ and $M = \pm 1$. In the exceptional case, $m(1) + m(-1) = 2$.

Proof.- By Eq. (19), it is clear that replacing M with $-M$ leaves U_{2n+1} fixed and changes only the sign of U_{2n} . Therefore to prove the result, it suffices to show in the case where $M > 0$ that U_n for $n > 1$ is a strictly increasing function of n . Since $\{U_n\}$ is non-degenerate, one has $MN(M^2 - 4N) \neq 0$.

If $N > 0$, then $\beta_1, \beta_2 = (M \pm \sqrt{\Delta})/2$ are positive real numbers with $\beta_1 > 1$. The function $f(x) = \sqrt{\Delta}^{-1}(\beta_1^x - \beta_2^x)$ has derivative $f'(x) = \sqrt{\Delta}^{-1}(\beta_1^x \log \beta_1 - \beta_2^x \log \beta_2) > 0$ and so is strictly increasing. Since $U_n = f(n)$ by Eq. (7), the assertion is proved in this case.

If $N < 0$, then by Eq. (19) one has

$$U_n = \sum_{i=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n-i-1}{i} M^{n-1-2i} (-N)^i,$$

and so it suffices to observe that the $\binom{n-i-1}{i}$ for $i > 0$ and $i \leq \lfloor \frac{n-1}{2} \rfloor$ are strictly increasing functions of n .

4.-

Throughout the rest of this paper, it is implicitly assumed that $\Delta = M^2 - 4N < 0$.

The next result is a corollary of a theorem of Chowla, Dunton and Lewis [3]; see [4, Lemma 1].

Lemma 2.- Let $\{V_n\}$ be a non-degenerate Lucas sequence of the second kind satisfying Eq. (4) with $M^2 - 4N < 0$. Then $V_n^2 = 1$ has at most one solution $n \geq 0$ except in the case $M = \pm 1, N = 2$. In the exceptional case, the only solutions are $n = 1$ and 4 .

Lemma 3.- Let $c \in \mathbb{N}^+$ and $\{U_n\}, \{U'_n\}$ be Lucas sequences of the first kind satisfying

$$U_{n+2} = M U_{n+1} - N U_n$$

$$U'_{n+2} = -M U'_{n+1} - N U'_n$$

where $M^2 - 4N < 0$ and either $M \neq \pm 1$ or $N \neq 2$.

(i) If $c \neq 1$ or $M \neq \pm 1$, then at least one of the subsequences $\{U_{2n}\}, \{U_{2n+1}\}$ contains no number of absolute value c .

(ii) Suppose that both c and $-c$ occur at most once each in $\{U_n\}$. If $M \neq -1$ or $c \neq 1$, then both c and $-c$ occur at most once each in $\{U'_n\}$. If $M = -1 = -c$, then $U'_1 = U'_2 = 1$ are the only occurrences of 1 in $\{U'_n\}$ and -1 does not occur in $\{U'_n\}$.

Proof.- If $M \neq \pm 1$, then assertion (i) is clear since $M = U_2 | U_n$ precisely when n is even by Lemma 1(ii-iii). Suppose $M = \pm 1$ and $|U_{2n}| = |U_{2m+1}| = c$. Letting $k = (2n, 2m+1)$, one has by Lemma 1 that

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$$c = (U_{2n}, U_{2m+1}) = |U_k| |U_{2k}$$

and $U_{2k} | |U_{2n}| = c$. So $\pm c = U_{2k} = U_k V_k$ and hence $V_k = \pm 1$. By Lemma 2, $k = 1$ and $c = |U_k| = 1$.

For the second assertion, note that by Eq. (19) one has

$$(27) \quad U'_n = (-1)^{n-1} U_n$$

for $n \geq 0$. Thus $U_{2n+1} = U'_{2n+1}$ and $U_{2n} = -U'_{2n}$ for $n \geq 0$. If $M \neq \pm 1$ or $c \neq 1$, then the second assertion is therefore a consequence of the first. If $M = 1$, then $U_1 = U_2 = 1$ and so the hypothesis of assertion (ii) does not hold when $c = 1$. Finally, if $M = -1$ and $c = 1$, then by hypothesis, $U_1 = -U_2 = 1$ are the only occurrences of ± 1 in $\{U_n\}$. Therefore, by Eq. (27), $U'_1 = U'_2 = 1$ are the only occurrences of ± 1 in $\{U'_n\}$.

Proposition 2.— Let $\{U_n\}$ be a non-degenerate Lucas sequence of the first kind satisfying Eq. (1) with $\Delta = M^2 - 4N < 0$. Let $d \in \mathbb{N}^+$ and p be a prime with $p^t || U_d$, $p^u || N^d - 1$, $p^v || M$, and $p^{e+1} || U_{d+1} - 1$ where $e = \delta_{2p}$ is the Kronecker δ and $2e < w = \min(u, t+v) < 2t$.

(i) If $u \neq t + v$, then the subsequences $\{U_{nd+1}\}$ and $\{U_{nd-1}\}$ both have multiplicity one.

(ii) Let $h = \max(0, u + 1 - w - k + e - f)$ where $p^k || d$, and f is 1 if $p = 2$, $u = t + v$ and 0 otherwise. Then for every $c \in \mathbb{N}^+$, at least one of c and $-c$ does not occur in the union $\{U_{np^h d-1}\} \cup \{U_{np^h d+1}\}$.

Proof.— By Eq. (17), one has for every positive integer g that

$$U_{g+1}^2 = U_{g+2} U_g + N^g, \quad U_{g-1}^2 = U_g U_{g-2} + N^{g-2},$$

and so by the recurrence relation (1) ,

$$(28) \quad (NU_{g-1}+1)(NU_{g-1}-1) = N^2 U_{g-1}^2 - 1 = N^2 U_g U_{g-2} + N^g - 1$$

$$= -N U_g^2 + N U_{g-1} M U_g + (N^g - 1)$$

$$(29) \quad (U_{g+1}-U_1)(U_{g+1}+1) = U_{g+1}^2 - 1 = U_{g+2} U_g + N^{g-1}$$

$$= -N U_g^2 + U_{g+1} M U_g + (N^{g-1})$$

Further, by Eq. (12) ,

$$(30) \quad U_{2g-1} - U_{g-1} = U_g^2 - (NU_{g-1}+1)U_{g-1}$$

Suppose g is a multiple of d . Since $p^{e+1} | U_{d+1} - 1$, one has

$$1 + N U_{d-1} \equiv U_{d+1} + N U_{d-1} = M U_d \equiv 0 \pmod{p^{e+1}},$$

and so by Eq. (16) , $1 + N U_{g-1} \equiv 0 \pmod{p^{e+1}}$ and $p^e || N U_{g-1} - 1$. Finally, Eq. (14) and $p^{e+1} | U_{d+1} - 1$ imply $p^{e+1} | U_{g+1} - 1$ and $p^e || U_{g+1} + 1$.

For assertion (i), let $g = d$. By Eqs (28, 29, 30) and the assumption that $u \neq t + v$, one has

$$p^{w-e} || N U_{d-1} + 1, U_{d+1} - U_1, U_{2d-1} - U_{d-1}.$$

Further, the assumption that $2e < w < 2t$ implies

$$w - e < 2 \min(w - e, t) - e,$$

and so assertion (i) follows from Theorem 1 applied with $q = d$ and $K \geq \min(w-e, t)$.

For assertion (ii), let $g = p^h d$, so that $p^{h+u} || N^{g-1}$ and $p^{t+h} || U_g$ by Lemma 1. By Eqs. (28, 29), one has

$$p^{w+h+f-e} \mid N U_{g-1}^{+1}, U_{g+1}^{-1},$$

and so by Eqs. (14, 15) and the fact that $w+h+f-e \leq 2(t+h)$,

$$U_{ng+1} \equiv U_{g+1}^n \equiv 1 \pmod{p^{w+h+f-e}}$$

$$U_{ng-1} \equiv (-N U_{g-1})^{n-1} U_{g-1} \equiv U_{g-1} \pmod{p^{w+h+f-e}}$$

for all n . If $U_{g-1} \not\equiv -1 \pmod{p^{w+h+f-e}}$, then assertion (ii) follows from these congruences. If $U_{g-1} \equiv -1 \pmod{p^{w+h+f-e}}$, then

$$1 - N \equiv 1 + U_{g-1} N \equiv 0 \pmod{p^{w+h+f-e}},$$

and so $p^{w+h+f-e+k} \mid N^d - 1$. It follows by the definition of u that

$w + h + f - e + k \leq u$ which is contrary to the definition of h . This proves the proposition.

Parts (ii) and (iii) of the last theorem stated in the introduction are very special cases of the next result.

Corollary 1. - Let $\{U_n\}$ be a Lucas sequence of the first kind satisfying Eq. (1) with $M^2 - 4N < 0$. Suppose $2^s \parallel M$ and $2^r \parallel N - \epsilon$ where $\epsilon = \pm 1$, $r \geq 2$, and $s \geq 1$.

(i) The subsequence $\{U_{2n}\}$ is of multiplicity one. If $r + 1 \neq 2s$, then the subsequences $\{U_{4n+1}\}$ and $\{U_{4n+3}\}$ are also of multiplicity one.

(ii) If $r < 2s$, then for all $n \geq 0$ one has

$$U_{4n+1} \equiv 1 \pmod{2^{r+1}} \quad \text{and} \quad U_{4n+3} \equiv -\epsilon + 2^r \pmod{2^{r+1}}.$$

In particular, if $r + 1 < 2s$, then $m(c) + m(-c) \leq 1$ for odd integers c .

(iii) If either $\epsilon = 1$ and $r + 1 \neq 2s$ or else $\epsilon = -1$ and $r + 1 < 2s$, then the sequence $\{U_n\}$ is of multiplicity one.

Proof.— Apply Proposition 2 with $p = 2$ and $d = 4$. Since

$$2^{s+1} \mid U_4 = M(M^2 - 2N), \quad 2^{r+2} \mid N^4 - 1, \quad \text{and} \quad 2^s \mid M,$$

the parameters are $t = s + 1$, $u = r + 2$, and $v = s$. Further,

$$U_5 = M^4 - 3M^2N + N^2 \equiv 1 \pmod{4}$$

and $2e < w = \min(r + 1, 2s) + 1 < 2t$. Proposition 2 (i) shows that $\{U_{4n+1}\}$ and $\{U_{4n+3}\}$ are of multiplicity one whenever $r + 1 \neq 2s$.

Theorem 2 (iii) applied with $d = 4$, $v = 1$ shows that the subsequence $\{U_{4n}\}$ is of multiplicity one. By Lemma 1, $2^{s+1} \mid U_{2n}$ if and only if n is even; hence the subsequences $\{U_{4n+2}\}$ and $\{U_{4n}\}$ have no elements in common. To complete the proof of the first assertion, it therefore suffices to show that $\{U_{4n+2}\}$ is of multiplicity one. By Eq. (22), the sequence of $a_n = U_{2n}/U_2$ is a Lucas sequence of the first kind satisfying the recurrence relation

$$a_{n+2} = V_2 a_{n+1} - N^2 a_n, \quad a_0 = 0, \quad a_1 = 1$$

where $V_2 = M^2 - 2N \equiv 2 \pmod{4}$ and $2^{r+1} \mid N^2 - 1$. By the last paragraph, it follows that $\{a_{4n+1}\}$ and $\{a_{4n+3}\}$ are each of multiplicity one. Since the sequence $\{a_n\}$ reduced modulo 4 consists of repetitions of the segment 0, 1, 2, 3 (mod 4), the two subsequences $\{a_{4n+1}\}$ and $\{a_{4n+3}\}$ have no elements in common. Thus the union $\{a_{4n+1}\} \cup \{a_{4n+3}\}$ has multiplicity one. Since one has

$$\{U_{4n+2}\} = \{U_{8n+2}\} \cup \{U_{8n+6}\} = \{U_2 a_{4n+1}\} \cup \{U_2 a_{4n+3}\},$$

the subsequence $\{U_{4n+2}\}$ is also of multiplicity one, and the first assertion is proved.

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If $r < 2s$, then $U_3 = M^2 - N \equiv -\epsilon + 2^r \pmod{2^{r+1}}$, and so $-N U_3 \equiv 1 \pmod{2^{r+1}}$. By Eqs. (15,14) and the inequality $r + 1 \leq 2s$, one has

$$U_{4n-1} \equiv (-N U_3)^{n-1} U_3 \equiv U_3 \equiv -\epsilon + 2^r \pmod{2^{r+1}}$$

$$U_{4n+1} \equiv U_5^n = (M^4 - 3M^2 N + N^2)^n \equiv N^{2n} \equiv 1 \pmod{2^{r+1}}.$$

Since by Lemma 1, U_n is odd precisely when n is odd, assertion (ii) follows from these congruences and the first assertion. Assertion (iii) follows from the first two assertions and the observation that when $\epsilon = 1$, the sequence $\{U_n\}$ reduced modulo 4 consists of repetitions of the segment $0, 1, M, -1 \pmod{4}$. This completes the proof.

Proposition 3.— Let $\{U_n\}$ be a non-degenerate Lucas sequence of the first kind satisfying Eq. (1) with $\Delta = M^2 - 4N < 0$. Suppose that $p^t \mid U_3$, $p^u \mid M^3 + 1$, and $w = \min(u, t)$ where p is a prime and $e = \delta_{2p}$ is the Kronecker δ . If $u \neq t$, $t+e$ and either $w > 2e$ or $w = u = 2$, then the recurrence $\{U_n\}$ has multiplicity one.

Proof.— Since $U_3 = M^2 - N$, one has

$$1 + N U_2 = 1 + N M = (1 + M^3) - M U_3 \equiv 0 \pmod{p^w}.$$

By Theorem 2 (iii) applied with $d = 3$, $v = 1$, the sequence $\{U_{3n}\}$ has multiplicity one. Further, the parameter K of Theorem 1 with $q = 3$ satisfies $K \geq w > e$. Since

$$U_4 - U_1 = M^3 - 2MN - 1 = 2 M U_3 - (1 + M^3),$$

$$U_5 - U_2 = M^4 - 3M^2 N + N^2 - M = U_3^2 - M(1 + M^3) + M^2 U_3,$$

the p -adic order of $U_4 - U_1$ and $U_5 - U_2$ are $\min(t+e, u)$ and w respectively.

It follows by Theorem 1, that the multiplicities of the subsequences $\{U_{3n+1}\}$ and $\{U_{3n+2}\}$ are both one. Since the sequence $\{U_n\}$ reduced modulo p^{e+1} consists of repetitions of the segment $0, 1, -1 \pmod{p^{e+1}}$, a given integer can occur in at most one of the subsequences $\{U_{3n}\}$, $\{U_{3n+1}\}$, $\{U_{3n+2}\}$. This proves the proposition.

Corollary 2.— Let $\{U_n\}$ be a non-degenerate Lucas sequence of the first kind satisfying Eq. (1) with $M^2 - 4N < 0$. Suppose that $p^t \mid U_3$, $p^u \mid M^3 - 1$, and $w = \min(u, t)$ where p is a prime and $e = \delta_{2p}$ is the Kronecker δ . Assume that $u \neq t$, $t + e$, and either $w > 2e$ or $w = u = 2$. If $M \neq 1$, then the sequence $\{U_n\}$ has multiplicity one. If $M = 1$, then $U_1 = U_2 = 1$ are the only occurrences of 1, the integer -1 does not occur in $\{U_n\}$, and $m(c) \leq 1$ for all $c \neq 1$.

Proof.— This is a consequence of Proposition 3 and Lemma 3.

The next result is the third theorem of the introduction.

Corollary 3.— Let $\{U_n\}$ be a non-degenerate Lucas sequence of the first kind satisfying Eq. (1) with $\Delta = M^2 - 4N < 0$, $M = \pm 1$, and $N \neq 2, 3$ or 5 . If $M = -1$, then the sequence $\{U_n\}$ has multiplicity one. If $M = 1$, then $U_1 = U_2 = 1$ are the only occurrences of 1, the integer -1 does not occur in $\{U_n\}$, and $m(c) \leq 1$ for all $c \neq 1$.

Proof.— This follows from Proposition 3 and Corollary 2 by taking for p the largest prime divisor of $U_3 = M^2 - N = 1 - N$. The hypotheses are satisfied except when $1 - N = -1, -2$, or -4 .

Remark.— The exceptional where $M = \pm 1$ and $N = 2, 3, 5$ have been treated. By Lemma 3, it suffices to treat the case $M = 1$. In the case $M = 1$, $N = 2$, Skolem, Chowla and Lewis [10] showed that

$$U_1 = U_2 = -U_3 = -U_5 = -U_{13} = 1$$

are the only solutions of $U_n^2 = 1$; Townes [12] completed the result by showing that $U_4 = U_8 = -3$ are the only occurrences of -3 and that no integer $\neq \pm 1, -3$ occurs more than once in $\{U_n\}$. In Alter and Kubota [1], it was shown that in the case $M = 1, N = 3$, the only occurrences of 1 are $U_1 = U_2 = U_5$, that -1 does not occur in $\{U_n\}$, and that $m(c) \leq 1$ for all $c \neq 1$. Finally, Alter (unpublished) has shown that in the case $M = 1, N = 5$, the only occurrences of 1 are $U_1 = U_2 = U_7$, that -1 does not occur in $\{U_n\}$, and that $m(c) \leq 1$ for all $c \neq 1$.

The next result contains part (i) of the last theorem stated in the introduction.

Corollary 4. - Let $\{U_n\}$ be a non-degenerate Lucas sequence of the first kind satisfying Eq. (1) with $M^2 - 4N < 0$. Suppose $2^s \parallel M - \epsilon$, $2^r \parallel N - 1$ where $\epsilon = \pm 1$, $s \geq 2$, $r \geq 3$, and $s \neq r, r+1$. If $M \neq 1$, then the sequence $\{U_n\}$ is of multiplicity one. If $M = 1$, then $U_1 = U_2 = 1$ are the only occurrences of 1 , $m(-1) = 0$, and $m(c) \leq 1$ for all $c \neq 1$. If $r+1 < s$, then for every odd positive integer c , one has $m(c) + m(-c) \leq 1$ except that $m(1) + m(-1) = 2$ in case $M = \pm 1$.

Proof. - Apply Proposition 3 and Corollary 2 with $p = 2$ and $u = s$. Since

$$2^t \parallel U_3 = (M^2 - 1) - (N - 1) \equiv 2^{s+1} - 2^r \pmod{2^{\min(r, s+1)+1}},$$

one has $t \geq \min(s+1, r) \geq 3$, and so $w > 2$ or $w = s = 2$. Also, $u \neq t, t+1$ since $s \neq r, r+1$ respectively. The above mentioned results therefore show the first two assertions.

If $r+1 < s$, then the first two assertions imply that the subsequence $\{U_{2^m n}\}$ for $m > 0$ is of multiplicity one. By Eq. (22), the subsequence $\{U_{2^k n}\}$ for $k \geq 0$ satisfies

$$U_{2^k(n+2)} = V_{2^k} U_{2^k(n+1)} - N^{2^k} U_{2^k n}$$

where $\{V_n\}$ is the Lucas sequence of the second kind satisfying the same recurrence relation as does $\{U_n\}$. If one defines $r(k)$, $s(k)$, and $\epsilon(k)$ by $2^{r(k)} \parallel N^{2^k} - 1$, $2^{s(k)} \parallel V_{2^k} - \epsilon(k)$, and $\epsilon(0) = \epsilon$, $\epsilon(k) = -1$ for $k > 0$, then evidently $r(k) = r+k$ and further $r(k) \leq s(k)$. In fact, the assertion is clear for $k = 0$, for $k = 1$, one has

$$V_2 + 1 = (M^2 - 1) - 2(N - 1) \equiv 0 \pmod{2^{r+1}},$$

and by induction using Eq. (8),

$$(31) \quad V_{2^k} = V_{2^{k-1}}^2 - 2N^{2^{k-1}} = (V_{2^{k-1}}^2 - 1) - 2(N^{2^{k-1}} - 1) - 1 \equiv -1 \pmod{2^{r+k}}.$$

Proposition 2 (ii) applied to $\{U_{2^k n}\}$ with the parameters $p = 2, d = 3$, $u = r+k$, $v = 0$, $t = \min(s(k)+1, r(k)) = r+k$, $w = r+k$, and $e = f = 1$ shows that the union $\{U_{3 \cdot 2^{k+1} n - 2^k}\} \cup \{U_{3 \cdot 2^{k+1} n + 2^k}\}$ cannot contain both an integer and its additive inverse. Further by Lemma 3, if $V_{2^k} \neq \pm 1$ (resp. $V_{2^k} = \pm 1$), then the intersection

$$\{|U_{2^{k+1} n}\} \cap \{|U_{2^{k+1} n + 2^k}\}$$

is empty (resp. contains only $|U_{2^k}|$). Finally, $2 \mid U_{3n}$ for all $n \geq 0$ by Lemma 1.

If c is an odd positive integer with $m(c) + m(-c) \neq 0$, let k be the least non-negative integer for which there is an n with $2^k \mid n$ and $|U_n| = c$. If $V_{2^k} \neq \pm 1$ or $c \neq |U_{2^k}|$, then by Lemma 1 and the last paragraph, all occurrences of c and $-c$ lie in

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$$\{U_{2^{k+1}n+2^k}\} \cap (\{U_{3n+1}\} \cup \{U_{3n+2}\}) \\ = \{U_{2^{k+1} \cdot 3n+2^k}\} \cup \{U_{2^{k+1} \cdot 3n-2^k}\} ,$$

and so $m(c) + m(-c) = 1$. If $V_{2^k} = \pm 1$ and $c = |U_{2^k}|$, then by Eqs. (7,8), one has $|U_{2^{k+1}}| = |U_{2^k} V_{2^k}| = c$, and so $m(c) + m(-c) = 2$. By Eq. (31), $V_{2^k} \neq 1$ for $k > 0$, and by Lemma 2 $V_{2^k} = \pm 1$ can happen for at most one value of $k \geq 0$. Therefore, if $V_1 = M = \pm 1$, then $m(1) + m(-1) = 2$ and $m(c) + m(-c) \leq 1$ for all odd $c > 1$. The proof would be complete if we could show that $V_k \neq -1$ for $k > 0$.

One has $V_k \neq -1$ for $k > 0$. In fact, if $V_2 = M^2 - 2n = -1$, then $N = (M^2 - 1)/2 + 1 \equiv (\text{mod } 2^s)$ and so $s \leq r$ contrary to hypothesis. If $V_4 = -1$, then by Eq. (11), one has

$$-1 = V_4 = MU_4 - 2NU_3 = -M^4 + 2(M^2 - N)^2 = -U_2^4 + 2U_3^2 .$$

Thus $x = U_2$, $y = U_3$ is a solution of the diophantine equation $x^4 - 2y^2 = 1$. By Ljunggren [8], it follows that U_2 or U_3 is zero. Thus $\{U_n\}$ has an infinite number of zeros by Lemma 1; this is contrary to the non-degeneracy of $\{U_n\}$, [4]. Finally, if $V_{2^k} = -1$ with $k \geq 3$, then Eq. (31) shows that $x = V_{2^{k-1}}$, $y = N^{2^{k-3}}$ are a solution of the diophantine equation $x^2 - 2y^4 = -1$. A well known theorem of Ljunggren [7] and Eq. (31) imply that $(V_{2^{k-1}}, N^{2^{k-3}})$ is either $(-1, 1)$ or $(239, 13)$. The first possibility implies that $\{U_{2^{k-1}n}\}$ and hence $\{U_n\}$ is degenerate. The second possibility implies that $k = 3$, $N = 13$, and

$$V_2^2 = V_4 + 2N^2 = 239 + 2 \cdot 13^2 = 577$$

which is absurd since 577 is non-square. This completes the proof.

5.-

The next three lemmas are applications of Theorem 3 preliminary to the proof of the first theorem of the introduction.

Lemma 4.- Let $\{U_n\}$ be a non-degenerate Lucas sequence of the first kind satisfying Eq. (1) with $\Delta = M^2 - 4N < 0$ and $2 \nmid MN$. Then

$$U_{6n+1} \equiv 1 \pmod{4} \quad \text{and} \quad U_{6n+5} \equiv -N \pmod{4}$$

for all $n \geq 0$; further, each subsequence $\{U_{6n+1}\}$, $\{U_{6n-1}\}$ contains at most two occurrences of 1 and -1. If $M \neq \pm 1$, then all occurrences of +1 and -1 lie in these two subsequences. In particular, if $M \neq \pm 1$, then $m(-1) = 0$ when $N \equiv 3 \pmod{4}$ and $m(1)$, $m(-1) \leq 2$ when $N \equiv 1 \pmod{4}$.

Proof.- U_3 is even, $U_2 = M$ and U_4 are odd; therefore by Eq. (12)

$$U_7 = U_4^2 - NU_3^2 \equiv 1 \pmod{4}, \quad \text{and} \quad U_5 = U_3^2 - NU_2^2 \equiv -N \pmod{4}.$$

By Eqs. (13, 14), it follows that $U_{6n+1} \equiv 1 \pmod{4}$ and $U_{6n+5} \equiv -N \pmod{4}$. Further, using Eq. (10) to check the multiplicative order mod 4 of the roots of the companion polynomial, one can apply Theorem 3 with $p = 2$ and $q = 6$ ($q = 3$ if $M \equiv -N \equiv 3 \pmod{4}$) to show that $\{U_{6n+1}\}$ and $\{U_{6n-1}\}$ have multiplicity at most two. Finally, by Lemma 1, $2 \mid U_{3n}$ and $M = U_2 \mid U_{2n}$ for all $n \geq 0$; therefore, if $M \neq \pm 1$, then all occurrences of ± 1 must lie in $\{U_{6n-1}\} \cup \{U_{6n+1}\}$.

Lemma 5.- Let $\{U_n\}$ be a non-degenerate Lucas sequence of the first kind satisfying Eq. (1) with $\Delta = M^2 - 4N < 0$. If $9 \mid M$, then $m(1)$, $m(-1) \leq 2$.

Proof.- If β_i for $i = 1, 2$ are the roots of the companion polynomial, then by Eq. (10), one has $\beta_1^2 \equiv -N \pmod{9}$ and $\beta_1^4 \equiv N^2 \pmod{9}$. Thus $\beta_1^k \equiv 1 \pmod{9}$ where $k = 4, 6, 12, 6, 12, 2$ in case $N \equiv 1, 2, 4, 5, 7, 8 \pmod{9}$ respectively. The

sequence $\{U_n\}$ reduced modulo 9 consists of repetitions of the following segments

0,1,0,8	if $N \equiv 1 \pmod{9}$
0,1,0,7,0,4	if $N \equiv 2 \pmod{9}$
0,1,0,5,0,7,0,8,0,4,0,2	if $N \equiv 4 \pmod{9}$
0,1,0,4,0,7	if $N \equiv 5 \pmod{9}$
0,1,0,2,0,4,0,8,0,7,0,5	if $N \equiv 7 \pmod{9}$
0,1	if $N \equiv 8 \pmod{9}$

Thus each of the integers 1 and -1 can lie in at most one subsequence $\{U_{kn+i}\}$, $0 \leq i < k$. Applying Theorem 3 with $p = 3$, $r = k$, and $s = 1$ gives the result.

Lemma 6. - If $\{U_n\}$ is a non-degenerate Lucas sequence of the first kind satisfying Eq. (1) with $\Delta = M^2 - 4N < 0$ and $M = \pm 3$, then $m(1)$, $m(-1) \leq 2$.

Proof. - Since $\Delta < 0$, $N > 2$ and so there is a largest prime divisor p of N . Suppose $p^t \parallel N$ and let d be the multiplicative order of $M \pmod{p^t}$. By Eq. (20), one knows that U_n can be 1 only if $n \equiv 1 \pmod{d}$ and U_n can be -1 only if d is even and $n \equiv d/2 + 1 \pmod{d}$.

If $d = 1$, then by the definition of p^t and d , we have $p^t = 2$ or 4 and hence $N = 2$ or 4 . Since $N > 2$, we have $N = 4$. If $M = \pm 3$ and $N = 4$, then the sequence $\{U_n\}$ reduced modulo 3 (resp. 5) consists of repetitions of the segment 0,1,0,2 (mod 3) (resp. 0,1, ± 3 , 0, ± 3 , 4, 0, 4, ± 2 , 0, ± 2 , 1 (mod 5)). Therefore, U_n can be 1 only if $n \equiv 1 \pmod{12}$ and it can be -1 only if $n \equiv 7 \pmod{12}$. Applying Theorem 3 with $p = 5$, $r = 3$, and $s = 4$ gives $m(1)$, $m(-1) \leq 2$. In particular, we may assume $d > 1$.

Since Theorem 3 gives the result in the contrary case, one can also assume that no prime larger than 3 divides U_d . By Lemma 1, we know U_n is a multiple of 3 (resp. is even) precisely when n is even (resp. is a multiple of 3). Suppose $2^u || d$ and define

$$v = \begin{cases} \text{ord}_3 d & \text{if } N \text{ is odd} \\ 0 & \text{otherwise} \end{cases} \quad \text{and } f = d 2^{-u} 3^{-v}.$$

Since $U_f | U_d$ by Lemma 1 and $2, 3 \nmid U_f$, one has $U_f = \pm 1$. If $U_f = 1$, then by the first paragraph of the proof, $d 2^{-u} 3^{-v} = f \equiv 1 \pmod{d}$ and so $d | 2^u 3^v$. If $U_f = -1$, then d is even and $d 2^{-u} 3^{-v} = f \equiv 1 + d/2 \pmod{d}$ and so again $d | 2^u 3^v$. Since $2^u 3^v | d$, one has in all cases that $d = 2^u 3^v$.

Suppose $u \geq 2$. Since $U_4 | U_d$ by Lemma 1, we know that U_4 is divisible by no prime larger than 5. But $U_4 = M(M^2 - 2N) = \pm 3(9 - 2N)$ is clearly odd and exactly divisible by 3. Thus $9 - 2N = \epsilon$ where $\epsilon = \pm 1$, and hence $N = (9 - \epsilon)/2 = 4$ or 5 . Now $N = 4$ is impossible since $p^t = 4$ and $d = 2$ in this case. Thus $N = 5$, $M = \pm 3$, and we have $m(1), m(-1) \leq 2$ by Lemma 4.

Suppose $d = 2$. Since $M^2 = 9 \equiv 1 \pmod{p^t}$, we have $p^t | 8$ and so $N = 4$ or 8 as $N > 2$. The case $N = 4$ having already been treated, we may assume $N = 8$ and $M = \pm 3$. The sequence $\{U_n\}$ reduced modulo 4 consists of 0 followed by repetitions of the segment $1, \pm 3 \pmod{4}$. Since $3 | U_{2n}$ for all $n \geq 0$ by Lemma 1, it follows that $m(-1) = 0$. By Eq. (22) with $r = 2$ and $V_2 = M^2 - 2N = -7$, one has

$$U_{2n+1} \equiv V_2 U_{2n-1} \equiv \dots \equiv V_2^{n-1} U_3 = (-7)^{n-1} \pmod{N^2}$$

for $n > 0$. Since -7 has multiplicative order 8 modulo $N^2 = 64$, it follows that U_{2n+1} can be 1 only if $n = 0$ or $n \equiv 1 \pmod{8}$. In particular, in

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order to prove $m(1) \leq 2$ it suffices to show that the subsequence $\{U_{8n+3}\}$ is of multiplicity one. Using Eq. (12), one obtains

$$U_4 = M(M^2 - 2N) = \mp 21 \equiv 0 \pmod{7}, \quad U_3 = M^2 - N = 1,$$

$$U_7 = U_4^2 - NU_3^2 \equiv -N \pmod{7^2},$$

$$1 + NU_7 \equiv 1 - N^2 = -63 \equiv 5 \cdot 7 \not\equiv 0 \pmod{7^2},$$

$$U_{11} - U_3 = (U_4 U_8 - NU_3 U_7) - U_3 \equiv -U_3(1 + NU_7) \not\equiv 0 \pmod{7^2}.$$

Applying Theorem 1 with $p = 7$, $q = 8$, $i = 3$, and $K = 1$, one sees that $\{U_{8n+3}\}$ is indeed of multiplicity one.

The above cases exhaust that in which $d = 2^{u_3 v}$ is a power of 2. By the definition of v , we may assume N is odd and $3 \nmid d$. If $6 \mid d$, then by the first paragraph of the proof, both 1 and -1 can each occur in at most one subsequence $\{U_{6n+i}\}$, $0 \leq i < 6$. By Lemma 4, it follows that $m(1)$, $m(-1) \leq 2$. If $9 \mid d$, then $U_9 \mid U_d$ by Lemma 1, and so U_9 is divisible by no prime larger than 3. By Eqs. (7,8), one has

$$U_9 = U_3(\beta_1^6 + \beta_1^3 \beta_2^3 + \beta_2^6) = U_3(V_3^2 - N^3).$$

Also $V_3 = M(M^2 - 3N) = \pm 3(9 - 3N) \equiv 0 \pmod{6}$ implies that $V_3^2 - N^3$ is neither even nor divisible by 3. Therefore $V_3^2 - N^3 = \epsilon$ where $\epsilon = \pm 1$. This is a special case of the Catalan equation; by theorems of Lebesgue [5] and Chao K o [2], the only solutions are

$$V_3 = \pm 3, N = 2, \epsilon = 1 \quad \text{or} \quad V_3 = \pm 1, 0.$$

Since N is odd, it follows that $V_3 = 0$ and $N = \pm 1$ contrary to the assumption that $\Delta = M^2 - 4N < 0$.

The remaining case is $d = 3$. Since N is odd, $\pm 27 = M^3 \equiv 1 \pmod{p^t}$ and so $N = p^t = 13$ if $M = 3$, and $N = p^t = 7$ if $M = -3$. If $M = 3$ and $N = 13$ then $m(1)$, $m(-1) \leq 2$ by Lemma 4. If $M = -3$ and $N = 7$, then Lemma 4 shows that $m(-1) = 0$ and $\{U_{6n+1}\}$ contains at most two occurrences of 1. By the first paragraph of the proof and the assumption that $d = 3$, -1 does not occur in $\{U_n\}$ and 1 does not occur in $\{U_{6n-1}\}$. Thus $m(1) \leq 2$ and the proof of the lemma is complete.

The next result contains the first two theorems stated in the introduction.

Theorem 4.— Let $\{U_n\}$ be a Lucas sequence of the first kind satisfying Eq. (1) with $\Delta = M^2 - 4N < 0$. The multiplicity of $\{U_n\}$ is at most two except when $M = 1$, $N = 3, 5$ or $M = \pm 1$, $N = 2$. More precisely, if $c > 1$ is a positive integer, then $m(c) + m(-c) \leq 2$, and the same inequality holds with $c = 1$ except possibly in the following cases.

- (a) $M = \pm 1$ and $N = 2, 3$, or 5 .
- (b) $M \neq \pm 1$, $N \equiv 2 \pmod{48}$, and for every odd prime divisor p_1 of N (resp. p_2 of M), the multiplicative order d_1 of $M \pmod{p_1}$ (resp. d_2 of $-N \pmod{p_2}$) satisfies $2^3 \parallel d_1$ (resp. $2^2 \parallel d_2$). In this case, $U_1 = 1$ is the only occurrence of 1, every occurrence of -1 lies in the subsequence $\{U_{8n+5}\}$, and every odd prime divisor p_1 of N (resp. p_2 of M) satisfies $p_1 \equiv 1 \pmod{8}$ (resp. $p_2 \equiv 1 \pmod{4}$).

Proof.— Let $\{V_n\}$ be the Lucas sequence of the second kind which satisfies the same recurrence relation as does $\{U_n\}$. One cannot have $U_m = 0$ for any $m > 0$ since this would imply by Lemma 1 that $\{U_n\}$ has an infinity of zeros contrary to the non-degeneracy of $\{U_n\}$, [4]. Let c be any non-zero integer occurring in $\{U_n\}$, and f be the least positive integer with $c \mid U_f$. By Lemma 1, $U_f = \pm c$

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and all occurrences of c and $-c$ lie in the subsequence $\{U_{fn}\}$. In particular, $m(U_f)$ (resp. $m(-U_f)$) is equal to the number of times 1 (resp. -1) occurs in the sequence $b_n = U_{fn}/U_f$. By Eq. (22), $\{b_n\}$ is a Lucas sequence of the first kind satisfying the recurrence relation

$$b_{n+2} = V_f b_{n+1} - N^f b_n, \quad b_0 = 0, \quad b_1 = 1.$$

Further, if $c \neq \pm 1$, then $f > 1$ and hence $N^f \neq 2, 3, 5$ and $N^f \not\equiv 2 \pmod{4}$. Therefore, we are reduced to showing that $m(1), m(-1) \leq 2$ except in case (a) above, that $m(1) + m(-1) \leq 2$ except in cases (a) and (b), and that the assertions of case (b) hold.

To show that $m(1), m(-1) \leq 2$ except when $M = \pm 1$, $N = 2, 3, 5$ it suffices in the case where M is a multiple of a prime greater than 3 (resp. $9 \nmid M$, $M = \pm 3$, $M = \pm 1$) to apply Theorem 3 (resp. Lemma 5, Lemma 6, Corollary 3). In case $M = \pm 1$, $N \neq 2, 3, 5$, one obtains the stronger assertion $m(1) + m(-1) \leq 2$. This leaves the case where M is even; here Theorem 3 applied with $p = 2$ and $q = 4$ shows the multiplicity of the subsequence $\{U_{4n+1}\}$ is at most 2, and therefore $m(1) + m(-1) \leq 2$ by Corollary 1 (i,ii). In particular, $\{U_n\}$ has multiplicity at most 2 unless $M = \pm 1$, $N = 2, 3, 5$.

Suppose that both M and N are odd and $M \neq \pm 1$. By Lemma 4, all occurrences of 1 and -1 lie in the subsequences $\{U_{6n-1}\}$ and $\{U_{6n+1}\}$. Further, one has by Eqs. (19,12) that

$$U_6 = M(M^2 - N)(M^2 - 3N) \equiv M(1-N)(1+N) \equiv 0 \pmod{8}$$

$$8 \nmid N^6 - 1, \quad 2 \mid U_3, \quad 3 \nmid U_4, \quad \text{and}$$

$$U_7 = U_4^2 - NU_3^2 \equiv 1 \pmod{4}.$$

Therefore, Proposition 2 (i,ii) applied with $p = 2$, $d = 6$ shows that either $m(1)$, $m(-1) \leq 1$ or else $m(-1) = 0$; and so $m(1) + m(-1) \leq 2$.

If $4 \mid N$ and $M \neq \pm 1$, then 1 and -1 cannot occur in $\{U_{2n}\}$ by Lemma 1 and $U_{2n+1} \equiv 1 \pmod{4}$ by Eq. (20). Hence $m(-1) = 0$ and so $m(1) + m(-1) \leq 2$.

Having treated the above cases, we may assume that $M \neq \pm 1$, $N \equiv 2 \pmod{4}$ and hence that 1 and -1 do not occur in $\{U_{2n}\}$. By Eq. (22) with $r = 2$ and $s = 1$, one has

$$U_{2n+1} \equiv V_2 U_{2n-1} \equiv \dots \equiv V_2^{n-1} U_3 = (M^2 - 2N)^{n-1} (M^2 - N) \equiv 3 \pmod{4}$$

for $n \geq 1$. Therefore $U_1 = 1$ is the only occurrence of 1 in $\{U_n\}$. With p_i and d_i as in the statement, Eq. (21) shows that U_n can be -1 only when d_1 and d_2 are even, $n \equiv 1 + d_1/2 \pmod{d_1}$, and $n \equiv 1 + d_2 \pmod{2d_2}$.

Since $2 \parallel N$ and M is odd, $V_2 = M^2 - 2N \equiv 5 \pmod{8}$. Therefore V_2 is divisible by an odd prime p , and we have $p \nmid U_4 = U_2 V_2$ and $p \nmid M$. By Theorem 2 (ii) applied with $d = 4$, it follows that -1 occurs at most once in the subsequence $\{U_{4n+3}\}$. Similarly, if $U_3 = M^2 - N$ is divisible by an odd prime p , then the same result applied with $d = 3$ and p shows that -1 occurs at most once in the subsequence $\{U_{3n-1}\}$.

Suppose that $3 \mid M$. With $p_2 = 3$ and $d_2 = 1$ or 2 depending on whether or not $N \equiv 2 \pmod{3}$, we see that $m(-1) = 0$ if $N \equiv 2 \pmod{3}$, and that -1 occurs only in the subsequence $\{U_{4n+3}\}$ if $N \equiv 1 \pmod{3}$. Therefore $m(-1) \leq 1$ and $m(1) + m(-1) \leq 2$.

Suppose that $3 \nmid M$. With $p_1 = 3$ and $d_1 = 1$ or 2 depending on whether or not $M \equiv 1 \pmod{3}$, we see that $m(-1) = 0$ since -1 does not occur in $\{U_{2n}\}$.

Suppose that $3 \nmid M$ and $3 \mid N-1$. Then $3 \mid U_3$ and so -1 occurs at most once

in $\{U_{3n-1}\}$. By Eqs. (12,14),

$$U_{6n+1} \equiv U_7^n = (U_4^2 - NU_3^2)^n \equiv (U_4^2)^n \equiv 1 \pmod{3}.$$

By Lemma 4, it follows that $m(-1) \leq 1$, and so $m(1) + m(-1) \leq 2$.

The remaining case is $3 \nmid M$ and $N \equiv 2 \pmod{3}$. Let p_i and d_i be as in the statement. By the criterion of the fifth paragraph of the proof, all occurrences of -1 in $\{U_n\}$ lie in the following subsequences.

none	if d_1 or d_2 is odd
$\{U_{2n}\}$	if $2 \parallel d_1$
$\{U_{4n+3}\}$	if $4 \parallel d_1$ or $2 \parallel d_2$
$\{U_{8n+5}\}$	if $8 \parallel d_1$ or $4 \parallel d_2$
$\{U_{8n+1}\}$	if $16 \mid d_1$ or $8 \mid d_2$.

In the first three case, $m(-1) \leq 1$ and so $m(1) + m(-1) \leq 2$. In the fifth case, $m(-1) = 0$ and hence $m(1) + m(-1) = 1$ since by Eqs. (14,12) and the fact that $3 \mid U_4$, one has

$$U_{8n+1} \equiv U_9^n = (U_5^2 - NU_4^2)^n \equiv U_5^{2n} \equiv 1 \pmod{3}.$$

Finally, in the fourth case, $p_1 \equiv 1 \pmod{8}$ and $p_2 \equiv 1 \pmod{4}$ since $d_1 \mid p_1 - 1$ and $d_2 \mid p_2 - 1$. In particular, since N is positive, $2 \parallel N$, and $3 \mid N-2$, we have $N \equiv 2 \pmod{48}$. This completes the proof of the Theorem.

6.- Open questions.

In view of Theorem 4, it is natural to make the following conjecture.

Conjecture 1.- If $\{U_n\}$ is a non-degenerate Lucas sequence of the first kind satisfying Eq. (1) with either $M \neq \pm 1$ or $N \neq 2, 3, 5$, then $m(1) + m(-1) \leq 2$.

Using Theorem 2 (ii) and Theorem 4, it is straightforward to check by considering the various possibilities of $M \pmod{5}$ and $M \pmod{7}$ that the following is true.

Proposition 4 .- If $\{U_n\}$ is a non-degenerate Lucas sequence of the first kind satisfying Eq. (1) with $M \neq \pm 1$ and either $N \equiv \pm 1 \pmod{5}$ or $N \equiv 6 \pmod{7}$, then -1 occurs at most once in $\{U_n\}$.

Applying this result and Theorem 4 to check the various values of $N \equiv 2 \pmod{48}$, one obtains

Corollary 5.- The above conjecture is true for all $N \leq 1200$ with the possible exception of $N = 578$.

Conjecture 2.- If $\{U_n\}$ is a non-degenerate Lucas sequence of the first kind, then with the possible exception of finitely many integers c , one has

$$m(c) + m(-c) \leq 1.$$

F. Beukers has announced to the author progress on both of the above conjectures.

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