

Extensions of simple cohomological Mackey functors

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Abstract: This is a report on some recent joint work with Radu Stancu, to appear in [4]. It is an expanded version of a talk given at the RIMS workshop *Cohomology of finite groups and related topics*, February 18-20, 2015.

1. Cohomological Mackey functors

1.1. Let G be a finite group, and k be a commutative ring. There are many equivalent definitions of *Mackey functors* for G over k . For the “naive” one, this is an assignment $H \mapsto M(H)$ of a k -module $M(H)$ to any subgroup H of G , together with k -linear maps

$$M(H) \xrightarrow{t_H^K} M(K) \xrightarrow{r_H^K} M(H), \quad M(H) \xrightarrow{c_{x,H}} M(xH)$$

whenever $H \leq K \leq G$ and $x \in G$, subject to a list of compatibility conditions, e.g. *transitivity* of transfers and restrictions, or the *Mackey formula* (see [6] for details).

A Mackey functor M is called *cohomological* if

$$\forall H \leq K \leq G, \quad t_H^K \circ r_H^K = |K : H| \text{Id}_{M(K)}.$$

The cohomological Mackey functors for G over k form a category $\mathbf{M}_k^c(G)$.

1.2. Examples :

- Let V be a kG -module. The *fixed points functor* FP_V is defined by $M(H) = V^H$, for any $H \leq G$, and by

$$\forall H \leq K \leq G, \quad r_H^K : V^K \hookrightarrow V^H, \quad t_H^K = \text{Tr}_H^K : V^H \rightarrow V^K,$$

and by $c_{x,H}(v) = x \cdot v$, for $x \in G$.

More generally, for $n \in \mathbb{N}$, the cohomology functor $H^n(-, V)$ is a cohomological Mackey functor.

- Let k be a field of characteristic p , let G be a finite p -group. The *simple cohomological Mackey functors* for G over k are the functors $S_X = S_X^G$, where $X \leq G$ (up to G -conjugation), defined by

$$\forall H \leq G, \quad S_X(H) = \begin{cases} k & \text{if } H =_G X, \\ \{0\} & \text{otherwise.} \end{cases}$$

1.3. Yoshida's Theorem

- Let $\mathbf{perm}_k(G)$ denote the full subcategory of $kG\text{-Mod}$ consisting of finitely generated *permutation* kG -modules.
- Let $\mathbf{Fun}_k(G)$ denote the category of (contravariant) k -linear functors from $\mathbf{perm}_k(G)$ to $k\text{-Mod}$.
- If $M \in \mathbf{M}_k^c(G)$, the functor $\tilde{M} : V \mapsto \text{Hom}_{\mathbf{M}_k^c(G)}(FP_V, M)$ is an object of $\mathbf{Fun}_k(G)$.

1.4. Theorem [Yoshida [7]] : *The functor $M \mapsto \tilde{M}$ is an equivalence of categories from $\mathbf{M}_k^c(G)$ to $\mathbf{Fun}_k(G)$.*

1.5. The (cohomological) Mackey algebra

- [Thévenaz-Webb [6]] The (cohomological) Mackey functors for G over k are exactly the modules over the (cohomological) *Mackey algebra*.
- Consider the Hecke algebra $Y_k(G) = \text{End}_{kG}(\bigoplus_{H \leq G} kG/H)$. This k -algebra is called *the Yoshida algebra* of G over k . It is isomorphic to the cohomological Mackey algebra. In other words, *the category $\mathbf{M}_k^c(G)$ is equivalent to $Y_k(G)\text{-Mod}$.*
- The algebra $Y_k(G)$ is a free k -module of rank $\sum_{H, K \leq G} |H \backslash G/K|$. In particular, when k is a field, the algebra $Y_k(G)$ is a finite dimensional k -algebra.

2. Complexity

Let k be a field, and A be a finite dimensional k -algebra. Then every finitely generated A -module M admits a resolution

$$\cdots \rightarrow P_n \rightarrow P_{n-1} \rightarrow \cdots \rightarrow P_1 \rightarrow P_0 \rightarrow M \rightarrow 0$$

by finitely generated projective A -modules.

2.1. Definition : *The module M has polynomial growth if there exists such a resolution and numbers c, d, e such that $\forall n \in \mathbb{N}, \dim_k P_n \leq cn^d + e$. The lower bound of such d 's is called the complexity of M .*

The module M has exponential growth if for any such resolution, there exist numbers $c > 0, d > 1$, and e such that $\forall n \in \mathbb{N}, \dim_k P_n \geq cd^n + e$.

The module M has intermediate growth in all other cases.

2.2. Lemma [Link with extensions] : *Let A be a finite dimensional algebra over a field k , and M be a finitely generated A -module.*

1. *If*

$$\cdots \rightarrow P_n \rightarrow P_{n-1} \cdots \rightarrow P_0 \rightarrow M \rightarrow 0$$

is a minimal projective resolution of M , then

$$P_n \cong \bigoplus_{S \in \text{Irr}(A)} P_S^{\dim_k \text{Ext}_A^n(M, S) / \dim_k \text{End}_A(S)},$$

where $\text{Irr}(A)$ is a set of representatives of isomorphism classes of simple A -modules, and P_S denotes a projective cover of S .

2. *In particular M has polynomial growth $\iff \forall S \in \text{Irr}(A), \exists (c, d, e)$ such that $\forall n \in \mathbb{N}, \dim_k \text{Ext}_A^n(M, S) \leq cn^d + e$.*
3. *Let $0 \rightarrow L \rightarrow M \rightarrow N \rightarrow 0$ be a short exact sequence of finitely generated A -modules. If any two of L, M, N have polynomial growth, so does the third.*

2.3. Definition [Poco groups] : *Let k be a field of positive characteristic p . A finite group G is called a poco group over k if any finitely generated cohomological Mackey functor for G over k has polynomial growth.*

2.4. Theorem [B. [3]] : *Let G be a finite group, and k be a field of characteristic $p > 0$. The following conditions are equivalent:*

1. *The group G is a poco group over k .*
2. *Let S be a Sylow p -subgroup of G . Then :*
 - *If $p > 2$, the group S is cyclic.*
 - *If $p = 2$, the group S has sectional rank at most 2.*

2.5. Remark : *A 2-group has sectional rank at most 2 if and only if it is cyclic or metacyclic (Blackburn [2], Andersen-Oliver-Ventura [1]).*

3. Construction of functors

3.1. *Let k be a field of characteristic $p > 0$, and G be a finite group. By*

Yoshida's equivalence $\mathbf{M}_k^c(G) \cong \mathbf{Fun}_k(G)$, cohomological Mackey functors for G over k can be viewed as functors

$$\mathbf{perm}_k(G) \longrightarrow k\text{-Mod} \quad .$$

When H is another finite group, any k -linear functor

$$F : \mathbf{perm}_k(H) \longrightarrow \mathbf{perm}_k(G)$$

induces a functor

$$\mathbf{M}_k^c(G) \cong \mathbf{Fun}_k(G) \longrightarrow \mathbf{Fun}_k(H) \cong \mathbf{M}_k^c(H) \quad .$$

3.2. In particular, when U is a (finite) (G, H) -biset, the functor

$$t_U : W \in \mathbf{perm}_k(H) \mapsto kU \otimes_{kH} W \in \mathbf{perm}_k(G)$$

induces a functor $L_U : \mathbf{M}_k^c(G) \rightarrow \mathbf{M}_k^c(H)$. Similarly, the functor

$$h_U : W' \in \mathbf{perm}_k(G) \mapsto \mathrm{Hom}_{kG}(kU, W') \in \mathbf{perm}_k(H)$$

induces a functor $R_U : \mathbf{M}_k^c(H) \rightarrow \mathbf{M}_k^c(G)$.

3.3. Properties

- The functors L_U and R_U are *exact*.
- As t_U is left adjoint to h_U , the functor L_U is *left adjoint* to R_U .
- Let U' be another finite (G, H) -biset. Then

$$L_{U \sqcup U'} \cong L_U \oplus L_{U'}, \quad R_{U \sqcup U'} \cong R_U \oplus R_{U'} \quad .$$

- Let Id_G denote the identity (G, G) -biset. Then L_{Id_G} and R_{Id_G} are isomorphic to *the identity functor*.
- If K is another finite group, and V is an (H, K) -biset, then

$$L_V \circ L_U \cong L_{U \times_H V}, \quad R_U \circ R_V \cong R_{U \times_H V} \quad .$$

3.4. Examples

- Let H be a subgroup of G , and U denote the set G , as a (G, H) -biset. Then $L_U \cong \text{Res}_H^G$, and $R_U \cong \text{Ind}_H^G$.
- Let H be a subgroup of G , and U denote the set G , as an (H, G) -biset. Then $L_U \cong \text{Ind}_H^G$, and $R_U \cong \text{Res}_H^G$.
- Let $N \trianglelefteq G$, let $H = G/N$, and let U denote the set H , as a (G, H) -biset. Then $L_U = \rho_{G/N}^G$, and $R_U = j_{G/N}^G$.
- Let $N \trianglelefteq G$, let $H = G/N$, and let U denote the set H , as an (H, G) -biset. Then $L_U = i_{G/N}^G$, and $R_U = \rho_{G/N}^G$.
- Let $f : G \rightarrow H$ be a group isomorphism, and U denote the set H , as a (G, H) -biset. Then $L_U \cong \text{Iso}(f)$ and $R_U \cong \text{Iso}(f^{-1})$.

3.5. Sketch of proof of Theorem 2.4

Recall that k is a field of characteristic $p > 0$, that G is a finite group, and S is a Sylow p -subgroup of G .

- Use the functors Ind_S^G and Res_S^G to reduce to the case where $G = S$ is a p -group.
- Let (B, A) be a *section* of G (i.e. $A \trianglelefteq B \leq G$). The set G/A is a $(G, B/A)$ -biset, and the set $A \backslash G$ is a $(B/A, G)$ -biset. The corresponding functors $L_{G/A}$, $R_{G/A}$, $L_{A \backslash G}$ and $R_{A \backslash G}$ allow for a reduction to the case where G is *elementary abelian*.
- The case of cyclic groups and Klein four group was settled by M. Samy Modeliar ([5]). In particular, these groups are poco groups.
- Describe the subfunctor structure of $\text{Ind}_H^G S_1^H$, leading to long exact sequences of Ext groups. These sequences show that the functor S_1^G has exponential growth if $G \cong (C_p)^m$, when $p > 2$ and $m \geq 2$, or $p = 2$ and $m \geq 3$.
- Use induction on the order of a 2-group G , to complete the case $p = 2$.

4. Presentation of some Ext algebras

Let p be a prime number, and $G \cong (C_p)^n$, $n \geq 1$.

- Let $X \leq G$ with $|X| = p$. Then there exists a unique non split extension $\alpha_X^G : 0 \rightarrow S_1^G \rightarrow \begin{pmatrix} S_X^G \\ S_1^G \end{pmatrix} \rightarrow S_X^G \rightarrow 0$ in $\mathbf{M}_{\mathbb{F}_p}^c(G)$.

Let $\gamma_X^G \in \text{Ext}_{\mathbf{M}_{\mathbb{F}_p}^c(G)}^2(S_1^G, S_1^G)$ denote the class of the splice

$$\alpha_X^G(\alpha_X^G)^* : 0 \rightarrow S_1^G \rightarrow \begin{pmatrix} S_X^G \\ S_1^G \end{pmatrix} \rightarrow \begin{pmatrix} S_1^G \\ S_X^G \end{pmatrix} \rightarrow S_1^G \rightarrow 0 \quad .$$

- When $p > 2$ and $\varphi : G \rightarrow \mathbb{F}_p$ is a group homomorphism, let U_φ^G be the vector space $\mathbb{F}_p \oplus \mathbb{F}_p$, on which $g \in G$ acts by $g(x, y) = (x + \varphi(g)y, y)$. There is a unique (cohomological) Mackey functor T_φ^G for G over $\mathbb{F}_p$ such that $T_\varphi(H) = \{0\}$ if $1 < H \leq G$, and $T_\varphi(1) \cong U_\varphi^G$. It fits in an extension

$$0 \rightarrow S_1^G \rightarrow U_\varphi^G \rightarrow S_1^G \rightarrow 0$$

in $\mathbf{M}_{\mathbb{F}_p}^c(G)$. Let $\tau_\varphi^G \in \text{Ext}_{\mathbf{M}_{\mathbb{F}_p}^c(G)}^1(S_1^G, S_1^G)$ denote the class of this extension.

4.1. The algebra $\mathcal{E}_k = \text{Ext}_{\mathbf{M}_k^c(G)}^*(S_1^G, S_1^G)$

4.2. Theorem [B. Stancu [4]] : *Let k be a field of characteristic $p > 0$, and $G \cong (C_p)^n$. Let $\mathcal{E}_k$ denote the algebra $\text{Ext}_{\mathbf{M}_k^c(G)}^*(S_1^G, S_1^G)$. Then:*

1. *The extension of scalars from $\mathbb{F}_p$ to k induces an isomorphism of k -algebras $\mathcal{E}_k \cong k \otimes_{\mathbb{F}_p} \mathcal{E}_{\mathbb{F}_p}$.*
2. *The algebra $\mathcal{E}_{\mathbb{F}_p}$ is generated by the elements γ_X^G , where $X \leq G$ with $|X| = p$, together, when $p > 2$, with the elements τ_φ^G , where $\varphi : G \rightarrow \mathbb{F}_p^+$.*

4.3. Presentation of $\mathcal{E}_k$ for $p = 2$

4.4. Theorem [B. [3]] : *Let k be a field of characteristic 2, and $G \cong (C_2)^m$. Then the graded algebra $\mathcal{E}_k = \text{Ext}_{\mathbf{M}_k^c(G)}^*(S_1^G, S_1^G)$ admits the following presentation:*

- *The generators γ_x are indexed by the elements x of $G - \{0\}$. They have degree 2.*
- *The relations are the following:*
 1. *If $H < G$ with $|G : H| = 2$, then $\sum_{x \notin H} \gamma_x = 0$.*
 2. *If x and y are distinct elements of $G - \{0\}$, then*

$$[\gamma_x + \gamma_y, \gamma_{x+y}] = 0 \quad .$$

4.5. Presentation of $\mathcal{E}_k$, for $p > 2$

4.6. Theorem [B. Stancu [4]] : *Let k be a field of characteristic $p > 2$, and $G \cong (C_p)^m$. Then the graded algebra $\mathcal{E}_k = \text{Ext}_{\mathbf{M}_k^e(G)}^*(S_1^G, S_1^G)$ admits the following presentation:*

1. *The generators are the elements γ_X in degree 2, for $X \leq G$ such that $|X| = p$, and the elements τ_φ in degree 1, for $\varphi \in \text{Hom}(G, \mathbb{F}_p^+)$.*
2. *The relations are the following:*
 - (a) $\tau_{\varphi+\psi} = \tau_\varphi + \tau_\psi$, for any φ, ψ in $\text{Hom}(G, \mathbb{F}_p^+)$.
 - (b) • *If $p \geq 5$, then $\tau_\varphi^2 = 0$ and $[\tau_\varphi, \sum_{X \not\leq \text{Ker} \varphi} \gamma_X] = 0$, for any φ in $\text{Hom}(G, \mathbb{F}_p^+)$.*
 • *If $p = 3$, then $\tau_\varphi^2 = - \sum_{X \not\leq \text{Ker} \varphi} \gamma_X$, for any $\varphi \in \text{Hom}(G, \mathbb{F}_p^+)$.*
 - (c) $[\gamma_X, \tau_\varphi] = 0$, for any $\varphi \in \text{Hom}(G, \mathbb{F}_p^+)$, for any $X \leq \text{Ker} \varphi$ with $|X| = p$.
 - (d) $[\gamma_X, \sum_{\substack{Y \leq Q \\ |Y|=p}} \gamma_Y] = 0$, for any $Q \leq G$ with $|Q| = p^2$ and any $X \leq Q$ with $|X| = p$.

4.7. Corollary : *The Poincaré series of $\mathcal{E}_k$ is equal to*

$$P(t) = \frac{1}{(1-t^2)(1-3t^2)(1-7t^2) \cdots (1-(2^{m-1}-1)t^2)}$$

when $p = 2$, and to

$$\frac{1}{(1-t)(1-t-(p-1)t^2)(1-t-(p^2-1)t^2) \cdots (1-t-(p^{m-1}-1)t^2)}$$

when p is odd.

4.8. Corollary : *Let k be a field of characteristic $p > 0$. When G is an elementary abelian p -group, one can compute explicitly all the extension groups between any two simple cohomological Mackey functors for G over k .*

Acknowledgements: I wish to thank Professor Fumihito Oda for his invitation, and Kinki University, Osaka for support. I also thank Professor Akihiko Hida for the opportunity of giving a talk at the RIMS workshop *Cohomology of finite groups and related topics*, February 18-20, 2015.

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