

DIFFERENTIABLE SINGULAR COHOMOLOGY

RELATED TO FOLIATION

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Introduction

Let $(M, \mathcal{F})$ be a C^∞ -foliation of codimension q on a paracompact Hausdorff manifold of dimension n . $T(M)$ be the tangent bundle of M and $F = T(\mathcal{F})$ the subbundle of $T(M)$ consisting of tangent vectors of leaves of $\mathcal{F}$. Let V denote the normal bundle of $\mathcal{F}$ with respect to a Riemannian metric on M . Then we have the splitting of $T(M)$ into a Whitney sum:

$$T(M) = F \oplus V$$

and the dual splitting of cotangent bundle,

$$T^*(M) = V^* \oplus F^*.$$

On a local foliation chart, $(x, u): U \rightarrow \mathbb{R}^p \times \mathbb{R}^q$ one chooses differential forms

$$\theta_j = dx_j + \sum_{\alpha=1}^q a_{j\alpha} du_\alpha \quad 1 \leq j \leq p,$$

$$v_\alpha = \partial/\partial u_\alpha - \sum_{j=1}^p a_{j\alpha} \partial/\partial x_j \quad 1 \leq \alpha \leq q$$

such that $\{\theta_1, \dots, \theta_p, du_1, \dots, du_q\}$, $\{\partial/\partial x_1, \dots, \partial/\partial x_p, v_1, \dots, v_q\}$ are dual bases of $T_m^*(M)$ and $T_m(M)$, $m \in U$.

Let $A^{r,s}$ be the vector space of differential forms which are locally

$$\sum a_{i_1 \dots i_r j_1 \dots j_s} du_{i_1} \dots du_{i_r} \theta_{j_1} \dots \theta_{j_s}.$$

Let $A(M)$ denote the vector space of C^∞ -differential forms on M .

Then we have

$$A(M) = \bigoplus A^{r,s}.$$

The exterior derivative of an $\omega \in A^{r,s}$ is of the form

$$d\omega = d_1 \omega + d_2 \omega + d_{\mathcal{F}} \omega$$

where $d_1 \omega \in A^{r+2,s-1}$, $d_2 \omega \in A^{r+1,s}$ and $d_{\mathcal{F}} \omega \in A^{r,s+1}$ are

uniquely determined.

From the relation $d^2 = 0$, it follows that

$$d_1^2 = 0, d_{\mathcal{F}}^2 = 0, \dots.$$

In particular $d_{\mathcal{F}}: A^{r,s} \rightarrow A^{r,s+1}$ defines a cohomology vector space $H_{\mathcal{F}DR}^{r,s}(M)$ with transverse degree r and leaf degree s .

Let $C_{\mathcal{F}}^{\infty}$ denote the sheaf of germs of real valued C^{∞} -functions on M constant along leaves of $\mathcal{F}$ and $\check{H}^s(M; C_{\mathcal{F}}^{\infty})$, the Čech cohomology vector space of M with the coefficient sheaf $C_{\mathcal{F}}^{\infty}$. We have already a de Rham type isomorphism

$$H_{\mathcal{F}DR}^{0,s}(M) \cong H^s(M; C_{\mathcal{F}}^{\infty}),$$

as a part of the Dolbeault isomorphism for foliation. (Cf., e.g., [V, Théorème 3,2], [S, Theorem 4.2].)

In the present paper, we establish a singular cohomology version of this isomorphism. Let $C_{*}^{\mathcal{F}}(M; R)$ be the chain complex with the coefficient group R generated by differentiable singular simplexes in leaves of $\mathcal{F}$. Then we introduce a differentiable singular cochain complex $C_{\mathcal{F}}^{*}(M, R)$ for $C_{*}^{\mathcal{F}}(M; R)$. (See Section 2 below.)

Let $H_{\mathcal{F}D}^s(M; R)$ denote the cohomology vector space of $C_{\mathcal{F}D}^{*}(M; R)$ and $\Lambda: A^{0,s} \rightarrow C_{\mathcal{F}D}^s(M; R)$, the homomorphism defined by the integration

$$\int_{\sigma_s^{\mathcal{F}}} \omega$$

of $\omega \in A^{0,s}$ on a simplex $\sigma_s^{\mathcal{F}} \in C_s^{\mathcal{F}}(M; \mathbb{R})$.

MAIN THEOREM. Λ induces an isomorphism

$$\Lambda^*: H_{\mathcal{F}DR}^{0,s}(M) \xrightarrow{\cong} H_{\mathcal{F}D}^s(M; \mathbb{R}).$$

(See Section 3.)

This relation can be used to give Weil operators [HH,

DEFINITION 1.6] a meaning as cohomology classes (cf. [S.

Theorem 5.4]).

All manifolds, maps and foliations are assumed to be of class C^∞ and manifolds are without boundaries.

1. Differentiable singular chains in leaves

Let $(M, \mathcal{F})$ be the C^∞ -foliation of codimension q on the paracompact Hausdorff manifold of dimension n . Let $\sigma_s^{\mathcal{F}}$ be a C^∞ -singular s -simplex such that the image of $\sigma_s^{\mathcal{F}}$ is contained in a leaf of $\mathcal{F}$.

Let $C_s^{\mathcal{F}}(M;R)$ denote the vector space over R with the basis $\{\sigma_s^{\mathcal{F}}\}$. Then we have obviously

$$\partial C_s^{\mathcal{F}}(M;R) \subset C_{s-1}^{\mathcal{F}}(M;R)$$

for the boundary operator ∂ , and obtain a chain complex

$$C_*^{\mathcal{F}}(M;R): \dots \xrightarrow{\partial} C_s^{\mathcal{F}}(M;R) \xrightarrow{\partial} C_{s-1}^{\mathcal{F}}(M;R) \xrightarrow{\partial} \dots \xrightarrow{\partial} C_0^{\mathcal{F}}(M;R) \rightarrow 0,$$

$$C_0^{\mathcal{F}}(M;R) = \sum_{m \in M} R_m \quad (R_m \cong R).$$

Let $(M', \mathcal{F}')$ be a codimension q foliation on a manifold M' of dimension n' . Let $f: M \rightarrow M'$ be a C^∞ -map which is transverse to $\mathcal{F}'$ and preserves leaves, i.e., for each leaf L of $\mathcal{F}$, there is a leaf L' of $\mathcal{F}'$ such that $f(L) \subset L'$. One can see that $\mathcal{F}$ coincides with the pullback $f^*\mathcal{F}'$ of $\mathcal{F}'$. f induces a chain map

$$f_{\#}: C_*^{\mathcal{F}}(M;R) \rightarrow C_*^{\mathcal{F}'}(M';R).$$

Let $f_0, f_1: M \rightarrow M'$ be C^∞ -maps transverse to $\mathcal{F}'$ so that $f_0^*\mathcal{F}' = f_1^*\mathcal{F}' = \mathcal{F}$. If there is a C^∞ -map $H: M \times R \rightarrow M'$ transverse to $\mathcal{F}'$ such that

$$f_i(m) = H(m, i) \quad i = 0, 1,$$

$$H^*\mathcal{F}' = \pi^*\mathcal{F}$$

where $\pi: M \times R \rightarrow M$ is the first factor projection, then f_0, f_1 are called C^∞ -homotopic by leaf preserving map and denoted by $f_0 \approx_{\mathcal{F}} f_1$. H is called leaf preserving C^∞ -homotopy of f_0 and f_1 .

Let $H_s^{\mathcal{F}}(M; R)$ denote the s -dimensional homology vector space of $C_*^{\mathcal{F}}(M; R)$. The chain map $f_\#: C_*(M; R) \rightarrow C_*^{\mathcal{F}'}(M'; R)$ induces the homomorphism of homology vector spaces, $f_*: H_s^{\mathcal{F}}(M; R) \rightarrow H_s^{\mathcal{F}'}(M'; R)$.

Since affine simplexes are differentiable ones and the homotopy $H: M \times R \rightarrow M'$ preserves leaves, prism operators are well defined in $C_*^{\mathcal{F}}(M; R)$ and one obtains,

PROPOSITION 1.1. If $f_0, f_1: M \rightarrow M'$ are homotopic by leaf preserving map, then $f_{0\#}$ and $f_{1\#}$ are chain homotopic and hence we have

$$f_{0\#} = f_{1\#}: H_s^{\mathcal{F}}(M; R) \rightarrow H_s^{\mathcal{F}'}(M'; R).$$

Again since affine simplexes are differentiable ones, as well as prism operator, one can construct subdivision operators

$Sd_s^{\mathcal{F}}: (C_s^{\mathcal{F}}(M;R) \rightarrow C_s^{\mathcal{F}}(M;R)$ which are chain maps and one obtains the chain homotopies $\Phi_s^{\mathcal{F}, Sd}: C_s^{\mathcal{F}}(M;R) \rightarrow C_{s+1}^{\mathcal{F}}(M;R)$ between the identity operator and $Sd_s^{\mathcal{F}}$ by the usual manner (see, e.g., [G, p.651]).

An open set $X \subset M$ has the restricted foliation of $\mathcal{F}$ denoted by the same symbol. $C_s^{\mathcal{F}}(X; \mathcal{F})$ has an excision property as follows:

PROPOSITION 1.2. If X_1 and X_2 are open sets of M , then the natural inclusion map $\iota: C_*^{\mathcal{F}}(X_1;R) + C_*^{\mathcal{F}}(X_2;R) \rightarrow C_*^{\mathcal{F}}(X_1 \cup X_2;R)$ gives a chain homotopy equivalence.

Let $\lambda_k: X_1 \cap X_2 \rightarrow X_k$ and $\mu_k: X_k \rightarrow X_1 \cup X_2$ be the natural inclusion maps for $k = 0, 1$, which induce chain maps

$$\lambda_{k\#}: C_*^{\mathcal{F}}(X_1 \cap X_2;R) \rightarrow C_*^{\mathcal{F}}(X_k;R) \text{ and } \mu_{k\#}: C_*^{\mathcal{F}}(X_k;R) \rightarrow C_*^{\mathcal{F}}(X_1 \cup X_2;R).$$

We define chain maps

$$\lambda: C_*^{\mathcal{F}}(X_1 \cap X_2;R) \rightarrow C_*^{\mathcal{F}}(X_1;R) \oplus C_*^{\mathcal{F}}(X_2;R),$$

$$\mu: C_*^{\mathcal{F}}(X_1;R) \oplus C_*^{\mathcal{F}}(X_2;R) \rightarrow C_*^{\mathcal{F}}(X_1 \cup X_2;R)$$

by $\lambda(c) = (\lambda_{1\#}c, -\lambda_{2\#}c)$ and $\mu(c_1, c_2) = \mu_{1\#}c_1 + \mu_{2\#}c_2$. One

obtains a short exact sequence of chain complexes,

$$0 \rightarrow C_*^{\mathcal{F}}(X_1 \cap X_2; R) \xrightarrow{\lambda} C_*^{\mathcal{F}}(X_1; R) \oplus C_*^{\mathcal{F}}(X_2; R) \xrightarrow{\mu} C_*^{\mathcal{F}}(X_1; R) + C_*^{\mathcal{F}}(X_2; R) \rightarrow 0.$$

Let λ_* and μ_* be homology homomorphisms induced by λ and μ respectively.

COROLLARY 1.3. If X_1 and X_2 are open sets of M , then we have the Mayer-Vietoris exact sequence of $H_*^{\mathcal{F}}$:

$$\begin{aligned} \dots \rightarrow H_i^{\mathcal{F}}(X_1 \cap X_2; R) &\xrightarrow{\lambda_*} H_i^{\mathcal{F}}(X_1; R) \oplus H_i^{\mathcal{F}}(X_2; R) \\ &\xrightarrow{\mu_*} H_i^{\mathcal{F}}(X_1 \cup X_2; R) \xrightarrow{\partial_*} H_{i-1}^{\mathcal{F}}(X_1 \cap X_2; R) \rightarrow \dots, \end{aligned}$$

where ∂_* is the connecting homomorphism.

2. Differentiable singular cochains restricted leaves

Let Δ^s be the standard s -simplex and $D^q(\varepsilon) \subset R^q$ be an open ε -ball around the origin for sufficiently small number $\varepsilon > 0$ and $\hat{\sigma}_s^{\mathcal{F}}: D^q(\varepsilon) \times \Delta^s \rightarrow M$ any differentiable map such that

$$\hat{\sigma}_s^{\mathcal{F}}\{x\} = \hat{\sigma}_s^{\mathcal{F}} \big|_{\{x\} \times \Delta^s} \in C_s^{\mathcal{F}}(M; R),$$

for each $x \in D^q(\varepsilon)$. An s -cochain ξ for $C_s^{\mathcal{F}}(M;R)$ is called differentiable with $\mathcal{F}$ if the value $\xi(\hat{\sigma}_s^{\mathcal{F}}(x))$ is differentiable with respect to x and $\hat{\sigma}_s^{\mathcal{F}}$ is called an ε -thickening of $\sigma_s^{\mathcal{F}}$ if $\sigma_s^{\mathcal{F}} = \hat{\sigma}_s^{\mathcal{F}}(0)$.

Let δ denote the usual coboundary operator for cochains of $C_s^{\mathcal{F}}(M;R)$. We denote the set of differentiable cochains of $C_s^{\mathcal{F}}(M;R)$ by $C_{\mathcal{F}D}^s(M;R)$. This is a vector subspace of the s -cochain vector space $C_s^{\mathcal{F}}(M;R)$.

Let $e_i: \Delta^{s-1} \rightarrow \Delta^s$, $i = 0, \dots, s$ be the standard face map and $\hat{\sigma}_s^{\mathcal{F}}$, a differentiable ε -thickening of $\sigma_s^{\mathcal{F}}$. Then the map $\hat{\partial}_i \sigma_s^{\mathcal{F}}: D^q(\varepsilon) \times \Delta^{q-1} \rightarrow M$ defined by

$$\hat{\partial}_i \sigma_s^{\mathcal{F}} = \hat{\sigma}_s^{\mathcal{F}} \cdot (\text{id}_{D^q(\varepsilon)} \times e_i)$$

is obviously a C^∞ ε -thickening of $\partial_i \sigma_s^{\mathcal{F}}$. So, if $\xi \in C_{\mathcal{F}D}^{s-1}(M;R)$

and $\varepsilon > 0$ is sufficiently small, then $\xi(\hat{\partial}_i \sigma_s^{\mathcal{F}}(x))$ is

differentiable with respect to $x \in D^q(\varepsilon)$ for $i = 0, \dots, s$.

Therefore we have

$$\begin{aligned}
\delta \xi(\hat{\sigma}_s^{\mathcal{F}}(x)) &= \xi(\partial \hat{\sigma}_s^{\mathcal{F}}(x)) \\
&= \sum_{i=0}^s (-1)^i \xi(\partial_i(\hat{\sigma}_s^{\mathcal{F}}(x))) \\
&= \sum_{i=0}^s (-1)^i \xi(\widehat{\partial_i \sigma}_s^{\mathcal{F}}(x)).
\end{aligned}$$

The last formula shows $\delta \xi(\hat{\sigma}_s^{\mathcal{F}}(x))$ is differentiable with respect to $x \in D^q(\varepsilon)$, and so we have $\delta \xi \in C_{\mathcal{F}D}^s(M; R)$. Thus one obtains,

LEMMA 2.1. $C_{\mathcal{F}D}^*(M; R) = \{C_{\mathcal{F}D}^s(M; R), \delta\}$ is a cochain complex.

$C_{\mathcal{F}D}^*(M; R)$ is called the differentiable singular cochain complex for the foliation $(M, \mathcal{F})$. In the rest of this section we introduce cochain maps induced by transverse C^∞ -maps for foliations, cochain homotopies induced by leaf preserving C^∞ -homotopies between transverse C^∞ -maps and cochain homotopies between cochain subdivision operators. These are obtained by checking that the images satisfy differentiability condition for ε -thickenings of differentiable singular simplexes contained in leaves.

Let $(M, \mathcal{F})$ and $(M', \mathcal{F}')$ be C^∞ -foliations of the same codimension q and $f: M \rightarrow M'$ a C^∞ -map transverse to $\mathcal{F}'$ such that $f^* \mathcal{F}' = \mathcal{F}$.

LEMMA 2.2. f induces a cochain map $f^\#: C_{\mathcal{F}, D}^*(M'; R) \rightarrow C_{\mathcal{F}, D}^*(M; R)$.

Let $f_0, f_1: M \rightarrow M'$ be C^∞ -maps transverse to $\mathcal{F}'$ so that $f_0^* \mathcal{F}' = f_1^* \mathcal{F}' = \mathcal{F}$. Assume that $f_0 \stackrel{\sim}{\mathcal{F}}, \mathcal{F} f_1$ by a leaf preserving C^∞ -homotopy $H: M \times R \rightarrow M'$.

LEMMA 2.3. If we have $f_0 \stackrel{\sim}{\mathcal{F}}, \mathcal{F} f_1$, then $f_0^\#, f_1^\#: G_{\mathcal{F}, D}^*(M'; R) \rightarrow C_{\mathcal{F}, D}^*(M; R)$ are cochain homotopic.

Let $Sd^{\mathcal{F}} = \{Sd_s^{\mathcal{F}}\}: C_*^{\mathcal{F}}(M; R) \rightarrow C_*^{\mathcal{F}}(M; R)$ be the subdivision operator and $\Phi^{\mathcal{F}, Sd} = \{\Phi_s^{\mathcal{F}, Sd}\}: C_*^{\mathcal{F}}(M; R) \rightarrow C_{*+1}^{\mathcal{F}}(M; R)$ the chain homotopy operator between the identity operator and $Sd^{\mathcal{F}}$. We define cochain map $Sd_{\mathcal{F}} = \{Sd_{\mathcal{F}}^s\}: C_{\mathcal{F}}^*(M; R) \rightarrow C_{\mathcal{F}}^*(M; R)$ by the formula,

$$(Sd_{\mathcal{F}}^s \xi)(\sigma_{\mathcal{F}}^s) = \xi(Sd_s^{\mathcal{F}} \sigma_s^{\mathcal{F}}), \quad \xi \in C_{\mathcal{F}}^s(M; R), \quad \sigma_s^{\mathcal{F}} \in C_s^{\mathcal{F}}(M; R)$$

and also define a homomorphism

$\Phi_{\mathcal{F}, \text{Sd}} = \{\Phi_{\mathcal{F}, \text{Sd}}^s\}: C_{\mathcal{F}}^*(M; R) \rightarrow C_{\mathcal{F}}^{*-1}(M; R)$ by the formula,

$$(\Phi_{\mathcal{F}, \text{Sd}}^s)(\sigma_{s-1}^{\mathcal{F}}) = \xi(\Phi_{s-1}^{\mathcal{F}, \text{Sd}} \sigma_{s-1}^{\mathcal{F}}), \quad \xi \in C_{\mathcal{F}}^s(M; R),$$

$$\sigma_{s-1}^{\mathcal{F}} \in C_{s-1}(M; R).$$

LEMMA 2.4. $\text{Sd}_{\mathcal{F}}$ is a cochain map $C_{\mathcal{F}D}^*(M; R) \rightarrow C_{\mathcal{F}D}^*(M; R)$ and

$\Phi_{\mathcal{F}, \text{Sd}}$ is a cochain homotopy $C_{\mathcal{F}D}^*(M; R) \rightarrow C_{\mathcal{F}D}^{*-1}(M; R)$ between the identity map and $\text{Sd}_{\mathcal{F}}$.

3. Differentiable singular cohomology for foliation

The cohomology vector space $H_{\mathcal{F}D}^*(M; R) = \bigoplus_{s \geq 0} H_{\mathcal{F}D}^s(M; R)$ of the differentiable singular cochain complex $C_{\mathcal{F}D}^*(M; R)$ in LEMMA 2.1 is called the differentiable singular cohomology for the foliation $(M, \mathcal{F})$.

Let $(M', \mathcal{F}')$ be another foliation and $f: M \rightarrow M'$, a C^∞ -map transverse to $\mathcal{F}'$ such that $f^* \mathcal{F}' = \mathcal{F}$. By LEMMA 2.2, f induces a homomorphism of differentiable singular cohomology vector spaces:

$$f^*: H_{\mathcal{F}', D}^*(M'; R) \rightarrow H_{\mathcal{F}D}^*(M; R).$$

Let $f_0, f_1: M' \rightarrow M$ be C^∞ -maps transverse to $\mathcal{F}'$ so that $f_0^* \mathcal{F}' = f_1^* \mathcal{F}' = \mathcal{F}$. Assume that $f_0 \approx_{\mathcal{F}} f_1$. Then by LEMMA 2.3, we have

$$f_0^* = f_1^*: H_{\mathcal{F}, D}^*(M'; R) \rightarrow H_{\mathcal{F}, D}^*(M; R).$$

If X_1 and X_2 are open sets of M , then the natural inclusion map $\iota: C_{*}^{\mathcal{F}}(X_1; R) + C_{*}^{\mathcal{F}}(X_2; R) \rightarrow C_{*}^{\mathcal{F}}(X_1 \sqcup X_2; R)$ induces a cochain map

$$\iota^{\#}: C_{\mathcal{F}, D}^*(X_1 \sqcup X_2; R) \rightarrow C_{\mathcal{F}, D}^*(X_1; R) + C_{\mathcal{F}, D}^*(X_2; R).$$

By making use of a chain homotopy equivalence of PROPOSITION 1.2 and LEMMA 2.4, one obtains,

LEMMA 3.1. $\iota^{\#}$ is a cochain homotopy equivalence.

Let $X_k \subset M$ $k = 1, 2$ be open sets and let $\lambda_k: X_1 \cap X_2 \rightarrow X_k$ and $\mu_k: X_k \rightarrow X_1 \sqcup X_2$ be the natural inclusion maps. They induce cochain maps $\lambda_k^{\#}: C_{\mathcal{F}, D}^*(X_k; R) \rightarrow C_{\mathcal{F}, D}^*(X_1 \cap X_2; R)$ and $\mu_k^{\#}: C_{\mathcal{F}, D}^*(X_1 \sqcup X_2; R) \rightarrow C_{\mathcal{F}, D}^*(X_k; R)$.

We define cochain maps

$$\lambda^{\#}: C_{\mathcal{F}, D}^*(X_1; R) \oplus C_{\mathcal{F}, D}^*(X_2; R) \rightarrow C_{\mathcal{F}, D}^*(X_1 \cap X_2; R),$$

$$\mu^\#: C_{\mathcal{F}D}^*(X_1; R) + C_{\mathcal{F}D}^*(X_2; R) \rightarrow C_{\mathcal{F}D}^*(X_1; R) \oplus C_{\mathcal{F}D}^*(X_2; R)$$

by $\lambda^\#(\xi_1, \xi_2) = \lambda_1^\#(\xi_1) - \lambda_2^\#(\xi_2)$ and $\mu^\#(\xi) = (\mu_1^\#(\xi), \mu_2^\#(\xi))$.

One obtains a short exact sequence of cochain complexes

$$\begin{aligned} 0 \rightarrow C_{\mathcal{F}D}^*(X_1; R) + C_{\mathcal{F}D}^*(X_2; R) &\xrightarrow{\mu^\#} C_{\mathcal{F}D}^*(X_1; R) \oplus C_{\mathcal{F}D}^*(X_2; R) \\ &\xrightarrow{\lambda^\#} C_{\mathcal{F}D}^*(X_1 \sqcup X_2; R) \rightarrow 0. \end{aligned}$$

Let λ^* and μ^* be cohomology homomorphism induced by $\lambda^\#$ and $\mu^\#$ respectively. By LEMMA 3.1, we obtain,

THEOREM 3.2. If X_1 and X_2 are open sets of M , then we have the Mayer-Vietoris exact sequence of $H_{\mathcal{F}D}^*$:

$$\begin{aligned} \dots \rightarrow H_{\mathcal{F}D}^i(X_1 \sqcup X_2; R) &\xrightarrow{\mu^*} H_{\mathcal{F}D}^i(X_1; R) \oplus H_{\mathcal{F}D}^i(X_2; R) \\ &\xrightarrow{\lambda^*} H_{\mathcal{F}D}^i(X_1 \sqcap X_2; R) \xrightarrow{\delta^*} H_{\mathcal{F}D}^{i+1}(X_1 \sqcup X_2; R) \rightarrow \dots \end{aligned}$$

where δ^* is the connecting homomorphism.

We call codimension q foliation $(M, \mathcal{F})$ $\mathcal{F}$ -contractible if there exists a q -dimensional submanifold N transverse to $\mathcal{F}$ and a map $f: M \rightarrow N \subset M$ transverse to $\mathcal{F}$ which is C^∞ -homotopic to the identity map id_M by leaf preserving map, i.e., $f \sim_{\mathcal{F}} \text{id}_M$.

Let $\mathcal{U} = \{U_\alpha\}$ be an open cover of M . If an intersection of finite open sets of $\mathcal{U}$ is $\mathcal{F}$ -contractible, then we call $\mathcal{U}$ an $\mathcal{F}$ -simple cover of $(M, \mathcal{F})$. If $(M, \mathcal{F})$ is a foliation on a paracompact Hausdorff manifold, then by [S, Lemma 4.1], every open cover $\mathcal{U}$ of M admits a refinement $\mathcal{U}' = \{U'_i\}$ which is $\mathcal{F}$ -simple.

Moreover, by taking sufficiently small neighborhood of foliation chart as U'_i , one can assume $\bar{U}'_i$ is compact. One constructs, by induction, an increasing sequence $\{V_j\}$ of open sets in M such that $\bar{V}_j$ is compact, $\bar{V}_j \subset V_{j+1}$ and $\bigcup_j V_j = M$.

The integral operator $\Lambda: A^{0,s} \rightarrow C^s_{\mathcal{F}D}(M; \mathbb{R})$ defined by

$$\Lambda(\omega)(\sigma^{\mathcal{F}}_s) = \int_{\sigma^{\mathcal{F}}_s} \omega, \quad \omega \in A^{0,s}, \quad \sigma^{\mathcal{F}}_s \in C^{\mathcal{F}}_s(M; \mathbb{R})$$

is a cochain map [S, Lemma 5.2] and defines a natural homomorphism,

$$\Lambda^*: H^{0,s}_{\mathcal{F}DR}(M) \rightarrow H^s_{\mathcal{F}D}(M; \mathbb{R}).$$

If $U'_i \subset M$ is on $\mathcal{F}$ -contractible open set, there exists a

q -dimensional submanifold $N \subset U'_1$ transverse to $\mathcal{F}$ and by LEMMA 2.3,

$$H_{\mathcal{F}D}^s(U'_1; R) = \begin{cases} C^\infty(N) & s = 0 \\ 0 & s > 0 \end{cases}$$

where $C^\infty(N)$ is the vector space of C^∞ -function on N . By [S, Corollary 3.2], Λ^* gives the isomorphism $H_{\mathcal{F}DR}^s(U'_1) \cong H_{\mathcal{F}D}^s(U'_1; R)$.

By making use of the Mayer-Vietoris sequence of $H_{\mathcal{F}D}^*$ obtained in THEOREM 3.2, that of $H_{\mathcal{F}DR}^*$ and the five lemma, we have,

LEMMA 3.3. For each j , Λ^* is an isomorphism of vector spaces: $H_{\mathcal{F}DR}^{0,s}(V_j) \cong H_{\mathcal{F}D}^s(V_j; R)$.

PROOF OF THE MAIN THEOREM. Both sequences $\{H_{\mathcal{F}DR}^{0,s}(V_j)\}$ and $\{H_{\mathcal{F}D}^s(V_j; R)\}$ satisfy the Mittag-Leffler condition since $\bar{V}_j \subset V_{j+1}$ is compact. Therefore, the isomorphism Λ^* of LEMMA 3.3 gives, by the arguments [M1, §§A.3-A.4] and [M2, Appendix §3], the isomorphism

$$H_{\mathcal{F}DR}^{0,s}(M) \cong H_{\mathcal{F}D}^s(M; R).$$

REFERENCES

- [G] M.J.Greenberg, Lectures on Algebraic Topology, Benjamin,
New York, 1976.
- [HH] J.Heitsch and S.Hurder, Secondary class, Weil operators
and the geometry of foliations, Journal of Differential
Geometry 20 (1984), 291-309.
- [M1] W.S.Massey, Homology and Cohomology Theory, Dekker,
New York, 1978.
- [M2] _____, Singular Homology Theory, Springer, New York,
1980.
- [S] H.Suzuki, An interpretation of the Weil operator $X(y_1)$,
Research Notes in Mathematics 131, Differential Geometry,
L.A.Cordero (Ed.), Pitman, London 1985, 228-244.
- [V] I.Vaisman, Variétés Riemanniennes feuilletées, Czech.
Math.J. 21(1971), 46-75.