

Positive Definite Kernels and Majorization

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1 Introduction

Definition 1.1 Let $f(t)$ be a real continuous function defined on I , and consider the functional calculus $f(X)$ for a Hermitian matrix X with eigenvalues in I .

- f is called an operator monotone function on I if $f(A) \leq f(B)$ whenever $A \leq B$ (of any order n).
- f is said to be operator decreasing if $-f$ is operator monotone.
- f is called an operator convex function on I if $f(sA + (1-s)B) \leq sf(A) + (1-s)f(B)$ ($0 < s < 1$) for every pair of bounded Hermitian operators A and B whose spectra are both in I .
- An operator concave function is likewise defined.

Definition 1.2 Let $K(t, s)$ be a real, continuous and symmetric function defined on $I \times I$.

- $K(t, s)$ is called a positive semi-definite kernel on I if

$$\iint_{I \times I} K(t, s) \phi(t) \phi(s) dt ds \geq 0 \quad (1)$$

for all real continuous functions ϕ with compact support in I .

Remark It is evident that $K(t, s)$ is positive semi-definite on I if and only if for each n and for all n points $t_i \in I$ the $n \times n$ matrices

$$(K(t_i, t_j))_{i,j=1}^n$$

are positive semi-definite.

- Suppose $K(t, s) \geq 0$ for every t, s in I . Then the kernel $K(t, s)$ is said to be infinitely divisible on I if $K(t, s)^r$ is a positive semi-definite kernel for every $r > 0$, i.e.,

$$\iint_{I \times I} K(t, s)^r \phi(t) \phi(s) dt ds \geq 0$$

- A kernel $K(t, s)$ is said to be conditionally positive semi-definite on I if $\iint_{I \times I} K(t, s) \phi(t) \phi(s) dt ds \geq 0$ for ϕ such that the support of ϕ is compact and $\int_I \phi(t) dt = 0$.
- A kernel $K(t, s)$ is said to be conditionally negative semi-definite on I if $-K(t, s)$ is conditionally positive semi-definite on I .

(**Löwner**) C^1 function f is operator monotone on I if and only if the Löwner kernel $K_f(t, s)$ defined by

$$K_f(t, s) = \frac{f(t) - f(s)}{t - s} \quad (t \neq s), \quad K_f(t, t) = f'(t),$$

is positive semi-definite on I . (**F. Krauss, J. Bendat- S. Shermann**)
 $g(t)$ is an operator convex function on I if and only if $g(t)$ is of class $C^2(I)$
 and for each $t_0 \in I$, the function $f(t)$ defined by

$$f(t) = \frac{g(t) - g(t_0)}{t - t_0} \quad (t \neq t_0), \quad f(t_0) = g'(t_0)$$

is operator monotone on I .

2 Operator convex functions

Proposition 2.1 Let $f(t)$ be an operator monotone (or decreasing) function on I . Then the indefinite integral $\int f(t)dt$ is an operator convex (or concave) function on I .

Example 2.1 $\int \log t dt = t \log t - t$, hence $t \log t$ and $\log \Gamma(t) = \int \frac{\Gamma'(t)}{\Gamma(t)} dt$ are both operator convex on $(0, \infty)$

But the converse is not true; $\frac{1}{t}$ on $(0, \infty)$ is a counter example.

Proposition 2.2 Let $g(t)$ be an operator convex function on $(0, \infty)$. Then $g'(\sqrt{t})$ is operator monotone there.

(Well-known) Let $f(t) \geq 0$ be defined on $[0, \infty)$. Then f is operator monotone $\Leftrightarrow f(t)$ is operator concave.

Theorem 2.3 Let $f(t)$ be defined on (a, ∞) with $a \geq -\infty$. Then

(i) $f(t)$ is operator decreasing $\Leftrightarrow f(t)$ is operator convex and $f(\infty) =$

$$\lim_{t \rightarrow \infty} f(t) < \infty;$$

- (ii) $f(t)$ is operator monotone $\Leftrightarrow f(t)$ is operator concave and $f(\infty) > -\infty$.

In (ii) the condition “ $f(\infty) > -\infty$ ” is indispensable; for instance, $f(t) = -t^2$ is operator concave on $(0, \infty)$ but not operator monotone there.

Corollary 2.4 Let $f(t)$ be defined on $(-\infty, b)$, where $b \leq \infty$. Then

- (i) $f(t)$ is operator monotone on $(-\infty, b) \Leftrightarrow f(t)$ is operator convex on $(-\infty, b)$ and $f(-\infty) < \infty$
- (ii) $f(t)$ is operator decreasing on $(-\infty, b) \Leftrightarrow f(t)$ is operator concave on $(-\infty, b)$ and $f(-\infty) > -\infty$.

Corollary 2.5 (Well-known) Let $f(t)$ be defined on $(-\infty, \infty)$. Then $f(t)$ is operator monotone on $(-\infty, \infty) \Leftrightarrow f(t) = at + b$ ($a \geq 0$).

How about the case of finite intervals? $\tan t$ is operator monotone on $(-\pi/2, \pi/2)$.

Proposition 2.6 Let $f(t)$ be an operator monotone function on a finite interval (a, b) . Then there is a decomposition of $f(t)$ such that

$$f(t) = f_+(t) + f_-(t) \quad (a < t < b)$$

where $f_+(t)$ and $f_-(t)$ are operator monotone on (a, ∞) and $(-\infty, b)$ respectively.

3 Löwner kernels

(Bhatia and Sano) Let $f(t)$ be a C^2 function on $[0, \infty)$ such that $f(t) \geq 0$ and $f(0) = f'(0) = 0$. Then f is operator convex on $[0, \infty) \Leftrightarrow$ the Löwner kernel $K_f(t, s)$ is conditionally negative semi-definite on $[0, \infty)$, where

$$K_f(t, s) = \frac{f(t) - f(s)}{t - s} \quad (t \neq s), \quad K_f(t, t) = f'(t),$$

Proposition 3.1 Let $f(t)$ be a C^1 function on (a, ∞) . Then

(i) $f(t)$ is operator convex on $(a, \infty) \Leftrightarrow$

the Löwner kernel $K_f(t, s)$ is conditionally negative semi-definite and $\lim_{t \rightarrow \infty} \frac{f(t)}{t} > -\infty$;

(ii) $f(t)$ is operator concave on $(a, \infty) \Leftrightarrow$ the Löwner kernel $K_f(t, s)$ is conditionally positive semi-definite and $\lim_{t \rightarrow \infty} \frac{f(t)}{t} < \infty$.

In (i) the condition “ $\lim_{t \rightarrow \infty} \frac{f(t)}{t} > -\infty$ ” is indispensable: in fact, the Löwner kernel $K_f(t, s) = -(t^2 + st + s^2)$ of $f(t) = -t^3$ is conditionally negative on $(0, \infty)$, but $f(t)$ is not operator convex there.

Theorem 3.2 Let $f(t)$ be C^1 function on (a, ∞) . Then the following hold:

(i) the Löwner kernel $K_f(t, s)$ is positive semi-definite on (a, ∞) if and only if $K_f(t, s)$ is conditionally positive semi-definite on (a, ∞) , $\lim_{t \rightarrow \infty} \frac{f(t)}{t} < \infty$, and $f(\infty) > -\infty$;

- (ii) $K_f(t, s)$ is negative semi-definite on (a, ∞) if and only if $K_f(t, s)$ is conditionally negative semi-definite on (a, ∞) , $\lim_{t \rightarrow \infty} \frac{f(t)}{t} > -\infty$, and $f(\infty) < \infty$.

Corollary 3.3 Let $f(t)$ be a C^1 function on $(-\infty, b)$. Then

- (i) $f(t)$ is operator convex on $(-\infty, b)$ if and only if the Löwner kernel $K_f(t, s)$ is conditionally positive semi-definite; $\lim_{t \rightarrow -\infty} \frac{f(t)}{t} < \infty$.
- (ii) $f(t)$ is operator concave on $(-\infty, b)$ if and only if the Löwner kernel $K_f(t, s)$ is conditionally negative semi-definite, and $\lim_{t \rightarrow -\infty} \frac{f(t)}{t} > -\infty$.

Corollary 3.4 Let $f(t)$ be C^1 function on $(-\infty, b)$. Then the following hold:

- (i) the Löwner kernel $K_f(t, s)$ is positive semi-definite on $(-\infty, b)$ if and only if $K_f(t, s)$ is conditionally positive semi-definite on $(-\infty, b)$, $\lim_{t \rightarrow -\infty} \frac{f(t)}{t} < \infty$, and $f(-\infty) < \infty$;
- (ii) the Löwner kernel $K_f(t, s)$ is negative semi-definite on $(-\infty, b)$ if and only if $K_f(t, s)$ is conditionally negative semi-definite on $(-\infty, b)$, $\lim_{t \rightarrow -\infty} \frac{f(t)}{t} > -\infty$, and $f(-\infty) > -\infty$.

4 Majorization and kernel functions

Definition 4.1 Let $h(t)$ and $g(t)$ be C^1 functions on I , and suppose that $g(t)$ is increasing. Then h is said to be majorized by g and denoted by

$h \preceq g$ on I if

$h(A) \leq h(B)$ whenever $g(A) \leq g(B)$ for A, B whose spectra are both in I .

- $f(t) \preceq t$ on $I \iff f(t)$ is operator monotone on I .

Definition 4.2 Let $h(t)$ and $g(t)$ be C^1 functions on I , and suppose that $g(t)$ is increasing. Then the kernel $K_{h,g}(t, s)$ defined by

$$K_{h,g}(t, s) = \frac{h(t) - h(s)}{g(t) - g(s)} \quad (s \neq t), \quad K_{h,g}(t, t) = \frac{h'(t)}{g'(t)}.$$

is continuous and symmetric.

- A Löwner kernel $K_f(t, s)$ can be written as $K_{f,t}(t, s)$.

Proposition 4.1 The following statements are equivalent:

- (i) The kernel $K_{h,g}(t, s)$ is positive semi-definite on I .
- (ii) There is an operator monotone function φ defined on $g(I)$ such that

$$h(t) = (\varphi \circ g)(t) \quad (t \in I).$$

- (iii) $h \preceq g$ on I .

Lemma 4.2 Let $h(t)$ and $g(t)$ be positive C^1 functions on an open interval I . Suppose $h(t)g(t)$ is increasing and its range is $(0, \infty)$. Then the kernel $K_{h,hg}$ is positive semi-definite on I if and only if so is the kernel $K_{g,hg}$.

Theorem 4.3 Let $h(t)$ and $g(t)$ be positive C^1 functions defined on I . Suppose g is increasing and its range is $(0, \infty)$. If the kernel $K_{h,g}$ is positive semi-definite on I , then

for $0 \leq i \leq n$, $0 \leq j \leq m$, $1 \leq m$, $i + j + 1 \leq n + m$

$$K_{h^i g^j, h^n g^m}(t, s) = \frac{h^i(t)g^j(t) - h^i(s)g^j(s)}{h^n(t)g^m(t) - h^n(s)g^m(s)}$$

is infinitely divisible.

Moreover, if f is a (not necessarily positive) C^1 function such that the kernel $K_{f,g}(t, s)$ is positive semi-definite, then the kernel

$$K_{g,efg}(t, s)$$

is infinitely divisible.

Example 4.1 (1). For $f(t) \preceq t$ on $(0, \infty)$

$$\frac{f(t)^i t^j - f(s)^i s^j}{f(t)^n t^m - f(s)^n s^m},$$

where $0 \leq i \leq n$, $0 \leq j \leq m$, $1 \leq m$, $i + j + 1 \leq n + m$,

$1 \leq n + 1 \leq m$,

$$\frac{1}{t+s} \text{ (Cauchy kernel)}, \quad \frac{t-s}{te^{-1/t} - se^{-1/s}}$$

are all infinitely divisible kernels on $(0, \infty)$.

(2). Consider a polynomial

$p(t) := \prod_{i=1}^n (t - a_i)$ with $a_1 \geq a_2 \geq \cdots \geq a_n$. Then the kernel

$$K_{t,p(t)}(t, s) = \frac{t-s}{p(t) - p(s)}$$

is infinitely divisible on (a_1, ∞)

Theorem 4.4 Let $h(t)$ and $g(t)$ be positive C^1 functions defined on an open interval (a, b) , where $-\infty \leq a < b \leq \infty$. Suppose the range of g is $(0, \infty)$. Then the following are equivalent:

- (i) the kernel $K_{h,g}$ is conditionally negative;
- (ii) there is an operator convex function φ defined on $(0, \infty)$ such that $\varphi(g(t)) = h(t)$ for $t \in (a, b)$.
- (iii)
$$\frac{h(t) - h(a+0)}{g(t)} \preceq g(t) \quad (a < t < b)$$