

A remark on characters of unitary
representations of semi-simple Lie groups

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Suppose we have a real semi-simple Lie group G and an irreducible unitary representation $(\rho, \mathcal{H})$ of G . It is then well known that the 'trace' of the representation can be defined in a natural way to be a distribution (and hence a hyperfunction) on G , and is called the character of the representation. We denote the character by χ .

The purpose of this note is to show that, if K is a maximal compact subgroup of G , the restriction $\chi|_K$ of χ does make sense to be a hyperfunction (in fact a distribution) on K in a very natural way, and that the fact is an easy corollary of a general result in hyperfunction theory.

For the reader's convenience we first quote some of general results in the theory of hyperfunctions ([1],[2],[3] and [4]).

Let M be a real analytic manifold, $\mathcal{A}_M$ be a sheaf of germs of real analytic functions. Then we can define the sheaf of germs of hyperfunctions over M , denoted by $\mathcal{B}_M$.

$\mathcal{B}_M$ satisfies following properties;

- i) $\mathcal{B}_M$ is a left $\mathcal{A}_M$ -Module. Moreover, if we denote by $\mathcal{D}_M$ the shaf of rings of germs of linear differential operators of finite order with real analytic coefficients, then $\mathcal{B}_M$ is a left $\mathcal{D}_M$ -module.
- ii) There is a canonical $\mathcal{D}_M$ -linear injection $\alpha: \mathcal{A}_M \rightarrow \mathcal{B}_M$
- iii) $\mathcal{B}_M$ is a flabby sheaf.
- iv) $\mathcal{B}_M$ contains a sheaf of germs of distributions in the sense of L. Schwartz.

Let T^*M be a cotangent bundle of M , $S^*M = (T^*M - M)/\mathbb{R}^+$ be the sphere bundle corresponding to T^*M , called cotangential sphere bundle of M , where $\mathbb{R}^+$ is a multiplicative group of positive real numbers. Fixing a local coordinate of M , the point of T^*M is represented by (x, η) , where x is a coordinate of M , and η is a cotangent vector. In this notation, we denote by $(x, \sqrt{-1}\eta^\infty)$ the corresponding point in S^*M . We denote by $\pi: S^*M \rightarrow M$ the natural projection. We can construct the sheaf $\mathcal{C}_M$ of S^*M . $\mathcal{C}_M$ is a sheaf describing singularities of hyperfunctions. $\mathcal{C}_M$ has following properties;

- v) $\mathcal{C}_M$ is a left $\pi^{-1}\mathcal{A}_M$ -Module. Moreover, $\mathcal{C}_M$ is a left $\pi^{-1}\mathcal{B}_M$ -Module.
- vi) There is a canonical $\mathcal{B}_M$ -linear homomorphism $\beta: \mathcal{B}_M \rightarrow \pi_*\mathcal{C}_M$, such that

$$0 \rightarrow \mathcal{A}_M \xrightarrow{\alpha} \mathcal{B}_M \xrightarrow{\beta} \pi_*\mathcal{C}_M \rightarrow 0$$

is exact.

- vii) $\mathcal{C}_M$ is a flabby sheaf.

Let $u(x)$ be a hyperfunction on M . We denote by $\text{sing. supp.}_M(u)$ the smallest closed subset of M such that u is real analytic in its complementary set. The support of $\beta(u) \in \Gamma(S^*M; \mathcal{C}_M)$ is denoted by $S\text{-}S(u)$, which is a closed subset of S^*M . By vi), we have $\pi(S\text{-}S(u)) = \text{sing. supp}(u)$.

We use following two theorems. Let N be a real analytic submanifold of M . T_N^*M is a conormal bundle

of N , that is, the kernel of $N \times T^*_M M \rightarrow T^*N$. We put

$$S_N^*M = (T_N^*M - N)/R^+. \quad S_N^*M \text{ is a closed subset of } S^*M.$$

Theorem A Let $u(x)$ be a hyperfunction on M satisfying $S - S(u) \cap S_N^*M = \emptyset$. Then we can canonically define the restriction $u|_N$ of u to N . $u|_N$ is a hyperfunction on N .

Theorem B Let $P(x, D_x)$ be a linear differential operator with real analytic coefficients of order m . Let $\sigma(P)$ be a principal symbol of P , which is a function on T^*M , homogeneous of degree m . We put

$$F = \{(x, \sqrt{-1}\eta) \in S^*M; \sigma(P)(x, \eta) = 0\}$$

This is a closed set of S^*M . Then, for every hyperfunction $u(x)$ on M , we have $S - S(u) \subset F \cup S - S(Pu)$. Especially, if Pu is a real analytic function, then we have $S - S(u) \subset F$.

Now, we apply the preceding theory to the theory of unitary representation.

Let G be a real semi-simple Lie group, $\mathfrak{g}$ be a Lie algebra of G . Let $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be a Cartan decomposition of $\mathfrak{g}$, K be a compact subgroup of G corresponding to the compact Lie subalgebra $\mathfrak{k}$. It is well known that the Killing form is positive definite on $\mathfrak{p}$ and negative definite on $\mathfrak{k}$. Let C be the Casimir operator of G , which is a bi-invariant linear differential operator on G of order 2. Let $(U, \mathfrak{h})$ be an irreducible unitary

representation of G . χ be its character. By the result of Harish-Chandra, χ is a distribution (and hence a hyperfunction) on G , satisfying;

- a) $\chi(gxg^{-1}) = \chi(x)$ for any $(g,x) \in G \times G$
- b) $C\chi = a\chi$ for some $a \in \mathbb{C}$

Harish-Chandra has also proved that unitary representation is uniquely determined by its character.

Our purpose is to show the following theorem;

Theorem Let $\chi(x)$ be a hyperfunction on G satisfying;

- (b) $C\chi(x) = a\chi(x)$ for some $a \in \mathbb{C}$. Then we can canonically define the restriction $\chi|_K$ of χ to the maximal compact subgroup K of G , which is a hyperfunction on K .

Proof

Let e be the neutral element of G . Then $T^*G_e = \mathfrak{g}^*$, $T_K^*G_e = \mathcal{P}^* \subset \mathfrak{g}^*$. Let $\sigma(C)$ be a principal symbol of C , which is bi-invariant. At e , $\sigma(C)$ can be considered as a quadratic form on $\mathfrak{g}^*$, which coincides with the dual form of the Killing form. Therefore $\sigma(C)$ is positive definite on $\mathcal{P}^*$. It follows that $\sigma(C)$ never vanishes on $T_K^*G_e$. Since $\sigma(C)$ is bi-invariant, $\sigma(C)$ vanishes nowhere on T_K^*G . By the theorem B, $S - S(\chi) \wedge S_K^*G = \phi$. Using the theorem A, $\chi|_K$ can be defined. q. e. d.

Remark.

Recently, L. Hörmander has proved an analogue of the theorem A and the theorem B in the category of distribution ([5]). Therefore, in Theorem, if χ is a distribution satisfying (b), $\chi|_K$ is also a distribution on K .

Reference

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