

# MODULAR FORMS WITH POLES ON HYPERPLANE ARRANGEMENTS

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This is a brief summary of a talk at the 2024 RIMS conference “Research on Automorphic Forms”, based on the paper [4] which is joint work with Haowu Wang. It is a continuation of our earlier work [3] which describes the structure of the algebra of modular forms for certain orthogonal groups of signature  $(2, n)$  related to root systems. Here, we consider algebras of meromorphic modular forms with certain particularly mild singularities.

## 1. ORTHOGONAL MODULAR FORMS

Let  $(L, \langle -, - \rangle)$  be an even integral lattice (so  $L$  is a free abelian group of finite rank, and  $\langle -, - \rangle$  is a nondegenerate, symmetric bilinear form on it) and assume further that  $L$  has signature  $(2, \ell)$  for some  $\ell \geq 3$ . Let  $L'$  be the dual lattice

$$L' = \{r \in L_{\mathbb{Q}} = L \otimes \mathbb{Q} : \langle r, b \rangle \in \mathbb{Z} \text{ for all } b \in L\}.$$

Fix an orientation on  $L_{\mathbb{R}} := L \otimes \mathbb{R}$ , and define the set

$$\mathcal{A} := \left\{ z \in L \otimes \mathbb{C} : \langle z, z \rangle = 0, \langle z, \bar{z} \rangle > 0, \{\operatorname{re}(z), \operatorname{im}(z)\} \text{ is oriented} \right\}.$$

Then  $\mathcal{D} := \mathcal{A}/\mathbb{C}^{\times}$  is an irreducible Hermitian symmetric space of Cartan type IV.

Let  $\Gamma \leq \mathrm{O}^+(L)$  be the integral spinor kernel (equivalently, the subgroup of  $\mathrm{O}(L)$  that preserves  $\mathcal{D}$  by multiplication) or a finite-index subgroup in it. A modular form of weight  $k$  is a holomorphic function

$$f : \mathcal{A} \longrightarrow \mathbb{C}$$

that satisfies

$$f(\gamma z) = f(z) \text{ for all } \gamma \in \Gamma \quad \text{and} \quad f(tz) = t^{-k} f(z) \text{ for all } t \in \mathbb{C}^{\times}.$$

No growth condition has to be imposed since we assume  $L$  has rank at least 5.

We will always assume that the lattice  $L$  splits in the form

$$L = U \oplus U \oplus K(-1),$$

where  $U$  is the hyperbolic plane ( $\mathbb{Z}^2$  with Gram matrix  $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ ), where  $K$  is positive-definite, and where  $K(-1)$  means that its bilinear form is multiplied by  $(-1)$ . In this case, one can represent modular forms by their Fourier–Jacobi expansions:

$$f = \sum_{n=0}^{\infty} \phi_n(\tau, z) e^{2\pi i n w},$$

where  $\tau, w \in \mathbb{H} = \{x + iy \in \mathbb{C} : y > 0\}$  and  $z \in K \otimes \mathbb{C}$ , and where each  $\phi_n$  is a Jacobi form of index  $n$  attached to the lattice  $K$ . See [1] for details.

## 2. HYPERPLANE ARRANGEMENTS

We continue to use the setup and notation of the previous section.

**Definition 1.** A **rational quadratic divisor**  $r^\perp$  is a divisor of the form

$$\{\mathbb{C}z : \langle z, r \rangle = 0\} \subseteq \mathcal{D}$$

for some  $r \in L'$  with  $\langle r, r \rangle > 0$ .

**Definition 2.** A **hyperplane arrangement**  $\mathcal{H}$  for  $\Gamma$  is a collection of rational quadratic divisors that satisfies:

- (1)  $\mathcal{H}$  is a finite union of  $\Gamma$ -orbits;
- and every nonempty intersection  $\mathcal{I} = r_1^\perp \cap \dots \cap r_n^\perp$  in  $\mathcal{H}$  satisfies the following conditions:
- (2)  $\dim \mathcal{I} > 1$ ;
- (3) if  $\dim \mathcal{I} = 2$ , then  $\mathcal{I}$  is anisotropic;
- (4) if  $\dim \mathcal{I} = 3$ , then  $\mathcal{I}$  contains no isotropic planes.

Here the dimension is taken in the naive sense (viewing  $\mathcal{I}$  as a linear subspace of  $L \otimes \mathbb{C}$ ). Items (2)-(4) are conditions “in codimension one”. For example, if  $\mathcal{H}$  is empty then these conditions are equivalent to requiring that the Baily–Borel boundary has no components of codimension one. The importance of this definition is due to Looijenga’s [2] construction of a toroidal compactification of the arrangement complement  $(\mathcal{D} \setminus \mathcal{H})/\Gamma$  generalizing the Baily–Borel compactification.

**Theorem 3.** *Let  $\mathcal{H}$  be a hyperplane arrangement. For any  $k \in \mathbb{Z}$ , let  $M_k^1(\Gamma; \mathcal{H})$  be the space of meromorphic modular forms of weight  $k$  whose poles are supported on  $\mathcal{H}$ . Then*

$$M^1(\Gamma; \mathcal{H}) = \bigoplus_{k \in \mathbb{Z}} M_k^1(\Gamma; \mathcal{H})$$

*is a finitely-generated graded ring, and any  $f \in M_k^1(\Gamma; \mathcal{H})$  has only poles of order at most  $k$ .*

*(In particular,  $M_k^1(\Gamma; \mathcal{H}) = \{0\}$  for  $k < 0$ , and  $M_0^1(\Gamma; \mathcal{H}) = \mathbb{C}$ .)*

## 3. MAIN RESULTS

In our paper [4] we consider lattices  $L = U \oplus U \oplus K(-1)$ , where  $K$  is a root lattice of ADE type of the form

$$K = A \oplus R = \left( \bigoplus_{j=1}^t A_{m_j} \right) \oplus R, \quad R \in \{A_m, D_m, E_6, E_7\}$$

subject to the constraints

$$\begin{aligned}
 (1) \quad & \sum_{j=1}^t (m_j + 1) + m \leq 11 \quad \text{if } R = D_m; \\
 (2) \quad & \sum_{j=1}^t (m_j + 1) + (m + 1) \leq 11 \quad \text{if } R = A_m; \\
 (3) \quad & \sum_{j=1}^t (m_j + 1) + (m + 2) \leq 11 \quad \text{if } R = E_m.
 \end{aligned}$$

(These constraints are necessary for our construction of generators of the algebra of modular forms.) For any such decomposition  $L = U \oplus U \oplus A(-1) \oplus R(-1)$  (which need not be unique), we construct a hyperplane arrangement  $\mathcal{H}_L = \mathcal{H}_A \cup \mathcal{H}_R$  for the discriminant kernel

$$\Gamma := \ker \left( \mathrm{O}^+(L) \longrightarrow \mathrm{O}(L'/L) \right)$$

as follows:

(i) If  $A = \bigoplus_{j=1}^t A_{m_j}$ , then  $\mathcal{H}_A = \bigcup_{j=1}^t \mathcal{H}_{A_{m_j}}$ , where  $\mathcal{H}_{A_{m_j}}$  is the union of all rational quadratic divisors  $r_j^\perp$ ,  $r_j \in A_{m_j}$  of the smallest possible norm,  $\langle r_j, r_j \rangle = \frac{m_j}{m_j+1}$ .

(ii)  $\mathcal{H}_R$  consists of a single orbit if  $R \in \{A_8, A_9, A_{10}, D_9, D_{10}, D_{11}\}$ , and is empty otherwise. In the nonempty case, there is a unique coset  $\gamma \in R'/R$  that contains no short vectors, i.e.

$$\delta_\gamma := \min \{ \langle x, x \rangle : x \in \gamma \} > 2,$$

and we define  $\mathcal{H}_R = \bigcup r^\perp$  where the union runs over  $r \in 2U \oplus R(-1)'$  with  $r|_{R'} \in \gamma$  and  $\langle r, r \rangle = \delta_\gamma - 2$ . (For example, when  $L = U \oplus U \oplus A_8(-1)$  with  $A = \{0\}$  and  $R = A_8$ , the arrangement  $\mathcal{H}_R$  is the Heegner divisor of discriminant  $1/9$ .)

The main result of [4] is:

**Theorem 4** (Theorem 1.2 of [4]). *Let  $K = A \oplus R$  be one of the 147 lattices satisfying Equations 1. Then  $\mathcal{H} = \mathcal{H}_A \cup \mathcal{H}_R$  is a hyperplane arrangement in the sense of Definition 2 and the algebra  $M_*^!(\Gamma; \mathcal{H})$  is freely generated by  $3 + \mathrm{rank}(K)$  meromorphic modular forms.*

In all cases, the generators of  $M_*^!(\Gamma; \mathcal{H})$  naturally fall into three classes depending on their leading Fourier–Jacobi coefficient:

(i) Eisenstein type: the leading Fourier–Jacobi coefficient is the classical Eisenstein series  $E_4$  or  $E_6$  and it occurs in index zero.

(ii) Abelian type: the leading Fourier–Jacobi coefficient is an abelian (or multiply periodic) function. These are meromorphic generators associated to the component  $A$ .

(iii) Jacobi type: the leading Fourier–Jacobi coefficients are related to the Wirthmüller generators [5] of the algebra of Weyl-invariant weak Jacobi forms attached to the root lattice  $R$ .

For the precise definition of the generators and their construction (which in part is by cases) we refer to [4].

#### 4. TWO EXAMPLES

We illustrate Theorem 4 with two examples, taking  $A = \{0\}$  and  $R$  to be either the  $A_n$  root lattice ( $n \leq 10$ ) or the  $D_n$  root lattice ( $n \leq 11$ ). In the following tables, the generators colored black can be chosen to be holomorphic, while the generators colored red cannot.

**Example 5.** The  $A_n$ -tower:  $L = U \oplus U \oplus A_n(-1)$  for  $n \leq 10$ . We have Eisenstein-type generators of weights 4 and 6 and Jacobi-type generators of weights

$$12, 10, 9, 8, \dots, 12 - (n + 1).$$

| $n$ | weights                  | $n$ | weights                                 |
|-----|--------------------------|-----|-----------------------------------------|
| 1   | 4, 6, 10, 12             | 6   | 4, 5, 6, 6, 7, 8, 9, 10, 12             |
| 2   | 4, 6, 9, 10, 12          | 7   | 4, 4, 5, 6, 6, 7, 8, 9, 10, 12          |
| 3   | 4, 6, 8, 9, 10, 12       | 8   | 3, 4, 4, 5, 6, 6, 7, 8, 9, 10, 12       |
| 4   | 4, 6, 7, 8, 9, 10, 12    | 9   | 2, 3, 4, 4, 5, 6, 6, 7, 8, 9, 10, 12    |
| 5   | 4, 6, 6, 7, 8, 9, 10, 12 | 10  | 1, 2, 3, 4, 4, 5, 6, 6, 7, 8, 9, 10, 12 |

**Example 6.** The  $D_n$ -tower. Let  $L = U \oplus U \oplus D_n(-1)$  for  $4 \leq n \leq 11$ . We have Eisenstein-type generators of weights 4 and 6 and Jacobi-type generators of weights

$$12 - n, 8, 10, 12 \text{ and } 18, 16, 14, \dots, 26 - 2n.$$

| $n$ | weights                            | $n$ | weights                                         |
|-----|------------------------------------|-----|-------------------------------------------------|
| 4   | 4, 6, 8, 8, 10, 12, 18             | 8   | 4, 4, 6, 8, 10, 10, 12, 12, 14, 16, 18          |
| 5   | 4, 6, 7, 8, 10, 12, 16, 18         | 9   | 3, 4, 6, 8, 8, 10, 10, 12, 12, 14, 16, 18       |
| 6   | 4, 6, 6, 8, 10, 12, 14, 16, 18     | 10  | 2, 4, 6, 6, 8, 8, 10, 10, 12, 12, 14, 16, 18    |
| 7   | 4, 5, 6, 8, 10, 12, 12, 14, 16, 18 | 11  | 1, 4, 4, 6, 6, 8, 8, 10, 10, 12, 12, 14, 16, 18 |

#### REFERENCES

- [1] Valery Gritsenko. Fourier-Jacobi functions in  $n$  variables. *Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI)*, 168(Anal. Teor. Chisel i Teor. Funktsii. 9):32–44, 187–188, 1988.
- [2] Eduard Looijenga. Compactifications defined by arrangements. II. Locally symmetric varieties of type IV. *Duke Math. J.*, 119(3):527–588, 2003.
- [3] Haowu Wang and Brandon Williams. On some free algebras of orthogonal modular forms. *Adv. Math.*, 373:107332, 22, 2020.
- [4] Haowu Wang and Brandon Williams. Modular forms with poles on hyperplane arrangements. *Algebr. Geom.*, 11(4):506–568, 2024.
- [5] Klaus Wirthmüller. Root systems and Jacobi forms. *Compositio Math.*, 82(3): 293–354, 1992.