

# UNIQUENESS AND EXISTENCE FOR SPIRAL CRYSTAL GROWTH

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In [2] Ohtsuka studied on a crystal growth of spirals and proposed us to use a level set method. Since the conventional level set method (see [1]) could not express spiral curves having the orientation, he modified the conventional method by using a sheet structure function.

Let  $\Omega$  be a bounded domain of  $\mathbb{R}^2$  with the smooth boundary and let  $B_{\rho_j}(a_j)$  be  $N$  screw dislocations in  $\Omega$ , which are disks small enough with  $a_j \in \Omega$  and  $\rho_j > 0$  so that  $\overline{B_{\rho_j}(a_j)} \subset \Omega$  and  $B_{\rho_j}(a_j) \cap B_{\rho_k}(a_k) = \emptyset$  if  $j \neq k$ . We denote

$$W = \Omega \setminus \left( \bigcup_{j=1}^N \overline{B_{\rho_j}(a_j)} \right).$$

The level set equation for his spiral crystal growth on  $W$  is

$$(1) \quad u_t - |\nabla(u - \theta)| \left\{ \operatorname{div} \frac{\nabla(u - \theta)}{|\nabla(u - \theta)|} + C \right\} = 0 \quad \text{in } W,$$

with the boundary condition of Neumann type

$$(2) \quad \langle \nu, \nabla(u - \theta) \rangle = 0 \quad \text{on } \partial W.$$

Here  $C$  is a constant,  $\nu$  is the unit normal vector of  $\partial W$  and  $\theta(x) = \sum_{j=1}^N m_j \arg(x - a_j)$  for  $m_j \in \mathbb{Z} \setminus \{0\}$ . We note that  $\theta(x)$  is a multi-valued function, but  $\nabla\theta$  is single-valued. When  $\Gamma_t$  is the spiral curve, it must be defined

$$\Gamma_t = \{x \in \overline{W} : u(t, x) - \theta(x) \equiv 0 \pmod{2\pi m\mathbb{Z}}\}.$$

Here  $m$  is the greatest common divisor of  $|m_j|$ . Ohtsuka proved the following results.

COMPARISON THEOREM. Let  $u$  and  $v$  be a viscosity subsolution and a supersolution of (1) and (2), respectively. If  $u^*(0, \cdot) \leq v_*(0, \cdot)$ , then we have  $u^*(t, x) \leq v_*(t, x)$  for any  $t > 0$ .

EXISTENCE THEOREM. For any given  $u_0 \in C(\overline{W})$  there exists a unique global-in-time viscosity solution  $u \in C([0, \infty) \times \overline{W})$  of (1) and (2) with initial data  $u(0, \cdot) = u_0$ .

This note is a short remark for the Ohtsuka's theory, that is, we would like to consider the uniqueness of  $\Gamma_t$ . It means that, for a given initial spiral  $\Gamma_0$ , we choose  $u_0$  an initial function satisfying

$$\Gamma_0 = \{x \in \overline{W} : u_0(x) - \theta(x) \equiv 0 \pmod{2\pi mZ}\},$$

the Existence Theorem says that there exists a unique solution  $u$ , but we can choose  $v_0$  an another initial function satisfying

$$\Gamma_0 = \{x \in \overline{W} : v_0(x) - \theta(x) \equiv 0 \pmod{2\pi mZ}\}$$

and we get a unique solution  $v$ . Our question is

$$\{x \in \overline{W} : u(t, x) - \theta(x) \equiv 0 \pmod{2\pi mZ}\}$$

and

$$\{x \in \overline{W} : v(t, x) - \theta(x) \equiv 0 \pmod{2\pi mZ}\}$$

are tracing the same spiral curve?

The paper [1] solved this uniqueness problem for the case of closed curves. The key step is to construct the order changing function satisfying  $u_0(x) \leq G(v_0(x))$ , when, generally,  $u_0$  and  $v_0$  are not maked order each other. Since  $G$  is nondecreasing, if  $v(t, x)$  is a viscosity supersolution, then  $G(v(t, x))$  is also a viscosity supersolution. By using the Comparison Theorem we see that  $u(t, x) \leq G(v(t, x))$ , which leads us to compair the level sets of  $u$  and  $v$ .

We try to extend this key idea to the spiral case. Applying the Ohtsuka's method in [2] we first introduce the covering space of  $\overline{W}$  like

$$\mathfrak{X} = \left\{ (x, \xi) \in \overline{W} \times \mathbb{R}^N : \xi = (\xi_1, \dots, \xi_N), (\cos \xi_j, \sin \xi_j) = \frac{x - a_j}{|x - a_j|} \right\}$$

and assume that

$$\left\{ (x, \xi) \in \mathfrak{X} : u_0(x) - \sum_{j=1}^N m_j \xi_j > 0 \right\} = \left\{ (x, \xi) \in \mathfrak{X} : v_0(x) - \sum_{j=1}^N m_j \xi_j > 0 \right\}.$$

We construct an order changing function  $G$  with

$$u_0(x) - \sum_{j=1}^N m_j \xi_j \leq G \left( v_0(x) - \sum_{j=1}^N m_j \xi_j \right) \quad \text{for } (x, \xi) \in \mathfrak{X}.$$

The important properties for  $G$  are nondecreasing and satisfying the periodical condition

(#)  $G(s) = G(s + 2\pi m_j) - 2\pi m_j$ . Basically,  $G$  is modified from

$$G_1(s) = \sup \{ (\tilde{u}_0(y, \eta))_+ : (y, \eta) \in \mathfrak{X}, \tilde{v}_0(y, \eta) \leq s \}.$$

Here  $\tilde{u}_0(y, \eta) = u_0(y) - \sum_{j=1}^N m_j \eta_j$ ,  $\tilde{v}_0(y, \eta) = v_0(y) - \sum_{j=1}^N m_j \eta_j$  and  $(a)_+ = \max\{a, 0\}$ .

Finally, we obtain

**INVARIANCE LEMMA.** *Let  $v$  be a viscosity supersolution with initial data  $v(0, \cdot) = v_0$  and define*

$$(3) \quad w(t, x) = G(v(t, x) - \theta(x)) + \theta(x)$$

*in the sense of some meaning in the covering space (because  $\theta(x)$  is multi-valued). Then we have  $w$  is a viscosity supersolution with  $w(0, \cdot) = w_0$ .*

The meaning of the definition (3) is the following: We denote that

$$\mathfrak{L} = \bigcup_{j=1}^N \mathfrak{L}_j, \quad \mathfrak{L}_j = \left\{ x \in \overline{W} : \frac{x - a_j}{|x - a_j|} = (-1, 0) \right\}$$

and  $\Theta_j(x) = \text{Arg}(x - a_j)$  is the principal value of the argument which is a function from  $\overline{W} \setminus \mathfrak{L}_j$  to  $(-\pi, \pi)$ . Then,  $\Theta(x) = \sum_{j=1}^N m_j \Theta_j(x)$  is a single-valued function with a jump discontinuity on  $\mathfrak{L}$ . However, since  $G$  is periodic like (#), we see that

$$g(x) = \begin{cases} G(f(x) - \Theta(x)) + \Theta(x) & \text{if } x \in \overline{W} \setminus \mathfrak{L}, \\ \lim_{\mathfrak{L} \ni y \rightarrow x} \{G(f(y) - \Theta(y)) + \Theta(y)\} & \text{if } x \in \mathfrak{L} \end{cases}$$

is continuous on  $\mathcal{L}$ .

We must discuss here about the construction of an initial function  $u_0$  for a given  $\Gamma_0$ , which gives us the existence result on the growth of  $\Gamma_t$ . The author hopes it will be stated in a forthcoming paper.

This research was started by Maki Nakagawa as the master's thesis [3] in a simple case, which is supervised by the author. After that, Takeshi Ohtsuka and the author have revised and completed it.

#### REFERENCES

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