

THE LIMITING ABSORPTION PRINCIPLE FOR ELASTIC WAVE PROPAGATION PROBLEMS IN PERTURBED STRATIFIED MEDIA $\mathbf{R}^3$

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ABSTRACT. We consider the self-adjoint operator governing the propagation of elastic waves in perturbed stratified media $\mathbf{R}^3$ with free boundary-interface conditions. In this paper we establish the limiting absorption principle for this self-adjoint operator in appropriate Hilbert space. The proof of the limiting absorption principle is based on the division theorem which is proved by means of eigenfunction expansions for the self-adjoint operator governing the propagation of elastic waves in unperturbed stratified media $\mathbf{R}^3$.

1. Introduction

In this paper we consider propagation problems of elastic waves in perturbed stratified media $\mathbf{R}^3$ with free boundary-interface conditions.

The object of this work is to establish a limiting absorption principle for the self-adjoint operator governing the propagation of elastic waves. The limiting absorption principle implies some significant spectral properties of the self-adjoint operator and gives a method of selecting steady-state solutions for the propagation problems of elastic waves.

The limiting absorption principle for acoustic wave propagation problems is studied by several authors. For example Ben-Artzi and Dermenjian and Guillot [2], Dermenjian and Guillot [3], [4], Weder [13] for stratified media, Phillips [9], Wilcox [14] for exterior domain.

Concerning elastic wave propagation problems, Dermenjian and Guillot [5] proved the limiting absorption principle in perturbed half space $\mathbf{R}_+^3$ by using so-called division theorem which is one of their main results. In this paper we shall prove the limiting absorption principle for elastic wave propagation problems in perturbed stratified media $\mathbf{R}^3$ using a corresponding division theorem. We prove the division theorem by using the representation of solutions by Lopatinski analysis and the eigenfunction expansion theorem established by [11]. Dermenjian and Guillot used the representation of solutions due to Dunford and Schwartz [6].

We start with the mathematical formulation of the elastic wave propagation problem in perturbed stratified media $\mathbf{R}^3$.

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Let Ω be an exterior domain in $\mathbf{R}^3 = \{x = (x', x_3) = (x_1, x_2, x_3); x_i \in \mathbf{R}\}$ whose boundary $\partial\Omega$ is compact. $\lambda(x)$ and $\mu(x)$ denote Lamé functions in Ω , and $\rho(x)$ denotes a density function in Ω . We assume that

$$(1.1) \quad 0 < m \leq \lambda(x) (\text{resp. } \mu(x), \rho(x)) \leq M \quad \text{for a.e. } x \in \bar{\Omega},$$

where

$$\lambda(x) = \begin{cases} \lambda_1(x), \\ \lambda_2(x), \end{cases} \quad \mu(x) = \begin{cases} \mu_1(x), \\ \mu_2(x), \end{cases} \quad \rho(x) = \begin{cases} \rho_1(x) & \text{for } x \in \Omega \cap \mathbf{R}_-^3, \\ \rho_2(x) & \text{for } x \in \Omega \cap \mathbf{R}_+^3, \end{cases}$$

and

$$(1.2) \quad (\lambda(x), \mu(x), \rho(x)) = \begin{cases} (\lambda_1, \mu_1, \rho_1) & \text{for } x \in \mathbf{R}_-^3, |x| > L, \\ (\lambda_2, \mu_2, \rho_2) & \text{for } x \in \mathbf{R}_+^3, |x| > L. \end{cases}$$

Here $\mathbf{R}_-^3 = \{x \in \mathbf{R}^3, x_3 < 0\}$, $\mathbf{R}_+^3 = \{x \in \mathbf{R}^3, x_3 > 0\}$, L is a fixed large real number, $\lambda_1, \lambda_2, \mu_1, \mu_2$ are certain quantities called the Lamé constants and ρ_1, ρ_2 are densities (cf. Figure 1).

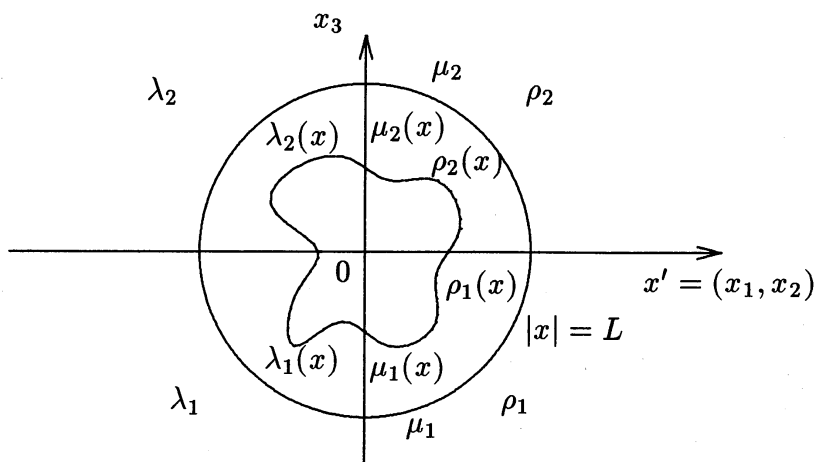


Figure 1 Perturbed Stratified Medium $\mathbf{R}^3$

Let $u(t, x) = {}^t(u_1(t, x), u_2(t, x), u_3(t, x)) \in \mathbf{R}^3$ be the displacement vector at time t and position x . The propagation problem of elastic waves in the perturbed stratified medium $\mathbf{R}^3$ is formulated as the following mixed problem:

$$(1.3) \quad \frac{\partial^2 u_k}{\partial t^2}(t, x) - \frac{1}{\rho(x)} \sum_{j=1}^3 \frac{\partial}{\partial x_j} \sigma_{kj} u(t, x) = 0, \quad x \in \Omega,$$

$$(1.4) \quad u(t, x)|_{\Omega \cap \{x_3 = -0\}} = u(t, x)|_{\Omega \cap \{x_3 = +0\}},$$

$$(1.5) \quad \sigma_{k3}(u(t, x))|_{\Omega \cap \{x_3 = -0\}} = \sigma_{k3}(u(t, x))|_{\Omega \cap \{x_3 = +0\}},$$

$$(1.6) \quad \sum_{j=1}^3 \sigma_{kj}(u(t, x)) \nu_j|_{\partial\Omega} = 0,$$

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$$(1.7) \quad u(0, x) = f(x), \quad \frac{\partial u}{\partial t}(0, x) = g(x),$$

where

$$(1.8) \quad \sigma_{kj}(u) = \lambda(\cdot)(\nabla \cdot u)\delta_{kj} + 2\mu(\cdot)\varepsilon_{kj}(u),$$

are symmetric stress tensors,

$$(1.9) \quad \varepsilon_{kj}(u) = \frac{1}{2} \left(\frac{\partial u_k}{\partial x_j} + \frac{\partial u_j}{\partial x_k} \right),$$

are symmetric strain tensors, and $\nu = (\nu_1, \nu_2, \nu_3)$ denotes the exterior normal at point $x \in \partial\Omega$. (1.4) and (1.5) are called free interface conditions, (1.6) is called a free boundary condition, and (1.7) is called an initial condition. Here 'free' means Neumann type, and these free interface and boundary conditions are appeared in practical situations.

Solutions to (1.3)-(1.7) with finite energy are associated with a Hilbert space and a self-adjoint operator as follows. Let

$$(1.10) \quad (\mathcal{A}u)_k = -\frac{1}{\rho(\cdot)} \sum_{j=1}^3 \frac{\partial}{\partial x_j} \sigma_{kj}(u).$$

Here $(\mathcal{A}u)_k$ has another expression

$$(1.11) \quad (\mathcal{A}u)_k = -\frac{1}{\rho(\cdot)} \sum_{j=1}^3 \frac{\partial}{\partial x_j} (c_{kjlh}(\cdot)\varepsilon_{lh}(u)),$$

where c_{kjlh} ($k, j, l, h = 1, 2, 3$) are called the stress-strain tensors given by

$$(1.12) \quad c_{kjlh}(\cdot) = \lambda(\cdot)\delta_{kj}\delta_{lh} + \mu(\cdot)\delta_{kh}\delta_{jl},$$

with the properties

$$(1.13) \quad c_{kjlh}(\cdot) = c_{jklh}(\cdot) = c_{lhkj}(\cdot)$$

and δ_{kj} is the Kronecker delta. By the condition (1.1), Lamé functions satisfy the conditions

$$(1.14) \quad 3\lambda(x) + 2\mu(x) > 0, \quad \mu(x) > 0, \quad \text{for a.e. } x \in \bar{\Omega},$$

so we have from Korn's inequality the following stability condition:

$$(1.15) \quad \sum_{k,j,l,h} c_{kjlh}(\cdot) s_{lh} \overline{s_{kj}} \geq c \sum_{k,j} |s_{kj}|^2, \quad c > 0$$

for all complex symmetric matrices (s_{kj}) , $s_{kj} = s_{jk} \in \mathbb{C}$ (cf. [8], [10]).

The Sobolev spaces on Ω are defined by

$$(1.16) \quad H^m(\Omega, \mathbb{C}^3) = \{u \in \mathbb{C}^3; D^\alpha u \in L^2(\Omega, \mathbb{C}^3), \text{ for } |\alpha| \leq m\},$$

where m is a non-negative integer and the multi-index notation is used for derivatives. $H^m(\Omega, \mathbb{C}^3)$ is a Hilbert space with inner product

$$(1.17) \quad (u, v)_m = \int_{\Omega} \sum_{|\alpha| \leq m} D^\alpha u(x) \cdot \overline{D^\alpha v(x)} dx,$$

where $u \cdot \bar{v}$ denotes the usual scalar product in $\mathbb{C}^3$: $u \cdot \bar{v} = \sum_{k=1}^3 u_k \bar{v}_k$.

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Definition 1.1. $u \in H^1(\Omega, \mathbf{C}^3) \cap \{\mathcal{A}u \in L^2(\Omega, \mathbf{C}^3)\}$ is said to satisfy the generalized free boundary-interface condition if and only if one has

$$(1.18) \quad \int_{\Omega} (\mathcal{A}u)_k \overline{v_k} \rho(x) dx - \int_{\Omega} (\lambda(x)(\nabla \cdot u)(\nabla \cdot \bar{v}) + 2\mu(x)\varepsilon_{kj}(u)\varepsilon_{kj}(\bar{v})) dx = 0$$

for every $v \in H^1(\Omega, \mathbf{C}^3)$.

We introduce the Hilbert space

$$(1.19) \quad \mathcal{H} = L^2(\Omega, \mathbf{C}^3, \rho(x)dx),$$

with inner product

$$(1.20) \quad (u, v)_{\mathcal{H}} = \int_{\Omega} u \cdot v \rho(x) dx.$$

Theorem 1.2. The following operator A in $\mathcal{H}$ with domain:

$$(1.21) \quad D(A) = \{u \in H^1(\Omega, \mathbf{C}^3) \cap \{\mathcal{A}u \in L^2(\Omega, \mathbf{C}^3)\}; u \text{ satisfies the generalized free boundary-interface condition (1.18)}\},$$

and action defined by

$$(1.22) \quad Au = \mathcal{A}u, \quad u \in D(A)$$

is a non-negative self-adjoint operator.

For a proof of Theorem 1.2 see [11].

Every $u \in D(A)$ satisfies the free interface conditions (1.4) and (1.5), and the free boundary condition (1.6), so the mixed problem (1.3)-(1.7) may be reformulated as the problem of finding a function $u : \mathbf{R} \rightarrow \mathcal{H}$ such that

$$(1.23) \quad \frac{d^2 u}{dt^2} + Au = 0, \quad \text{for } \forall t \in \mathbf{R},$$

$$(1.24) \quad u(0) = f, \quad \frac{du}{dt}(0) = g.$$

The operator A is non-negative and the spectral theory for self-adjoint operators implies that (1.23) and (1.24) has a (generalized) solution given by

$$(1.25) \quad u(t) = \left(\cos t A^{\frac{1}{2}}\right) f + \left(A^{-\frac{1}{2}} \sin t A^{\frac{1}{2}}\right) g, \quad t \in \mathbf{R}.$$

Let $E(u, K, t)$ be the restriction of the energy of u to a measurable subset K of Ω :

$$(1.26) \quad E(u, K, t) = \frac{1}{2} \left(\sum_{k=1}^3 \int_K \left| \frac{\partial u_k}{\partial t} \right|^2 \rho(x) dx \right)$$

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$$\begin{aligned}
& + \sum_{k,j=1}^3 \int_K (\lambda(x) |\nabla \cdot u|^2 + 2\mu(x) |\varepsilon_{kj}(u)|^2) dx \Big) \\
& = \left\| \frac{du}{dt} \right\|_{\mathcal{H}}^2 + \|A^{\frac{1}{2}} u\|^2
\end{aligned}$$

If $f \in D(A^{\frac{1}{2}})$, $g \in \mathcal{H}$, then $u \in D(A^{\frac{1}{2}})$, $\frac{du}{dt} \in \mathcal{H}$ and $E(u, K, t) < \infty$. In this case $u(t)$ is called a solution with finite energy.

In order to state our main theorem, we introduce several function spaces.

Let s_1, s_2 be two real numbers. Let $L^{2;s_1, s_2}(\Omega, \mathbf{C}^3)$ be the space of all measurable $\mathbf{C}^3$ valued functions on Ω defined by

$$(1.27) \quad L^{2;s_1, s_2}(\Omega, \mathbf{C}^3) = \{u; (1 + x_1^2 + x_2^2)^{\frac{s_1}{2}} (1 + x_3^2)^{\frac{s_2}{2}} u(x) \in L^2(\Omega, \mathbf{C}^3)\},$$

with the norm

$$(1.28) \quad \|u\|_{0;s_1, s_2}^2 = \int_{\Omega} (1 + x_1^2 + x_2^2)^{s_1} (1 + x_3^2)^{s_2} u(x) \cdot \overline{u(x)} dx.$$

We consider weighted Sobolev spaces $H^{m;s_1, s_2}(\Omega, \mathbf{C}^3)$ defined for any integer $m \geq 0$ by

$$(1.29) \quad H^{m;s_1, s_2}(\Omega, \mathbf{C}^3) = \{u; D^{\alpha} u \in L^{2;s_1, s_2}(\Omega, \mathbf{C}^3), |\alpha| \leq m\},$$

with the norm

$$(1.30) \quad \|u\|_{m;s_1, s_2}^2 = \sum_{|\alpha| \leq m} \|D^{\alpha} u\|_{0;s_1, s_2}^2.$$

We introduce

$$(1.31) \quad H^{1;s_1, s_2}(\Omega, \mathcal{A}, \mathbf{C}^3) = \{u \in H^{1;s_1, s_2}(\Omega, \mathbf{C}^3); \mathcal{A}u \in L^{2;s_1, s_2}(\Omega, \mathbf{C}^3)\},$$

with the norm

$$(1.32) \quad \|u\|_{\mathcal{A}, s_1, s_2}^2 = \|u\|_{1, s_1, s_2}^2 + \|\mathcal{A}u\|_{0, s_1, s_2}^2.$$

Finally let

$$(1.33) \quad H^{1;-s_1, -s_2}(\Omega, A, \mathbf{C}^3) = \{u \in H^{1;-s_1, -s_2}(\Omega, \mathcal{A}, \mathbf{C}^3);$$

u satisfy the generalized free boundary-interface condition (1.18)\}.

Let $R(z)$ be the resolvent of A . Then the limiting absorption principle which is our main result can be stated as follows:

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Main Theorem. Suppose $s_1 > \frac{1}{2}$ and $s_2 > \frac{1}{2}$. And suppose $\rho_1 = \rho_2$. If $\omega(> 0)$ is not an eigenvalue for A , then the following two limits exist in the uniform operator topology of $B(L^{2;s_1,s_2}(\Omega, \mathbb{C}^3), H^{1;-s_1,-s_2}(\Omega, A, \mathbb{C}^3))$:

$$(1.34) \quad R^\pm(\omega) = \lim_{\substack{z \rightarrow \omega \\ \pm \operatorname{Im} z > 0}} R(z).$$

The remainder of this paper is organized as follows. In Section 2, we consider the plane stratified media $\mathbf{R}^3$ with the planer interface $x_3 = 0$, which is defined by

$$(\lambda(x_3), \mu(x_3), \rho(x_3)) = \begin{cases} (\lambda_1, \mu_1, \rho_1) & \text{for } x_3 < 0, \\ (\lambda_2, \mu_2, \rho_2) & \text{for } x_3 > 0. \end{cases}$$

The self-adjoint operator A_0 governing the propagation of elastic waves in this unperturbed media is defined. A is considered as a perturbation of A_0 . We recall eigenfunction expansions for A_0 and state the limiting absorption principle for A_0 . Section 3 is devoted to the proof of the division theorem for A_0 . Finally in Section 4, we prove the limiting absorption principle for A , and give some properties of the spectrum of A .

2. Eigenfunction Expansions and the Limiting Absorption Principle for A_0

In this section, we consider the plane stratified medium $\mathbf{R}^3$ with the planar interface $x_3 = 0$, which is defined by

$$(2.1) \quad (\lambda(x_3), \mu(x_3), \rho(x_3)) = \begin{cases} (\lambda_1, \mu_1, \rho_1) & \text{for } x_3 < 0, \\ (\lambda_2, \mu_2, \rho_2) & \text{for } x_3 > 0. \end{cases}$$

Here $\lambda_1, \lambda_2, \mu_1, \mu_2$ are certain quantities called the Lamé constants and $\rho_1, \rho_2 > 0$ are the densities.

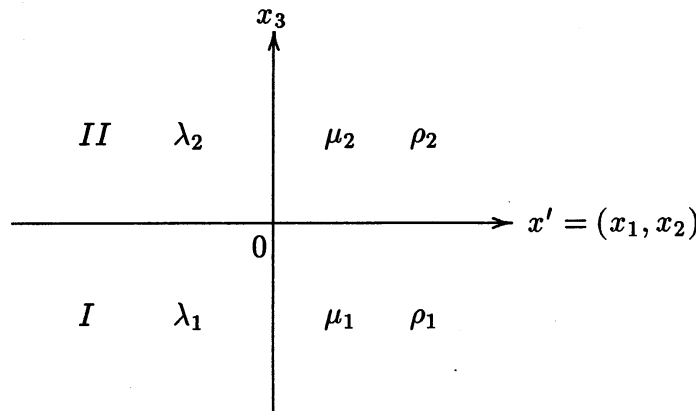


Figure 2 Unperturbed Stratified Medium $\mathbf{R}^3$

The propagation problem of elastic waves in this unperturbed stratified medium is formulated as the following mixed initial and interface value problem:

$$(2.2) \quad \frac{\partial^2 u}{\partial t^2}(t, x) + \mathcal{A}_0 u(t, x) = 0,$$

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$$(2.3) \quad u(t, x)|_{x_3=-0} = u(t, x)|_{x_3=+0},$$

$$(2.4) \quad \sigma_{k3}u(t, x)|_{x_3=-0} = \sigma_{k3}u(t, x)|_{x_3=+0},$$

$$(2.5) \quad u(0, x) = f(x), \quad \frac{\partial u}{\partial t}(0, x) = g(x),$$

where

$$(2.6) \quad \mathcal{A}_0 u = -\frac{\lambda(x_3) + \mu(x_3)}{\rho(x_3)} \nabla(\nabla \cdot u) - \frac{\mu(x_3)}{\rho(x_3)} \Delta u.$$

We introduce the Hilbert space

$$(2.7) \quad \mathcal{H}_0 = L^2(\mathbf{R}^3, \mathbf{C}^3, \rho(x_3) dx)$$

with inner product

$$(u, v)_{\mathcal{H}_0} = \int_{\mathbf{R}^3} u \cdot v \rho(x_3) dx.$$

Proposition 2.1. *The following the operator A_0 on $\mathcal{H}_0$ with domain*

$$\begin{aligned} D(A_0) = \{ & u \in H^2(\mathbf{R}_-^3, \mathbf{C}^3) \oplus H^2(\mathbf{R}_+^3, \mathbf{C}^3); \\ & u \text{ satisfies the interface conditions (1.2) and (1.3)} \\ & \text{in the sense of trace on } x_3 = 0 \} \end{aligned}$$

and action defined by

$$(2.8) \quad A_0 u = \mathcal{A}_0 u, \quad u \in D(A_0)$$

is a non-negative self-adjoint operator on $\mathcal{H}_0$.

Eigenfunction expansions for A_0 was developed in [11]. We give a brief review of the structure and properties of eigenfunctions and the expansion theorem.

Let $\eta' = (\eta_1, \eta_2) \in \mathbf{R}^2$ be the dual variables of $x' = (x_1, x_2)$ and let $F_{x'}$ denote the partial Fourier transformation with respect to x' . Let

$$(2.9) \quad U = \frac{1}{|\eta'|} \begin{pmatrix} \eta_1 & -\eta_2 & 0 \\ \eta_2 & \eta_1 & 0 \\ 0 & 0 & |\eta'| \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},$$

where U and C are unitary matrices and $|\eta'| = (\eta_1^2 + \eta_2^2)^{\frac{1}{2}}$ (cf. [5], [7]).

Proposition 2.2. *We have*

$$(2.10) \quad A_0 u = F_{\eta'}^{-1} U C (A_0^1(\eta') \oplus A_0^2(\eta')) (U C)^{-1} F_{x'} u \quad \text{for } u \in D(A_0),$$

where $A_0^1(\eta')$ and $A_0^2(\eta')$ are non-negative self-adjoint operators in $L^2(\mathbf{R}, \mathbf{C}^2, \rho(x_3) dx_3)$ and $L^2(\mathbf{R}, \mathbf{C}, \rho(x_3) dx_3)$ defined respectively as follows:

$$\begin{aligned} D(A_0^1(\eta')) = \left\{ \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \in H^2(\mathbf{R}_-, \mathbf{C}^2) \oplus H^2(\mathbf{R}_+, \mathbf{C}^2); \right. \\ \left. u|_{x_3=-0} = u|_{x_3=+0}, B_0^1 \left(\eta', \frac{d}{dx_3} \right) u \Big|_{x_3=-0} = B_0^1 \left(\eta', \frac{d}{dx_3} \right) u \Big|_{x_3=+0} \right\}, \end{aligned}$$

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$$A_0^1(\eta', \frac{d}{dx_3}) \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \frac{1}{\rho} \begin{pmatrix} -\mu \frac{d^2}{dx_3^2} + (\lambda + 2\mu)|\eta'|^2 & -i|\eta'|(\lambda + \mu)\frac{d}{dx_3} \\ -i|\eta'|(\lambda + \mu)\frac{d}{dx_3} & -(\lambda + 2\mu)\frac{d^2}{dx_3^2} + \mu|\eta'|^2 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix},$$

$$B_0^1\left(\eta', \frac{d}{dx_3}\right) \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} \mu \frac{d}{dx_3} & i|\eta'|\mu \\ i|\eta'|\lambda & (\lambda + 2\mu)\frac{d}{dx_3} \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix},$$

$$D(A_0^2(\eta')) = \{u \in H^2(\mathbf{R}_-) \oplus H^2(\mathbf{R}_+);$$

$$u|_{x_3=-0} = u|_{x_3=+0}, \quad B_0^2\left(\eta', \frac{d}{dx_3}\right) u \Big|_{x_3=-0} = B_0^2\left(\eta', \frac{d}{dx_3}\right) u \Big|_{x_3=+0} \},$$

$$A_0^2\left(\eta', \frac{d}{dx_3}\right) u = -\frac{\mu}{\rho} \frac{d^2 u}{dx_3^2} + \frac{\mu}{\rho} |\eta'|^2 u,$$

$$B_0^2\left(\eta', \frac{d}{dx_3}\right) u = \mu \frac{d}{dx_3} u,$$

where $\lambda = \lambda(x_3)$, $\mu = \mu(x_3)$ and $\rho = \rho(x_3)$.

The Lopatinski determinant $\Delta(\eta', \zeta)$ for $A_0^1(\eta')$ is given as follows:

$$\Delta(\eta', \zeta) = |\eta'|^6 \text{Dis}(z),$$

where $\text{Dis}(z)$ is given in [11 (3.2)] as $D(z)$. The squares of propagation speeds of shear (SV, SH) and pressure (P) waves are given by

$$(2.11) \quad c_{s_i}^2 = \frac{\mu_i}{\rho_i}, \quad c_{p_i}^2 = \frac{\lambda_i + 2\mu_i}{\rho_i}, \quad (i = 1, 2),$$

respectively. $\text{Dis}(z)$ has the only one real zero c_{St} when either $\text{Dis}(c_{s_1}^2) > 0$ or $\text{Dis}(c_{s_1}^2) = 0$ under some restriction if $c_{s_1} < c_{s_2}$ (see [11 Theorem 6.5]). If $c_{s_1} < c_{s_2}$, then we must replace $\text{Dis}(c_{s_1}^2)$ by $\text{Dis}(c_{s_2}^2)$. Then the zero of $\Delta(\eta', \zeta)$ is $c_{St}^2 |\eta'|^2$ and the origin of the Stoneley wave with speed c_{St} propagating along the interface $x_3 = 0$ in the elastic space $\mathbf{R}^3$.

Let $\eta = (\eta_1, \eta_2, \xi) = (\eta', \xi)$. $c_j^2 |\eta|^2$ ($j \in M = \{s_1, p_1, s_2, p_2\}$) and $c_k^2 |\eta|^2$ ($k \in N = \{s_1, s_2\}$) are the eigenvalues of $A_0^1(\eta')$ and $A_0^2(\eta')$, respectively. We obtain explicit expression of generalized eigenfunctions $\psi_{1j}^\pm(x_3, \eta)$, $\psi_{1j}^{St}(x_3, \eta)$ ($j \in M$) for $A_0^1(\eta')$ and $\psi_{2k}^\pm(x_3, \eta)$ ($k \in N$) for $A_0^2(\eta')$ (see [11 (4.9)-(4.20), (4.21)-(4.22), (5.8)-(5.13), respectively]).

Using these generalized eigenfunctions for $A_0^1(\eta')$ and $A_0^2(\eta')$, we define generalized eigenfunctions for A_0 as follows:

$$(2.12) \quad \psi_{1j}^\pm(x, \eta) = \frac{1}{2\pi} e^{i(x_1 \eta_1 + x_2 \eta_2)} \text{UC}(\psi_{1j}^\pm(x_3, \eta) \oplus O_{1 \times 1}), \quad j \in M,$$

$$(2.13) \quad \psi_{1j}^{St}(x, \eta) = \frac{1}{2\pi} e^{i(x_1 \eta_1 + x_2 \eta_2)} \text{UC}(\psi_{1j}^{St}(x_3, \eta) \oplus O_{1 \times 1}), \quad j \in M,$$

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$$(2.14) \quad \psi_{2k}^{\pm}(x, \eta) = \frac{1}{2\pi} e^{i(x_1 \eta_1 + x_2 \eta_2)} \text{UC}(O_{2 \times 2} \oplus \psi_{2k}^{\pm}(x_3, \eta)), \quad k \in N,$$

where $O_{n \times n}$ denotes the $n \times n$ zero matrix.

Now we define the Fourier transform of $f \in \mathcal{H}$ with respect to these generalized eigenfunctions: $f \mapsto (\hat{f}_{1j}^{\pm}, \hat{f}_{1j}^{St}, \hat{f}_{2k}^{\pm})$,

$$(2.15) \quad \hat{f}_{1j}^{\pm}(\eta) = \text{l.i.m.}_{R \rightarrow \infty} \int_{|x| \leq R} \psi_{1j}^{\pm}(x, \eta)^* f(x) \rho(x_3) dx, \quad j \in M,$$

$$(2.16) \quad \hat{f}_{1j}^{St}(\eta) = \text{l.i.m.}_{R \rightarrow \infty} \int_{|x| \leq R} \psi_{1j}^{St}(x, \eta)^* f(x) \rho(x_3) dx, \quad j \in M,$$

$$(2.17) \quad \hat{f}_{2k}^{\pm}(\eta) = \text{l.i.m.}_{R \rightarrow \infty} \int_{|x| \leq R} \psi_{2k}^{\pm}(x, \eta)^* f(x) \rho(x_3) dx, \quad k \in N.$$

We then have the eigenfunction expansion theorem.

Theorem 2.3. *We assume that the real zero of $\Delta(\eta'; \zeta)$ exists.*

(1) For $f, g \in \mathcal{H}_0$,

$$(2.18) \quad (f, g) = \sum_{j \in M} \left(\int_{\mathbf{R}^3} \hat{f}_{1j}^{\pm}(\eta) \cdot \overline{\hat{g}_{1j}^{\pm}(\eta)} d\eta + \int_{\mathbf{R}^3} \hat{f}_{1j}^{St}(\eta) \cdot \overline{\hat{g}_{1j}^{St}(\eta)} d\eta \right) \\ + \sum_{k \in N} \int_{\mathbf{R}^3} \hat{f}_{2k}^{\pm}(\eta) \cdot \overline{\hat{g}_{2k}^{\pm}(\eta)} d\eta.$$

(2) For $f \in \mathcal{H}_0$,

$$(2.19) \quad f(x) = \sum_{j \in M} \text{l.i.m.}_{R \rightarrow \infty} \int_{|\eta| \leq R} \left(\psi_{1j}^{\pm}(x, \eta) \hat{f}_{1j}^{\pm}(\eta) + \psi_{1j}^{St}(x, \eta) \hat{f}_{1j}^{St}(\eta) \right) d\eta \\ + \sum_{k \in N} \text{l.i.m.}_{R \rightarrow \infty} \int_{|\eta| \leq R} \psi_{2k}^{\pm}(x, \eta) \hat{f}_{2k}^{\pm}(\eta) d\eta.$$

(3) For $f \in D(A_0)$,

$$(2.20) \quad A_0 f(x) = \sum_{j \in M} \text{l.i.m.}_{R \rightarrow \infty} \int_{|\eta| \leq R} \left(c_j^2 |\eta|^2 \psi_{1j}^{\pm}(x, \eta) \hat{f}_{1j}^{\pm}(\eta) + c_{St}^2 |\eta'|^2 \psi_{1j}^{St}(x, \eta) \hat{f}_{1j}^{St}(\eta) \right) d\eta \\ + \sum_{k \in N} \text{l.i.m.}_{R \rightarrow \infty} \int_{|\eta| \leq R} c_k^2 |\eta|^2 \psi_{2k}^{\pm}(x, \eta) \hat{f}_{2k}^{\pm}(\eta) d\eta.$$

(4) We define the mappings by

$$\Phi_{1j}^{\pm} : \mathcal{H}_0 \ni f \rightarrow \hat{f}_{1j}^{\pm}(\eta) \in L^2(\mathbf{R}_+^3, \mathbf{C}^3)(\xi > 0), \in L^2(\mathbf{R}_-^3, \mathbf{C}^3)(\xi < 0), \quad j \in M,$$

$$\Phi_{1j}^{St} : \mathcal{H}_0 \ni f \rightarrow \hat{f}_{1j}^{St}(\eta) \in L^2(\mathbf{R}^3, \mathbf{C}^3), \quad j \in M,$$

$$\Phi_{2k}^{\pm} : \mathcal{H}_0 \ni f \rightarrow \hat{f}_{2k}^{\pm}(\eta) \in L^2(\mathbf{R}_+^3, \mathbf{C}^3)(\xi > 0), \in L^2(\mathbf{R}_-^3, \mathbf{C}^3)(\xi < 0), \quad k \in N,$$

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and put

$$\Phi^\pm f = \left(\sum_{j \in M} \Phi_{1j}^\pm f, \sum_{j \in M} \Phi_{1j}^{St} f, \sum_{k \in N} \Phi_{2k}^\pm f \right).$$

Then we have

$$(2.21) \quad R(\Phi^\pm) = L^2(\mathbf{R}_\pm^3, \mathbf{C}^3) \oplus L^2(\mathbf{R}^3, \mathbf{C}^3) \oplus L^2(\mathbf{R}_\pm^3, \mathbf{C}^3).$$

This theorem implies that $\Phi^\pm$ are unitary operators in $\mathcal{H}_0$, and that the systems of generalized eigenfunctions $\{\psi_{1j}^+, \psi_{1j}^{St}, \psi_{2k}^+\}_{j \in M, k \in N}$ and $\{\psi_{1j}^-, \psi_{1j}^{St}, \psi_{2k}^-\}_{j \in M, k \in N}$ are complete, respectively.

Let $R_0(z)$ be the resolvent of A_0 . By using Theorem 2.3 and the operational calculus, we have for f and g in $C_0^\infty(\mathbf{R}^3, \mathbf{C}^3)$ and $z \in \mathbf{C} \setminus [0, \infty)$,

$$(2.22) \quad \begin{aligned} & (R_0(z)f, g)_{\mathcal{H}_0} \\ &= \sum_{j \in M} \left(\int_{\mathbf{R}_\pm^3} \frac{1}{c_j^2 |\eta|^2 - z} \hat{f}_{1j}^\pm(\eta) \cdot \overline{\hat{g}_{1j}^\pm(\eta)} d\eta + \int_{\mathbf{R}^3} \frac{1}{c_{St}^2 |\eta'|^2 - z} \hat{f}_{1j}^{St}(\eta) \cdot \overline{\hat{g}_{1j}^{St}(\eta)} d\eta \right) \\ &+ \sum_{k \in N} \int_{\mathbf{R}_\pm^3} \frac{1}{c_k^2 |\eta|^2 - z} \hat{f}_{2k}^\pm(\eta) \cdot \overline{\hat{g}_{2k}^\pm(\eta)} d\eta. \end{aligned}$$

By changing to polar coordinates and using continuity properties of Cauchy type integrals, we get

$$(2.23) \quad \begin{aligned} & \lim_{\substack{z \rightarrow \omega \\ \pm \operatorname{Im} z > 0}} (R_0(z)f, g)_{\mathcal{H}_0} \\ &= \sum_{j \in M} \left(\pm i \frac{\pi}{2\sqrt{\omega}} \int_{|\eta|=\frac{\sqrt{\omega}}{c_j}} \hat{f}_{1j}^\pm(\eta) \cdot \overline{\hat{g}_{1j}^\pm(\eta)} dS_j + \text{p.v.} \int_{\mathbf{R}_\pm^3} \frac{\hat{f}_{1j}^\pm(\eta) \cdot \overline{\hat{g}_{1j}^\pm(\eta)}}{c_j^2 |\eta|^2 - \omega} d\eta \right) \\ &+ \sum_{j \in M} \left(\pm i \frac{\pi}{2\sqrt{\omega}} \int_{\mathbf{R}} \int_{|\eta'|=\frac{\sqrt{\omega}}{c_{St}}} \hat{f}_{1j}^{St}(\eta) \cdot \overline{\hat{g}_{1j}^{St}(\eta)} dS' d\xi + \text{p.v.} \int_{\mathbf{R}^3} \frac{\hat{f}_{1j}^{St}(\eta) \cdot \overline{\hat{g}_{1j}^{St}(\eta)}}{c_{St}^2 |\eta'|^2 - \omega} d\eta \right) \\ &+ \sum_{k \in N} \left(\pm i \frac{\pi}{2\sqrt{\omega}} \int_{|\eta|=\frac{\sqrt{\omega}}{c_k}} \hat{f}_{2k}^\pm(\eta) \cdot \overline{\hat{g}_{2k}^\pm(\eta)} dS_k + \text{p.v.} \int_{\mathbf{R}_\pm^3} \frac{\hat{f}_{2k}^\pm(\eta) \cdot \overline{\hat{g}_{2k}^\pm(\eta)}}{c_k^2 |\eta|^2 - \omega} d\eta \right), \end{aligned}$$

where dS_j, dS', dS_k denote the surface element of the spheres $|\eta| = \frac{\sqrt{\omega}}{c_j}, |\eta'| = \frac{\sqrt{\omega}}{c_{St}}, |\eta| = \frac{\sqrt{\omega}}{c_k}$, respectively. Now we define generalized trace operators associated with A_0 . For any $\omega > 0$, put

$$(2.24) \quad E_{1j}^\pm(\omega) = \left\{ \eta \in \mathbf{R}_\pm^3, |\eta| = \frac{\sqrt{\omega}}{c_j} \right\},$$

$$(2.25) \quad E_{1j}^{St}(\omega) = \left\{ \eta \in \mathbf{R}^3, |\eta'| = \frac{\sqrt{\omega}}{c_{St}}, \xi \in \mathbf{R} \right\},$$

$$(2.26) \quad E_{2k}^\pm(\omega) = \left\{ \eta \in \mathbf{R}_\pm^3, |\eta| = \frac{\sqrt{\omega}}{c_k} \right\},$$

then we have

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Proposition 2.4. Suppose $s_1 > \frac{1}{2}$ and $s_2 > \frac{1}{2}$. For any $\omega > 0$ there exist generalized trace operators

$$(2.27) \quad \tau_{1j}^{\pm}(\omega) : L^{2;s_1,s_2}(\mathbf{R}^3, \mathbf{C}^3) \rightarrow L^2(E_{1j}^{\pm}(\omega)),$$

$$(2.28) \quad \tau_{1j}^{St}(\omega) : L^{2;s_1,s_2}(\mathbf{R}^3, \mathbf{C}^3) \rightarrow L^2(E_{1j}^{St}(\omega)),$$

$$(2.29) \quad \tau_{2k}^{\pm}(\omega) : L^{2;s_1,s_2}(\mathbf{R}^3, \mathbf{C}^3) \rightarrow L^2(E_{2k}^{\pm}(\omega)),$$

such that for any $f \in C_0^\infty(\mathbf{R}^3, \mathbf{C}^3)$:

$$(2.30) \quad \tau_{1j}^{\pm}(\omega)f(\eta) = \hat{f}_{1j}^{\pm}(\eta), \quad |\eta| = \frac{\sqrt{\omega}}{c_j},$$

$$(2.31) \quad \tau_{1j}^{St}(\omega)f(\eta) = \hat{f}_{1j}^{St}(\eta), \quad |\eta'| = \frac{\sqrt{\omega}}{c_{St}}, \quad \xi \in \mathbf{R},$$

$$(2.32) \quad \tau_{2k}^{\pm}(\omega)f(\eta) = \hat{f}_{2k}^{\pm}(\eta), \quad |\eta| = \frac{\sqrt{\omega}}{c_k}.$$

Furthermore for any $f \in L^2(\mathbf{R}^3, \mathbf{C}^3)$

$$(2.33) \quad \|\tau_{1j}^{\pm}(\omega)f\|_{L^2(E_{1j}^{\pm}(\omega))} \leq M(\omega)\|f\|_{0;s_1,s_2},$$

$$(2.34) \quad \|\tau_{1j}^{St}(\omega)f\|_{L^2(E_{1j}^{St}(\omega))} \leq M(\omega)\|f\|_{0;s_1,s_2},$$

$$(2.35) \quad \|\tau_{2k}^{\pm}(\omega)f\|_{L^2(E_{2k}^{\pm}(\omega))} \leq M(\omega)\|f\|_{0;s_1,s_2},$$

where $M(\omega)$ is a continuous function on $(0, \infty)$.

Then we have the limiting absorption principle for A_0 .

Theorem 2.5. Suppose $s_1 > \frac{1}{2}$ and $s_2 > \frac{1}{2}$. Then for any $\omega > 0$, the following two limits exist in the uniform operator topology of $B(L^{2;s_1,s_2}(\mathbf{R}^3, \mathbf{C}^3), H^{2;-s_1,-s_2}(\mathbf{R}^3, \mathbf{C}^3))$:

$$(2.36) \quad R_0^{\pm}(\omega) = \lim_{\substack{z \rightarrow \omega \\ \pm \operatorname{Im} z > 0}} R_0(z).$$

Finally we conclude this section with the following proposition.

Proposition 2.6. Suppose $s_1 > \frac{1}{2}$ and $s_2 > \frac{1}{2}$. Let $\omega > 0$ and $f \in L^{2;s_1,s_2}(\mathbf{R}^3, \mathbf{C}^3)$. Then the following statements are equivalent:

$$(2.37) \quad R_0^+(\omega)f = R_0^-(\omega)f,$$

$$(2.38) \quad \operatorname{Im} \int_{\mathbf{R}_\pm^3} R_0^+(\omega)f \cdot \bar{f}\rho(x_3) dx = 0,$$

$$(2.39) \quad \operatorname{Im} \int_{\mathbf{R}_\pm^3} R_0^-(\omega)f \cdot \bar{f}\rho(x_3) dx = 0,$$

$$(2.40) \quad \sum_{j \in M} \tau_{1j}^{\pm}(\omega)f = \sum_{j \in M} \tau_{1j}^{St}(\omega)f = \sum_{k \in N} \tau_{2k}^{\pm}(\omega)f = 0,$$

$$(2.41) \quad \sum_{j \in M} \tau_{1j}^{\pm}(\omega)\bar{f} = \sum_{j \in M} \tau_{1j}^{St}(\omega)\bar{f} = \sum_{k \in N} \tau_{2k}^{\pm}(\omega)\bar{f} = 0.$$

3. The Division Theorem for A_0

This section is devoted to the division theorem for A_0 . This theorem states that if the generalized traces of $f \in L^{2;s_1,s_2}(\mathbf{R}^3, \mathbf{C}^3)$ vanish on $E_{1j}^\pm(\omega)$, $E_{1j}^{St}(\omega)$, and $E_{2k}^\pm(\omega)$ then the function $u = R_0^\pm(\omega)f$ has a better decay at infinity than is expected from Theorem 2.4. The division theorem plays a role corresponding to radiation condition or uniqueness theorem such as Rellich theorem.

The proof of the division theorem is done along the line of proof by Dermenjian and Guillot [5]. They proved the division theorem for their problem using representations of solutions due to Dunford and Schwartz [6 Theorem XIII. 3.16]. But we prove our division theorem using the integral representation of solutions by means of Lopatinski analysis.

Let us recall (2.10). For any $z \in \mathbf{C} \setminus [0, \infty)$ let

$$(3.1) \quad R_0^1(z) = (A_0^1(\eta') - z)^{-1}, \quad R_0^2(z) = (A_0^2(\eta') - z)^{-1}.$$

Suppose $s_1 > \frac{1}{2}$, $s_2 > \frac{1}{2}$, and $f \in L^{2;s_1,s_2}(\mathbf{R}^3, \mathbf{C}^3)$. Let

$$(3.2) \quad g(\eta', x_3) = {}^t(g_1(\eta', x_3), g_2(\eta', x_3)) = (UC)^{-1} F_{x'} f(\eta', x_3),$$

where $g_1(\eta', x_3)$ and $g_2(\eta', x_3)$ are 2×1 and 1×1 vectors, respectively. Thus we have

$$(3.3) \quad g(\eta', x_3) \in L^{2;0,s_2}(\mathbf{R}^3, \mathbf{C}^3)$$

and

$$(3.4) \quad ((UC)^{-1} F_{x'} R_0^\pm(\omega) f)(\eta', x_3) = R_0^{1\pm}(\omega) g_1(\eta', x_3) \oplus R_0^{2\pm}(\omega) g_2(\eta', x_3).$$

Then we have the following theorem.

Theorem 3.1. Suppose $s_1 > \frac{1}{2}$, $s_2 > \frac{1}{2}$, and $\rho_1 = \rho_2$. Let $f \in L^{2;s_1,s_2}(\mathbf{R}^3, \mathbf{C}^3)$ and ω be a strictly positive number such that

$$(3.5) \quad \sum_{j \in M} \tau_{1j}^\pm(\omega) f = \sum_{j \in M} \tau_{1j}^{St}(\omega) f = \sum_{k \in N} \tau_{2k}^\pm(\omega) f = 0.$$

Then we have

$$(3.6) \quad R_0^+(\omega) f = R_0^-(\omega) f \in L^{2;s_1-1,\tilde{s}_2}(\mathbf{R}_\pm^3, \mathbf{C}^3),$$

and

$$(3.7) \quad \|R_0^\pm(\omega) f\|_{0;s_1-1,\tilde{s}_2} \leq M(\omega) \|f\|_{0;s_1,s_2},$$

where $M(\cdot)$ is a positive continuous function on $(0, \infty)$ depending only on s_1 , s_2 , and $\tilde{s}_2$. Here $\tilde{s}_2$ is a real number such that

$$(3.8) \quad \tilde{s}_2 < s_2 - 1.$$

This theorem is called division theorem for A_0 . The proof of this theorem will be a consequence of (3.4) and Propositions 3.2-3.4 below.

Remark 1. Continuity of $M(\cdot)$ is useful in proving that the positive eigenvalues of A can accumulate only at 0 and $+\infty$.

Remark 2. If $\rho_1 \neq \rho_2$, we have (3.6) with $\tilde{s}_2 < -\frac{1}{2}$ by (3.17) below. So we cannot use this result to prove Main Theorem, because $\tilde{s}_2$ is negative.

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3.1 The Division Theorem for $A_0^2(\eta')$

Let

$$(3.9) \quad v_2(\eta', x_3, z) = R_0^2(z)g_2(\eta', x_3).$$

$v_2(\eta', x_3, z)$ has a meaning for $z \in \mathbf{C} \setminus [0, \infty)$. $v_2(\eta', x_3, \omega)$ will be defined as the limit of $v_2(\eta', x_3, z)$ as z tends to ω such that $\text{Im}z > 0$; that is,

$$(3.10) \quad v_2(\eta', x_3, \omega) = R_0^{2+}(\omega)g_2(\eta', x_3).$$

Then we have

Proposition 3.2. Suppose $s_1 > \frac{1}{2}$, $s_2 > \frac{1}{2}$, and $\rho_1 = \rho_2$. Let $f \in L^{2;s_1,s_2}(\mathbf{R}^3, \mathbf{C}^3)$ and ω be a strictly positive number such that

$$(3.11) \quad \sum_{k \in N} \tau_{2k}^+(\omega) \bar{f} = 0 \quad \text{or} \quad \sum_{k \in N} \tau_{2k}^-(\omega) \bar{f} = 0.$$

Then

$$(3.12) \quad v_2(\cdot, \cdot, \omega) = R_0^{2+}(\omega)g_2 = R_0^{2-}(\omega)g_2 \in L^2(\mathbf{R}^3, (1+x_3^2)^{\delta-\frac{1}{2}} d\eta' dx_3)$$

and

$$(3.13) \quad \|v_2(\cdot, \cdot, \omega)\|_{L^2(\mathbf{R}^3, (1+x_3^2)^{\delta-\frac{1}{2}} d\eta' dx_3)} \leq M(\omega) \|f\|_{0,s_1,s_2},$$

where $M(\cdot)$ is a positive continuous function on $(0, \infty)$ depending only on δ , and δ is a real number such that $\delta < s_2 - \frac{1}{2}$.

Remark. The assertion of the first half of (3.12) follows immediately from Proposition 2.6.

Proof. The explicit integral representation of solution $v_2(\eta', x_3, z)$ is given in [11 (5.4) and (5.5)]. So we have the explicit expression of $v_2(\eta', x_3, \omega)$ by exchanging z , τ_{s_1} , τ_{s_2} to ω ,

$$(3.14) \quad \begin{aligned} \xi_{s_1} &= \lim_{\substack{z \rightarrow \omega \\ \text{Im}z > 0}} \tau_{s_1} = \lim_{\substack{z \rightarrow \omega \\ \text{Im}z > 0}} \sqrt{\frac{z}{c_{s_1}^2} - |\eta'|^2}, \\ \xi_{s_2} &= \lim_{\substack{z \rightarrow \omega \\ \text{Im}z > 0}} \tau_{s_2} = \lim_{\substack{z \rightarrow \omega \\ \text{Im}z > 0}} \sqrt{\frac{z}{c_{s_2}^2} - |\eta'|^2}, \end{aligned}$$

respectively.

Consider the case where the condition $\sum_{k \in N} \tau_{2k}^+(\omega) \bar{f} = 0$ is satisfied. We also prove (3.12) and (3.13) for $v_2^I(\eta', x_3, \omega)$. The other cases can be handled similarly.

By (2.32), (2.17) and (2.14), the condition $\sum_{k \in N} \tau_{2k}^+(\omega) \bar{f} = 0$ can be rewritten as follows:

$$(3.15) \quad \sum_{k \in N} \int_{-\infty}^{\infty} \psi_{2k}^+(y_3, \eta) g_2(\eta', y_3) \rho(y_3) dy_3 = 0 \quad \text{for } |\eta| = \frac{\sqrt{\omega}}{c_k}.$$

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In more detail, we have by (5.8)-(5.10) in [11],

$$(3.16) \quad \left\{ \int_{-\infty}^0 e^{i\xi_{s_1} y_3} - \frac{e^{-i\xi_{s_1} y_3}}{\Delta'(\eta', \omega)} (\rho_1 c_{s_1}^2 \xi_{s_1} - \rho_2 c_{s_2}^2 \xi_{s_2}) \right. \\ \left. + \int_0^{\infty} \frac{e^{i\xi_{s_2} y_3}}{\Delta'(\eta', \omega)} (2\rho_1 c_{s_1}^2 \xi_{s_1}) \right\} g_2(\eta', y_3) dy_3 = 0.$$

By substituting (3.16) multiplied by $e^{-i\xi_{s_1} x_3}$ into $v_2^I(\eta', x_3, \omega)$, we obtain

$$(3.17) \quad v_2^I(\eta', x_3, \omega) \\ = \frac{i}{2 c_{s_1}^2 \xi_{s_1}} \int_{-\infty}^{x_3} (e^{i\xi_{s_1} x_3} e^{-i\xi_{s_1} y_3} - e^{-i\xi_{s_1} x_3} e^{i\xi_{s_1} y_3}) g_2(\eta', y_3) dy_3 \\ + i \frac{\rho_2 - \rho_1}{\Delta'(\eta', \omega)} e^{-i\xi_{s_1} x_3} \int_0^{\infty} e^{i\xi_{s_2} y_3} g_2(\eta', y_3) dy_3, \quad x_3 < 0.$$

If $\rho_1 = \rho_2$, then the second term of the right-hand side of (3.17) is equal to 0, because $\Delta'(\eta', \omega)$ has no zero. Thus we may only estimate the first term of the right-hand side of (3.17).

Let $\chi_1(|\eta'|)$, $\chi_2(|\eta'|)$, $\chi_3(|\eta'|)$ be the characteristic function of $(0, \frac{\sqrt{\omega}}{c_{s_1}})$, $(\frac{\sqrt{\omega}}{c_{s_1}}, \frac{\sqrt{2\omega}}{c_{s_1}})$, $(\frac{\sqrt{2\omega}}{c_{s_1}}, \infty)$, respectively. Consider the case where $\chi_1(|\eta'|)v^I(\eta', x_3, \omega)$. The other cases can be handled similarly.

In the case where $\chi_1(|\eta'|)v^I(\eta', x_3, \omega)$, we have $0 \leq |\eta'| \leq \frac{\sqrt{\omega}}{c_{s_1}}$, so $\xi_{s_1} \geq 0$. From the inequalities

$$(3.18) \quad |e^{i\xi_{s_1} x_3} - e^{i\xi_{s_1} y_3}| \leq 2|\xi_{s_1}|^\gamma |x_3 - y_3|^\gamma \quad \text{for } 0 \leq \gamma \leq 1$$

and

$$(3.19) \quad \int_{-\infty}^{x_3} (1 + y_3^2)^{\gamma - s_2} \leq C(1 + x_3^2)^{-\alpha} \quad \text{for } 0 < \alpha < s_2 - \frac{1}{2} - \gamma,$$

it follows that

$$(3.20) \quad |\chi_1(|\eta'|)v^I(\eta', x_3, \omega)|^2 \leq C\xi_{s_1}^{2\gamma-2} (1 + x_3^2)^{-\alpha} \|g_2\|_{0;s_2}^2$$

for α such that $0 < \alpha < s_2 - \frac{1}{2} - \gamma$. Thus

$$(3.21) \quad \|\chi_1(|\eta'|)v^I(\eta', x_3, \omega)\|_{L^2(\mathbf{R}_-, (1+x_3^2)^{\delta-\frac{1}{2}} d\eta' dx_3)}^2 \\ \leq C\|g_2\|_{0;s_2}^2 \left(\int_{\mathbf{R}_-} (1 + x_3^2)^{\delta-\frac{1}{2}-\alpha} dx_3 \right) \left(\int_{0 \leq |\eta'| \leq \frac{\sqrt{\omega}}{c_{s_1}}} \frac{1}{\left(\sqrt{\frac{\omega}{c_{s_1}^2} - |\eta'|^2} \right)^{2-2\gamma}} d\eta' \right).$$

Consequently if $\gamma > 0$ and $\delta < \alpha < s_2 - \frac{1}{2} - \gamma$ then we obtain (3.12) and (3.13) for $0 \leq |\eta'| \leq \frac{\sqrt{\omega}}{c_{s_1}}$.

This completes the proof of proposition 3.2. $\square$

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3.2 The Division Theorem for $A_0^1(\eta')$

Let

$$(3.22) \quad v_1(\eta', x_3, z) = R_0^1(z)g_1(\eta', \cdot)(x_3),$$

where $z \in \mathbf{C} \setminus [0, \infty)$. $v_1(\eta', x_3, \omega)$ will be defined as the limit of $v_1(\eta', x_3, z)$ as z tends to ω such that $\text{Im}z > 0$; that is,

$$(3.23) \quad v_1(\eta', x_3, \omega) = R_0^{1+}(\omega)g_1(\eta', \cdot)(x_3).$$

Let $\chi_4(|\eta'|)$ and $\chi_5(|\eta'|)$ be the characteristic functions of $(0, \infty) \setminus \left[\frac{\sqrt{\omega}}{c_{St}} - \varepsilon, \frac{\sqrt{\omega}}{c_{St}} + \varepsilon \right]$ and $\left(\frac{\sqrt{\omega}}{c_{St}} - \varepsilon, \frac{\sqrt{\omega}}{c_{St}} + \varepsilon \right)$, respectively. Then we have the following propositions.

Proposition 3.3. Suppose $s_1 > \frac{1}{2}$, $s_2 > \frac{1}{2}$, and $\rho_1 = \rho_2$. Let $f \in L^{2;s_1,s_2}(\mathbf{R}^3, \mathbf{C}^3)$ and ω be a strictly positive number such that

$$(3.24) \quad \sum_{j \in M} \tau_{1j}^+(\omega) \bar{f} = 0 \quad \text{or} \quad \sum_{j \in M} \tau_{1j}^-(\omega) \bar{f} = 0.$$

Then we have

$$(3.25) \quad \begin{aligned} \chi_4(|\eta'|)v_1(\cdot, \cdot, \omega) &= \chi_4(|\eta'|)R_0^{1+}(\omega)g_1 = \chi_4(|\eta'|)R_0^{1-}(\omega)g_1 \\ &\in L^2(\mathbf{R}^3, \mathbf{C}^2, (1 + x_3^2)^{\delta-\frac{1}{2}} d\eta' dx_3). \end{aligned}$$

Proposition 3.4. Suppose $s_1 > \frac{1}{2}$, $s_2 > \frac{1}{2}$, and $\rho_1 = \rho_2$. Let $f \in L^{2;s_1,s_2}(\mathbf{R}^3, \mathbf{C}^3)$ and ω be a strictly positive number such that

$$(3.26) \quad \sum_{j \in M} \tau_{1j}^{St}(\omega) f = 0.$$

Then we obtain

$$(3.27) \quad \chi_5(|\eta'|)v_1(\cdot, \cdot, \omega) \in H^{s_1-1}(\mathbf{R}_{\eta'}^2, L^{2;\delta-\frac{1}{2}}(\mathbf{R}, \mathbf{C}^2, dx_3)).$$

From Propositions 3.2-3.4, we have

$$(3.28) \quad \begin{aligned} \chi_5(|\eta'|)(R_0^{1+}(\omega)g_1(\eta', \cdot) \oplus R_0^{2+}(\omega)g_2(\eta', \cdot))(x_3) \\ \in H^{s_1-1}(\mathbf{R}_{\eta'}^2, L^{2;\delta-\frac{1}{2}}(\mathbf{R}, \mathbf{C}^2, dx_3)), \end{aligned}$$

moreover

$$(3.29) \quad F_{\eta'}^{-1}(\text{UC})\chi_5(|\eta'|)(R_0^{1+}(\omega)g_1(\eta', \cdot) \oplus R_0^{2+}(\omega)g_2(\eta', \cdot))(x) \in L^{2;s_1-1,\delta-\frac{1}{2}}(\mathbf{R}^3, \mathbf{C}^3).$$

Thus Theorem 3.1 will be a consequence of Propositions 3.2-3.4 and (3.4).

4. The Limiting Absorption Principle for A

In this section we give a proof of the limiting absorption principle for A along the line of proof by Dermenjian and Guillot for their problem [5]. The key part of the proof is the following proposition.

Proposition 4.1. *For every $f \in L^{2;s_1,s_2}(\Omega, \mathbf{C}^3)$ and $z \in J^\pm(a, b) \setminus [a, b]$, we have*

$$(4.1) \quad \|R(z)f\|_{\mathcal{A}; -s_1, -s_2} \leq C\|f\|_{0;s_1,s_2},$$

where $[a, b]$ is any compact interval in $(0, \infty)$ which does not contain any eigenvalue of A and

$$(4.2) \quad J^\pm(a, b) = \{z \in \mathbf{C}; \operatorname{Re} z \in [a, b], \operatorname{Im} z \in [0, 1]\}.$$

Proof. We prove this proposition by contradiction. Fourth steps are needed.

Step 1. Suppose that (4.1) is false. Then there exist sequences $\{f_n\}_{n \geq 1}$ in $L^{2;s_1,s_2}(\Omega, \mathbf{C}^3)$ and $\{z_n\}_{n \geq 1}$ in $J^\pm(a, b) \setminus [a, b]$ such that

$$(4.3) \quad \|f_n\|_{0;s_1,s_2} = 1, \quad n \geq 1,$$

$$(4.4) \quad \|R(z_n)f_n\|_{\mathcal{A}; -s_1, -s_2} > n, \quad n \geq 1.$$

It follows that there exists a subsequence such that

$$(4.5) \quad \lim_{n \rightarrow \infty} z_n = \omega \in [a, b],$$

we denote it by the same symbol (cf. [14]). Put

$$(4.6) \quad u_n = \frac{R(z_n)f_n}{\|R(z_n)f_n\|_{\mathcal{A}; -s_1, -s_2}}, \quad n \geq 1,$$

$$(4.7) \quad F_n = \frac{f_n}{\|R(z_n)f_n\|_{\mathcal{A}; -s_1, -s_2}}, \quad n \geq 1.$$

Then we have

$$(4.8) \quad u_n \in D(A), \quad n \geq 1,$$

$$(4.9) \quad \|u_n\|_{\mathcal{A}; -s_1, -s_2} = 1, \quad n \geq 1,$$

$$(4.10) \quad \|F_n\|_{0;s_1,s_2} < \frac{1}{n}, \quad n \geq 1,$$

$$(4.11) \quad (\mathcal{A} - z_n)u_n = F_n, \quad n \geq 1.$$

From (4.9) and (1.32)

$$(4.12) \quad \|u_n\|_{1;-s_1,-s_2}^2 \leq 1.$$

Since $\{u_n\}_{n \geq 1}$ is a bounded sequence in $H^{1;-s_1,-s_2}(\Omega, \mathbf{C}^3)$, by Rellich theorem, we choose a subsequence of $\{u_n\}_{n \geq 1}$ that we denote by the same symbol such that

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$\{u_n\}_{n \geq 1}$ converges to a limit, denoted by u in $L^{2;-s'_1, -s'_2}(\Omega, \mathbf{C}^3)$, where $s'_1 > s_1$ and $s'_2 > s_2$. From (4.5), (4.10) and (4.11), it follows that

$$(4.13) \quad \mathcal{A}u = \omega u$$

in the distribution sense. So we get

$$(4.14) \quad \mathcal{A}u \in L^{2;-s'_1, -s'_2}(\Omega, \mathbf{C}^3).$$

Then we deduce from Korn's inequality

$$(4.15) \quad u \in H^{1;-s'_1, -s'_2}(\Omega, A, \mathbf{C}^3).$$

Step 2. In Step 2 and 3, we shall show that u belongs to $D(A)$.

Let $\phi(x)$ be a function in $C^\infty(\mathbf{R}^3)$ such that $\phi(x) = 1$ for $|x| > L + 2$ and $= 0$ for $|x| < L + 1$. Since $\phi u(x)$ is defined on $\Omega \cap \{|x| > L\}$, we put

$$(4.16) \quad \mu_1 \varepsilon_{13}(\phi u)|_{x_3=-0} = \mu_2 \varepsilon_{13}(\phi u)|_{x_3=+0} = h_1,$$

$$(4.17) \quad \mu_1 \varepsilon_{23}(\phi u)|_{x_3=-0} = \mu_2 \varepsilon_{23}(\phi u)|_{x_3=+0} = h_2,$$

$$(4.18) \quad \sigma_{33}(\phi u)|_{x_3=-0} = \sigma_{33}(\phi u)|_{x_3=+0} = h_3,$$

where

$$(4.19) \quad h = {}^t(h_1, h_2, h_3) \in H^{\frac{1}{2}}(\mathbf{R}^2, \mathbf{C}^3), \quad \text{supp } h \subset \{x \in \mathbf{R}^3; L + 1 < |x| < L + 2\}.$$

It follows from Lemma 5.1 in Dermenjian and Guillot [5] that there exists an extension $\tilde{u}$ of h belongs to $H^2(\mathbf{R}^3, \mathbf{C}^3)$ such that

$$(4.20) \quad \mu_1 \varepsilon_{13}(\tilde{u})|_{x_3=-0} = \mu_2 \varepsilon_{13}(\tilde{u})|_{x_3=+0} = h_1,$$

$$(4.21) \quad \mu_1 \varepsilon_{23}(\tilde{u})|_{x_3=-0} = \mu_2 \varepsilon_{23}(\tilde{u})|_{x_3=+0} = h_2,$$

$$(4.22) \quad \sigma_{33}(\tilde{u})|_{x_3=-0} = \sigma_{33}(\tilde{u})|_{x_3=+0} = h_3,$$

and

$$(4.23) \quad \text{supp } \tilde{u} \subset \{x \in \mathbf{R}^3; L < |x| < L + 3\}.$$

Putting

$$(4.24) \quad u' = \phi u - \tilde{u},$$

the support of u' is contained in $\{x \in \mathbf{R}^3; |x| > L\}$. u' in $H^{1;-s'_1, -s'_2}(\Omega, \mathcal{A}, \mathbf{C}^3)$ satisfies generalized free boundary-interface condition (1.18) for every v in $H^{1;s'_1, s'_2}(\Omega, \mathbf{C}^3)$. We have

$$(4.25) \quad u' = R_0(z_n)(\mathcal{A}_0 - z_n)u' \quad \text{for } n \geq 1,$$

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From (4.24), (4.5), (4.10), (4.11) and (4.13),

$$\begin{aligned}
 (4.26) \quad & (\mathcal{A}_0 - z_n)u'_n - (\mathcal{A}_0 - \omega)u' \\
 &= \phi(\mathcal{A}_0 - z_n)u_n + C\nabla\phi \cdot \nabla u_n + u_n\mathcal{A}_0\phi - (\mathcal{A}_0 - z_n)\tilde{u} \\
 &- \phi(\mathcal{A}_0 - \omega)u - C\nabla\phi \cdot \nabla u - u\mathcal{A}_0\phi + (\mathcal{A}_0 - \omega)\tilde{u} \\
 &= \phi F_n + C\nabla\phi \cdot \nabla(u_n - u) + (u_n - u)\mathcal{A}_0\phi + (z_n - \omega)\tilde{u}
 \end{aligned}$$

converges to 0 as $n \rightarrow \infty$ in $L^{2;s'_1, s'_2}(\mathbf{R}^3, \mathbf{C}^3)$ because the supports of $\nabla\phi$, $\mathcal{A}_0\phi$ and $\tilde{u}$ are compact.

From the sequence $\{z_n\}_{n \geq 1}$ there exists a subsequence we denote by the same symbol such that either $\text{Im}z_n > 0$ or $\text{Im}z_n < 0$. Suppose that $\text{Im}z_n > 0$. It follows from (4.25) and (4.26) that

$$(4.27) \quad u' = R_0^+(\omega)(\mathcal{A}_0 - \omega)u'$$

in $H^{2;-s'_1, -s'_2}(\mathbf{R}^3, \mathbf{C}^3)$ by Theorem 2.4.

Step 3. We shall show

$$\begin{aligned}
 (4.28) \quad & \sum_{j \in M} \tau_{1j}^\pm(\omega)[(\mathcal{A}_0 - \omega)u'] = \sum_{j \in M} \tau_{1j}^{St}(\omega)[(\mathcal{A}_0 - \omega)u'] \\
 &= \sum_{k \in N} \tau_{2k}^\pm(\omega)[(\mathcal{A}_0 - \omega)u'] = 0.
 \end{aligned}$$

Then it follows from Theorem 4.1 that $u' \in L^2(\mathbf{R}^3, \mathbf{C}^3)$ taking $s'_1 > 1$ and $s'_2 > 1$. Thus u belongs to $L^2(\Omega, \mathbf{C}^3, \rho(x)dx)$.

We denote by $\langle \cdot, \cdot \rangle_\rho$ the duality between $L^{2;-s'_1, -s'_2}(\Omega, \mathbf{C}^3, \rho(x)dx)$ and $L^{2;s'_1, s'_2}(\Omega, \mathbf{C}^3, \rho(x)dx)$. From Proposition 2.6 and (4.27) it is sufficient to show that

$$(4.29) \quad I = \left\langle \overline{R_0^+[(\mathcal{A}_0 - \omega)u']}, (\mathcal{A}_0 - \omega)u' \right\rangle_\rho = \langle \bar{u}', (\mathcal{A}_0 - \omega)u' \rangle_\rho$$

is a real number. Remark that the support of $(\mathcal{A}_0 - \omega)u'$ is contained in $|x| < L + 2$. Let χ be a function $\chi(x) \in C_0^\infty(\mathbf{R}^3, \mathbf{R})$ such that $\chi(x) = 1$ for $|x| < L + 2$. Then we have

$$(4.30) \quad I = \langle \chi \bar{u}', \mathcal{A}_0 u' \rangle_\rho - \omega |\chi u'|^2.$$

Thus it is sufficient to show that $\langle \chi \bar{u}', \mathcal{A}_0 u' \rangle_\rho$ is real. Since u' satisfies the generalized free boundary-interface condition, we have

$$\begin{aligned}
 (4.31) \quad I &= \int_{\mathbf{R}_-^3} \left(\lambda_1(\nabla \cdot u')(\bar{u}', \nabla \chi) + \mu_1 \sum_{k,j=1}^3 \varepsilon_{kj}(u') \left(\bar{u}'_k \frac{\partial \chi}{\partial x_j} + \bar{u}'_j \frac{\partial \chi}{\partial x_k} \right) \right) dx \\
 &+ \int_{\mathbf{R}_+^3} \left(\lambda_2(\nabla \cdot u')(\bar{u}', \nabla \chi) + \mu_2 \sum_{k,j=1}^3 \varepsilon_{kj}(u') \left(\bar{u}'_k \frac{\partial \chi}{\partial x_j} + \bar{u}'_j \frac{\partial \chi}{\partial x_k} \right) \right) dx \\
 &+ \int_{\mathbf{R}_-^3} \left(\lambda_1 |\chi(\nabla \cdot u')|^2 + 2\mu_1 \sum_{k,j=1}^3 |\chi \varepsilon_{kj}(u')|^2 \right) dx \\
 &+ \int_{\mathbf{R}_+^3} \left(\lambda_2 |\chi(\nabla \cdot u')|^2 + 2\mu_2 \sum_{k,j=1}^3 |\chi \varepsilon_{kj}(u')|^2 \right) dx,
 \end{aligned}$$

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where the third and fourth terms of the right-hand side of (4.31) are real numbers. Note that the first and second terms of the right-hand side of (4.31) are integrated on

$$\text{supp } \nabla \chi \in \{x \in \mathbf{R}^3; L+2 < |x| < L+3\}.$$

Consider $\langle \chi \bar{u}, \mathcal{A}u \rangle_\rho$. From (4.13) and (4.15), we have

$$J = \langle \chi \bar{u}, \mathcal{A}u \rangle_\rho = \langle \chi \bar{u}, \omega u \rangle_\rho = \omega |\chi u|^2.$$

On the other hand

$$\begin{aligned} J &= \left(\int_{\Omega_-} + \int_{\Omega_+} \right) \left(\lambda(x)(\nabla \cdot u)(\nabla \cdot \chi \bar{u}) + 2\mu(x) \sum_{k,j=1}^3 \varepsilon_{kj}(u) \varepsilon_{kj}(\chi \bar{u}) \right) dx \\ &= \int_{\mathbf{R}^3} \left(\lambda(x)(\nabla \cdot u)(\bar{u}, \nabla \chi) + \mu(x) \sum_{k,j=1}^3 \varepsilon_{kj}(u) \left(\bar{u}_k \frac{\partial \chi}{\partial x_j} + \bar{u}_j \frac{\partial \chi}{\partial x_k} \right) \right) dx + G, \end{aligned}$$

where G is a real number. On the support of $\nabla \chi$, we have

$$\lambda(x) = \begin{cases} \lambda_1, & \mu(x) = \begin{cases} \mu_1 & x \in \text{supp } \nabla \chi \cap \mathbf{R}_-^3, \\ \mu_2 & x \in \text{supp } \nabla \chi \cap \mathbf{R}_+^3. \end{cases} \end{cases}$$

So we obtain I is real, because J is real.

Step 4. Finally we prove

$$\|u\|_{\mathcal{A}; -s_1, -s_2} = 1.$$

On the sequence

$$u_n = \phi u_n - \tilde{u} + (1 - \phi)u_n + \tilde{u},$$

$(1 - \phi)u_n$ converges to $(1 - \phi)u$ in $H^{1; -s_1, -s_2}(\Omega, A, \mathbf{C}^3)$, and $\phi u_n - \tilde{u}$ converges to $\phi u - \tilde{u}$ in $H^{2; -s_1, -s_2}(\Omega, \mathbf{C}^3)$. So we have

$$\begin{aligned} \|u_n - u_m\|_{\mathcal{A}; -s_1, -s_2} &\leq \|(\phi u_n - \tilde{u}) - (\phi u_m - \tilde{u})\|_{\mathcal{A}; -s_1, -s_2} \\ &\quad + \|(1 - \phi)(u_n - u_m)\|_{\mathcal{A}; -s_1, -s_2} \\ &\rightarrow 0 \quad \text{as } n, m \rightarrow \infty. \end{aligned}$$

Thus

$$\|u\|_{\mathcal{A}; -s_1, -s_2} = \lim_{n \rightarrow \infty} \|u_n\|_{\mathcal{A}; -s_1, -s_2} = 1.$$

This completes the proof of Proposition 4.1. $\square$

Proposition 4.2. Let $f \in L^{2; s_1, s_2}(\Omega, \mathbf{C}^3)$ and $z \in J^\pm(a, b) \setminus [a, b]$. Then the mapping

$$T : z \rightarrow R(z)f$$

is uniformly continuous in $H^{1; -s_1, -s_2}(\Omega, A, \mathbf{C}^3)$.

This proposition can be proved similarly in [4, Section 3 Proposition 2].

Proof of Main Theorem. The mapping

$$T : z \rightarrow R(z)f$$

is extended from $J^\pm(a, b) \setminus [a, b]$ to $J^\pm(a, b)$, because of the completeness of $H^{1; -s_1, -s_2}(\Omega, A, \mathbf{C}^3)$ and of Proposition 4.2.

Therefore we prove Main Theorem. $\square$

Finally by Theorem 3.1 and Main Theorem, we have some properties of the spectrum of A .

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Theorem 4.3.

1. A has no continuous singular spectrum.
2. If $[a, b]$ is a compact interval contained in $(0, \infty)$, A can only have a finite number of eigenvalues in $[a, b]$, and each of these eigenvalues has a finite multiplicity.

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