

# On Construction of Continuous Functions with Cusp Singularities

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## 1 Introduction

In this paper, we study various constructions of continuous functions on  $\mathbf{R}$  which have the prescribed cusp singularities at each point. As applications, we get a generalization of the result given in our previous paper [7], which discuss the cusp singularities of the classical Weierstrass functions.

Let  $s$  be a positive number, which is not an integer and let  $x_0$  be a point in  $\mathbf{R}^n$ . Then a function  $f$  on  $\mathbf{R}^n$  belongs to the pointwise Hölder space  $C^s(x_0)$ , if there exists a polynomial  $P$  of degree less than  $s$  such that

$$|f(x) - P(x - x_0)| \leq C|x - x_0|^s$$

in a neighborhood of  $x_0$ . The pointwise Hölder exponent of a function  $f$  at a point  $x_0$  in  $\mathbf{R}^n$  is defined as

$$H(f, x_0) = \sup \{s > 0; f \in C^s(x_0)\}.$$

If a continuous function  $f$  does not belong to  $C^s(x_0)$  for every  $s > 0$ , then  $H(f, x_0) = 0$ .

However the pointwise Hölder exponent of a function  $f$  at a point  $x_0$  in  $\mathbf{R}^n$  is not stable under the pseudo-differential operators. Similarly it does not fully characterize the oscillatory behavior on a neighborhood of  $x_0$ . This implies that  $f \in C^s(x_0)$  cannot be characterized by size estimates on the wavelet coefficients of  $f$ .

Here let us recall the definition of the weak scaling exponent characterizing the local oscillatory behavior.

$\mathcal{S}_0(\mathbf{R}^n)$  denotes the closed subspace of the Schwartz class  $\mathcal{S}(\mathbf{R}^n)$  such that

$$\int_{\mathbf{R}^n} x^\alpha \psi(x) dx = 0$$

for every multi-index  $\alpha$  in  $\mathbf{Z}_+^n$ . Then a tempered distribution  $f$  belongs to  $\Gamma^s(x_0)$ , if for every  $\psi$  in  $\mathcal{S}_0(\mathbf{R}^n)$ , there exists a constant  $C(\psi)$  such that

$$\left| \int_{\mathbf{R}^n} f(x) \frac{1}{a^n} \psi\left(\frac{x-x_0}{a}\right) dx \right| \leq C(\psi) a^s, \quad 0 < a \leq 1.$$

The weak scaling exponent of a function  $f$  at a point  $x_0$  in  $\mathbf{R}^n$  is defined as

$$\beta(f, x_0) = \sup \{s \in \mathbf{R}; f \text{ locally belongs to } \Gamma^s(x_0)\}.$$

Since it is known that the pointwise Hölder space  $C^s(x_0)$  is contained in local  $\Gamma^s(x_0)$ , it is obvious that

$$H(f, x_0) \leq \beta(f, x_0).$$

Now we recall the definition of the two-microlocal spaces  $C_{x_0}^{s,s'}$ , which characterize this weak scaling exponent.

Let  $\varphi$  be a function in the Schwartz class  $\mathcal{S}(\mathbf{R}^n)$  such that

$$\hat{\varphi}(\xi) = \begin{cases} 1 & \text{on } |\xi| \leq \frac{1}{2} \\ 0 & \text{on } |\xi| \geq 1 \end{cases},$$

where  $\hat{\varphi}$  is the Fourier transform of  $\varphi$ . For every non-negative integer  $j$ , we define the convolution operator  $S_j(f) = f * \varphi_{\frac{1}{2^j}}$  where  $\varphi_a(x) = \frac{1}{a^n} \varphi(\frac{x}{a})$ , and the difference operator  $\Delta_j = S_{j+1} - S_j$ . Then

$$I = S_0 + \sum_{j=0}^{\infty} \Delta_j.$$

Let  $\psi = \varphi_{\frac{1}{2}} - \varphi$ . Then  $\psi \in \mathcal{S}_0(\mathbf{R}^n)$  and

$$\Delta_j(f) = f * \psi_{\frac{1}{2^j}}.$$

Let  $s$  and  $s'$  be two real numbers and  $x_0$  a point in  $\mathbf{R}^n$ . Then a tempered distribution  $f$  belongs to the two-microlocal spaces  $C_{x_0}^{s,s'}$ , if there exists a constant  $C$  such that

$$|S_0(f)(x)| \leq C(1 + |x - x_0|)^{-s'}$$

and

$$|\Delta_j(f)(x)| \leq C 2^{-js} (1 + 2^j |x - x_0|)^{-s'}$$

for every  $j \in \mathbf{Z}_+$  and  $x \in \mathbf{R}^n$ .

The following remarkable theorems with respect to the two-microlocal spaces  $C_{x_0}^{s,s'}$  and  $\Gamma^s(x_0)$  were given in [5].

**Theorem A** [5, Theorem 1.8.]. *Let  $s$  and  $s'$  be two real numbers and  $x_0$  a point in  $\mathbf{R}^n$  and let us assume two positive integers  $r$  and  $N$  satisfying*

$$r + s + \inf(s', n) > 0$$

and

$$N > \sup(s, s + s').$$

Let  $\psi$  be a function such that

$$|\partial^\alpha \psi(x)| \leq \frac{C(q)}{(1 + |x|)^q}, \quad |\alpha| \leq r, \quad q \geq 1$$

and

$$\int_{\mathbf{R}^n} x^\beta \psi(x) dx = 0, \quad |\beta| \leq N - 1.$$

If a function or a distribution  $f$  belongs to the two-microlocal spaces  $C_{x_0}^{s,s'}$ , then we have

$$\left| \int_{\mathbf{R}^n} f(x) \frac{1}{a^n} \overline{\psi\left(\frac{x-b}{a}\right)} dx \right| \leq C a^s \left(1 + \frac{|b-x_0|}{a}\right)^{-s'}, \quad 0 < a \leq 1, \quad |b-x_0| \leq 1.$$

**Theorem B** [5, Theorem 1.2.]. *Let  $s$  be a real number and let  $f$  be a function or a distribution defined on a neighborhood  $V$  of  $x_0$ .*

*Then  $f$  locally belongs to  $\Gamma^s(x_0)$  if and only if  $f$  locally belongs to the two-microlocal spaces  $C_{x_0}^{s,s'}$  for some  $s'$ .*

Several scientists have been interested in constructing irregular functions. The well-known example is the Weierstrass function [8]. It is an example of a nowhere differentiable continuous function. Hardy gave better estimates of the regularities for the Weierstrass function

$$\mathcal{W}_c(x) = \sum_{n=0}^{\infty} a^n \cos(b^n \pi x) \tag{1}$$

and its sine series

$$\mathcal{W}_s(x) = \sum_{n=0}^{\infty} a^n \sin(b^n \pi x), \tag{2}$$

where  $0 < a < 1$ ,  $b > 1$  and  $ab \geq 1$  [3]. He proved that these functions do not possess finite derivatives at each point  $x$  and showed more precisely that if  $ab > 1$  and  $\xi = \frac{\log(\frac{1}{a})}{\log b}$ , then these functions satisfy

$$\mathcal{W}_c(x+h) - \mathcal{W}_c(x) = O(|h|^\xi) \quad \text{and} \quad \mathcal{W}_s(x+h) - \mathcal{W}_s(x) = O(|h|^\xi)$$

for each  $x$ , but satisfy neither

$$\mathcal{W}_c(x+h) - \mathcal{W}_c(x) = o(|h|^\xi) \quad \text{nor} \quad \mathcal{W}_s(x+h) - \mathcal{W}_s(x) = o(|h|^\xi)$$

for any  $x$ .

Next let us recall the definition of the Takagi function [6]. Let  $\theta^*$  be the 1-periodic function such that

$$\theta^*(x) = \begin{cases} x & \text{if } 0 \leq x < \frac{1}{2} \\ 1-x & \text{if } \frac{1}{2} \leq x < 1 \end{cases}.$$

Then the Takagi function is defined by

$$\mathcal{T}(x) = \sum_{n=0}^{\infty} \frac{\theta^*(2^n x)}{2^n}.$$

It is another example of a nowhere differentiable continuous function.

Using the scaling exponents, Meyer defined two types of singularities of functions as follows [5]: a point  $x_0$  in  $\mathbf{R}^n$  is called a cusp singularity of a function  $f$ , when

$$H(f, x_0) = \beta(f, x_0) < \infty,$$

while a point  $x_0$  in  $\mathbf{R}^n$  is called an oscillating singularity of a function  $f$ , when

$$H(f, x_0) < \beta(f, x_0).$$

When a point  $x_0$  is a cusp singularity of a function  $f$ , the pointwise Hölder exponent can be found by computing the size estimates on the wavelet coefficients of  $f$  inside the influence cone. Using this fact, we construct continuous functions which have a prescribed cusp singularity at each point  $x_0$  in  $\mathbf{R}$ .

Daoudi and his team [2] studied the following problem which was raised by Lévy Véhel:

*Let  $s$  be a function from  $[0, 1]$  to  $[0, 1]$ . Under what conditions on  $s$  does there exist a continuous function  $f$  from  $[0, 1]$  to  $\mathbf{R}$  such that  $H(f, x) = s(x)$  for all  $x$  in  $[0, 1]$ ?*

They solved the problem as follows: "For a function  $s$  from  $[0, 1]$  to  $[0, 1]$ , there exist a continuous function  $f$  on  $[0, 1]$  such that  $H(f, x) = s(x)$  for all  $x$  in  $[0, 1]$  if and only if  $s$  is a function which can be represented as a limit inferior of a sequence of continuous functions

on  $[0, 1]$ ." Further, they constructed such  $f$  by various methods, - as the Weierstrass type function, using Schauder bases and using Iterated Function System.

On the other hand, Andersson [1] proved a similar characterization for a function  $s$  from  $\mathbf{R}$  to  $[0, \infty]$  and constructed  $f$  satisfying  $H(f, x) = s(x)$  for all  $x$  in  $\mathbf{R}$  by a method using orthogonal wavelets.

In the rest of the paper we study, for a given function on  $\mathbf{R}$ , various constructions of a function  $f$  satisfying

$$H(f, x) = \beta(f, x) = s(x), \quad x \in \mathbf{R},$$

using orthonormal wavelets in Section 2 and as the Weierstrass type function in Section 3.

## 2 Construction Using Orthonormal Wavelets

In this section, using orthonormal wavelets, we construct a continuous function which has a prescribed cusp singularity at each point in  $\mathbf{R}$ .

The following Lemma 1 is used in the proof of Theorems 1 and 2.

**Lemma 1.** *Let  $s$  be a function from  $\mathbf{R}$  to  $[0, \infty]$ , which is the lower limit of a sequence of real continuous functions  $\{t_l\}_{l \in \mathbf{N}}$ . Then there exists a sequence  $\{s_l\}_{l \in \mathbf{Z}_+}$  of infinitely differentiable non-negative functions with compact supports such that*

$$(i) \quad s(x) = \liminf_{l \rightarrow \infty} s_l(x), \quad x \in \mathbf{R},$$

(ii) *For each  $x_0$  in  $\mathbf{R}$ , there exists a positive integer  $l_0$  such that*

$$s_l(x) \geq \frac{1}{\sqrt{l+1}}, \quad l \geq l_0, \quad |x - x_0| \leq 1.$$

(iii) *There exists a sequence  $\{C_k\}_{k \in \mathbf{Z}_+} \subset (0, \infty)$  such that*

$$\sup_{x \in \mathbf{R}} |s_l^{(k)}(x)| \leq C_k l^{k+1}, \quad l \in \mathbf{Z}_+,$$

where  $s_l^{(k)}$  is the  $k$ -th derivative of  $s_l$ .

**Proof.** Let  $\eta$  be a non-negative infinitely differentiable function supported on  $[-1, 1]$  satisfying  $\eta(x) = 1$  if  $|x| \leq \frac{1}{4}$ ,  $\sup_{x \in \mathbf{R}} \eta(x) = 1$  and  $\int_{\mathbf{R}} \eta(x) dx = 1$ . If we put

$$\tilde{t}_l(x) = \eta\left(\frac{x}{l}\right) \min\left(\max\left(t_l(x), \frac{1}{\sqrt{l+1}}\right), l\right), \quad l \in \mathbf{N},$$

it is easy to see that  $\{\tilde{t}_l\}_{l \in \mathbf{N}}$  satisfies

$$\liminf_{l \rightarrow \infty} \tilde{t}_l(x) = s(x), \quad x \in \mathbf{R},$$

$$\tilde{t}_l(x) \geq \frac{1}{\sqrt{l+1}}, \quad |x| \leq \frac{l}{4},$$

$$\tilde{t}_l(x) = 0, \quad |x| \geq l$$

and

$$\sup_{x \in \mathbf{R}} \tilde{t}_l(x) \leq l.$$

Since each  $\tilde{t}_l$  is uniformly continuous, we can choose a strictly increasing sequence of positive integers  $\{p_l\}_{l \in \mathbf{N}}$  such that

$$\sup_{|x-y| \leq \frac{1}{p_l}} |\tilde{t}_l(x) - \tilde{t}_l(y)| \leq \frac{1}{l}, \quad l \in \mathbf{N}.$$

Under these circumstances, we define  $s_l(x)$  for  $l \in \mathbf{Z}_+$  and  $x \in \mathbf{R}$  by

$$s_l(x) = \begin{cases} 0 & \text{if } 0 \leq l < p_1 \\ \int_{\mathbf{R}} p_m \eta(p_m(x-y)) \tilde{t}_m(y) dy & \text{if } p_m \leq l < p_{m+1}, \quad m \in \mathbf{N}. \end{cases}$$

If we put  $C_k = \int_{\mathbf{R}} |\eta^{(k)}(x)| dx$  for  $k \in \mathbf{Z}_+$ , then  $\{s_l\}_{l \in \mathbf{Z}_+}$  satisfies the required properties (i), (ii) and (iii). To prove (i) we have

$$\begin{aligned} |s_l(x) - \tilde{t}_m(x)| &= \left| \int_{\mathbf{R}} p_m \eta(p_m(x-y)) (\tilde{t}_m(y) - \tilde{t}_m(x)) dy \right| \\ &\leq \sup_{|x-y| \leq \frac{1}{p_m}} |\tilde{t}_m(y) - \tilde{t}_m(x)| \int_{\mathbf{R}} \eta(y) dy \\ &\leq \frac{1}{m}, \quad p_m \leq l < p_{m+1}. \end{aligned}$$

This proves the desired result. To prove (ii) we choose  $m_0 \in \mathbf{N}$  such that  $\frac{m_0}{4} - \frac{1}{m_0} \geq |x_0| + 1$  and put  $l_0 = p_{m_0}$ . For a positive integer  $l \geq l_0$ , choose  $m \in \mathbf{N}$  such that  $p_m \leq l < p_{m+1}$ . Then if  $|x - x_0| \leq 1$ , we have

$$\begin{aligned} s_l(x) &= \int_{\mathbf{R}} p_m \eta(p_m(x-y)) \tilde{t}_m(y) dy \\ &\geq \inf_{|x-y| \leq \frac{1}{p_m}} \tilde{t}_m(y) \int_{\mathbf{R}} \eta(y) dy \end{aligned}$$

$$\begin{aligned}
&\geq \inf_{|y| \leq |x_0| + 1 + \frac{1}{m}} \tilde{t}_m(y) \\
&\geq \inf_{|y| \leq \frac{m}{4}} \tilde{t}_m(y) \\
&\geq \frac{1}{\sqrt{m+1}} \geq \frac{1}{\sqrt{l+1}}.
\end{aligned}$$

To prove (iii) we choose  $m \in \mathbf{N}$ , for a given  $l \in \mathbf{N}$ , such that  $p_m \leq l < p_{m+1}$ . Then we have

$$\begin{aligned}
|s_l^{(k)}(x)| &= \left| \int_{\mathbf{R}} p_m^{k+1} \eta^{(k)}(p_m(x-y)) \tilde{t}_m(y) dy \right| \\
&\leq p_m^k \sup_{|x-y| \leq \frac{1}{p_m}} \tilde{t}_m(y) \int_{\mathbf{R}} |\eta^{(k)}(y)| dy \\
&\leq C_k m p_m^k \leq C_k l^{k+1}.
\end{aligned}$$

■

**Theorem 1.** *Let  $s$  be a function from  $\mathbf{R}$  to  $[0, \infty]$ , which is the lower limit of a sequence of continuous functions. Then there exists a sequence  $\{s_l\}_{l \in \mathbf{Z}_+}$  of differentiable functions such that*

$$s(x) = \liminf_{l \rightarrow \infty} s_l(x), \quad x \in \mathbf{R} \quad (3)$$

and

$$\sup_{x \in \mathbf{R}} |s'_l(x)| \leq C_1 l^2, \quad l \in \mathbf{Z}_+. \quad (4)$$

Let  $\psi$  be an orthonormal wavelet in the Schwartz class  $\mathcal{S}(\mathbf{R})$ . If we define a continuous function  $f$  by

$$f(x) = \sum_{l=2}^{\infty} \sum_{m=0}^{\infty} c(l, m) \psi(2^l x - m),$$

where

$$c(l, m) = \min(2^{-ls_l(\frac{m}{2^l})}, 2^{-\frac{l}{\log l}}),$$

then we have

$$H(f, x_0) = \beta(f, x_0) = s(x_0)$$

at each point  $x_0$  in  $\mathbf{R}$ .

**Proof.** The existence of  $\{s_l\}_{l \in \mathbf{Z}_+}$  satisfying (3) and (4) follows from Lemma 1. Since

$$\begin{aligned} \lim_{j \rightarrow \infty} \sup_{|x-y| \leq 2^{-\frac{j}{(\log j)^2}}} |s_j(x) - s_j(y)| &\leq \lim_{j \rightarrow \infty} \sup_{x \in \mathbf{R}} |s'_j(x)| \sup_{|x-y| \leq 2^{-\frac{j}{(\log j)^2}}} |x-y| \\ &\leq C_1 \lim_{j \rightarrow \infty} j^2 2^{-\frac{j}{(\log j)^2}} \\ &= 0, \end{aligned}$$

$H(f, x_0) = s(x_0)$  at each point  $x_0 \in \mathbf{R}$  (cf. [1] p.441, proof of Theorem 1.). We only need to compute the value of  $\beta(f, x_0)$ .

Let us assume  $f$  locally belongs to  $\Gamma^s(x_0)$ . Then by Theorem B,  $f$  locally belongs to  $C_{x_0}^{s, s'}$  for some  $s' < 0$ . On the other hand,  $\psi \in \mathcal{S}_0(\mathbf{R})$  (cf. [4, 2. Corollary 3.7.]). By Theorem A, there exist two constants  $C \in (0, \infty)$  and  $\delta \in (0, \frac{1}{2})$  such that

$$\left| \int f(x) \frac{1}{a} \overline{\psi\left(\frac{x-b}{a}\right)} dx \right| \leq C a^s \left(1 + \frac{|b-x_0|}{a}\right)^{-s'}, \quad 0 < a \leq \delta, \quad |b-x_0| \leq \delta. \quad (5)$$

Let  $j_0$  be a positive integer such that  $\frac{1}{2^{j_0}} \leq \delta$ . For every  $j \geq j_0$ , there exists  $k_j \in \mathbf{Z}$  such that  $\frac{k_j}{2^j} \leq x_0 < \frac{k_j+1}{2^j}$  and we define  $a_j$  and  $b_j$  by  $a_j = \frac{1}{2^j}$  and  $b_j = \frac{k_j}{2^j}$ . Then  $|b_j - x_0| \leq a_j$  and by (5), we have

$$\left| \int f(x) 2^j \overline{\psi(2^j x - k_j)} dx \right| \leq \frac{C 2^{-s'}}{2^{js}}, \quad j \geq j_0. \quad (6)$$

We estimate the left hand side of (6) as follows:

$$\begin{aligned} \left| \int f(x) 2^j \overline{\psi(2^j x - k_j)} dx \right| &= \left| \sum_{l=2}^{\infty} \sum_{m=-\infty}^{\infty} c(l, m) \int \psi(2^l x - m) 2^j \overline{\psi(2^j x - k_j)} dx \right| \\ &= c(j, k_j). \end{aligned} \quad (7)$$

By (6) and (7),  $f \in \Gamma^s(x_0)$  implies

$$c(j, k_j) = \min(2^{-js_j(\frac{k_j}{2^j})}, 2^{-\frac{j}{\log j}}) \leq \frac{C 2^{-s'}}{2^{js}}, \quad j \geq j_0. \quad (8)$$

Observe that

$$\begin{aligned} \lim_{j \rightarrow \infty} \left| s_j\left(\frac{k_j}{2^j}\right) - s_j(x_0) \right| &\leq \lim_{j \rightarrow \infty} \sup_{x \in \mathbf{R}} |s'_j(x)| \left(x_0 - \frac{k_j}{2^j}\right) \\ &\leq C_1 \lim_{j \rightarrow \infty} \frac{j^2}{2^j} \\ &= 0. \end{aligned}$$

By (8), we have

$$\begin{aligned}
s &\leq \liminf_{j \rightarrow \infty} \max \left( s_j \left( \frac{k_j}{2^j} \right), \frac{1}{\log j} \right) \\
&= \liminf_{j \rightarrow \infty} s_j \left( \frac{k_j}{2^j} \right) \\
&= \liminf_{j \rightarrow \infty} s_j(x_0) + \lim_{j \rightarrow \infty} \left( s_j \left( \frac{k_j}{2^j} \right) - s_j(x_0) \right) \\
&= s(x_0).
\end{aligned}$$

Therefore  $\beta(f, x_0) \leq s(x_0) = H(f, x_0)$ . Since  $H(f, x_0) \leq \beta(f, x_0)$  is trivial, we have  $H(f, x_0) = \beta(f, x_0) = s(x_0)$ . ■

### 3 Use of Weierstrass Type Functions

In this section, we construct the Weierstrass type continuous function which has a prescribed cusp singularity at each point in  $\mathbf{R}$ .

We begin with the following lemma.

**Lemma 2.** *Let  $s \in [0, \infty]$ ,  $l_0 \in \mathbf{Z}_+$  and  $\{s_l\}_{l \in \mathbf{Z}_+} \subset \mathbf{R}$  be such that*

- (a)  $\liminf_{l \rightarrow \infty} s_l = s$ ,
- (b)  $s_l \geq \frac{1}{\sqrt{l+1}}$ ,  $l \geq l_0$ .

*Suppose  $\lambda > 1$  and  $\{\theta_l\}_{l \in \mathbf{Z}_+} \subset \mathbf{R}$  are chosen arbitrary.*

(i) *If  $m \in \mathbf{Z}_+$  and  $\{\alpha_l\}_{l \in \mathbf{Z}_+}$  is a bounded sequence in  $\mathbf{R}$  and if we define a continuous function  $f$  by*

$$f(x) = \sum_{l=0}^{\infty} \frac{\alpha_l l^m}{\lambda^{ls_l}} \sin(\lambda^l x + \theta_l), \quad x \in \mathbf{R},$$

*then we have*

$$H(f, x_0) \geq s$$

*at each point  $x_0$  in  $\mathbf{R}$ .*

(ii) *If we define a continuous function  $g$  by*

$$g(x) = \sum_{l=0}^{\infty} \frac{1}{\lambda^{ls_l}} \sin(\lambda^l x + \theta_l), \quad x \in \mathbf{R},$$

*then we have*

$$H(g, x_0) = \beta(g, x_0) = s$$

at each point  $x_0$  in  $\mathbf{R}$ .

**Proof.** (i) By (b),  $f$  is a continuous function on  $\mathbf{R}$  and hence we have only to show (i) when  $s > 0$ .

Let  $x_0 \in \mathbf{R}$  be fixed arbitrary.

First, we consider the case  $0 < s \leq 1$ . Let  $\varepsilon \in (0, s)$  be arbitrary. By (a), we can choose  $l_0 \in \mathbf{Z}_+$  such that  $s_l > s - \frac{\varepsilon}{2}$  for  $l \geq l_0$  and we put  $f_1(x) = \sum_{l=l_0}^{\infty} \frac{\alpha_l l^m}{\lambda^{ls_l}} \sin(\lambda^l x + \theta_l)$ . To show  $H(f, x_0) \geq s - \varepsilon$ , it suffices to show  $f_1 \in C^{s-\varepsilon}(x_0)$  since  $H(f - f_1, x_0) = \infty$  is obvious. Let  $x$  be a real number such that  $|x - x_0| < \frac{1}{\lambda^{l_0}}$  and choose  $N \in \mathbf{Z}_+$  such that  $\frac{1}{\lambda^{N+1}} \leq |x - x_0| < \frac{1}{\lambda^N}$ . Then we have

$$\begin{aligned} |f_1(x) - f_1(x_0)| &= \left| \sum_{l=l_0}^{\infty} \frac{\alpha_l l^m}{\lambda^{ls_l}} (\sin(\lambda^l x + \theta_l) - \sin(\lambda^l x_0 + \theta_l)) \right| \\ &\leq \left| \sum_{l=l_0}^{N-1} \frac{\alpha_l l^m}{\lambda^{ls_l}} (\sin(\lambda^l x + \theta_l) - \sin(\lambda^l x_0 + \theta_l)) \right| \\ &\quad + \left| \sum_{l=N}^{\infty} \frac{\alpha_l l^m}{\lambda^{ls_l}} (\sin(\lambda^l x + \theta_l) - \sin(\lambda^l x_0 + \theta_l)) \right| \\ &= A_1 + A_2. \end{aligned} \tag{9}$$

Observe first that there exists a constant  $M_1 \in (0, \infty)$  such that

$$|\alpha_l| l^m \leq M_1 \lambda^{\frac{ls_l}{2}}, \quad l \geq l_0. \tag{10}$$

To estimate  $A_1$  and  $A_2$  we use (10) to obtain

$$\begin{aligned} A_1 &\leq 2 \sum_{l=l_0}^{N-1} \frac{|\alpha_l| l^m}{\lambda^{ls_l}} \left| \cos \left( \frac{\lambda^l (x + x_0)}{2} + \theta_l \right) \sin \left( \frac{\lambda^l (x - x_0)}{2} \right) \right| \\ &\leq \sum_{l=l_0}^{N-1} |\alpha_l| l^m \lambda^{l(1-s_l)} |x - x_0| \\ &\leq M_1 \sum_{l=l_0}^{N-1} \lambda^{l(1-s+\varepsilon)} |x - x_0| \\ &= \frac{M_1 \lambda^{l_0(1-s+\varepsilon)} (\lambda^{(N-l_0)(1-s+\varepsilon)} - 1)}{\lambda^{1-s+\varepsilon} - 1} |x - x_0| \\ &\leq \frac{M_1 \lambda^{N(1-s+\varepsilon)}}{\lambda^{1-s+\varepsilon} - 1} |x - x_0| \\ &\leq \frac{M_1}{\lambda^{1-s+\varepsilon} - 1} |x - x_0|^{s-\varepsilon}, \\ A_2 &\leq 2 \sum_{l=N}^{\infty} \frac{|\alpha_l| l^m}{\lambda^{ls_l}} \left| \cos \left( \frac{\lambda^l (x + x_0)}{2} + \theta_l \right) \sin \left( \frac{\lambda^l (x - x_0)}{2} \right) \right| \end{aligned}$$

$$\begin{aligned}
&\leq 2 \sum_{l=N}^{\infty} \frac{|\alpha_l| l^m}{\lambda^{ls_l}} \\
&\leq 2M_1 \sum_{l=N}^{\infty} \frac{1}{\lambda^{l(s-\varepsilon)}} \\
&= \frac{2M_1}{\lambda^{N(s-\varepsilon)}} \frac{1}{1 - \frac{1}{\lambda^{s-\varepsilon}}} \\
&\leq \frac{2M_1 \lambda^{2(s-\varepsilon)}}{\lambda^{s-\varepsilon} - 1} |x - x_0|^{s-\varepsilon}.
\end{aligned}$$

The estimates for  $A_1$  and  $A_2$  with (9) show that there exists a constant  $M_2 \in (0, \infty)$  such that

$$|f_1(x) - f_1(x_0)| \leq M_2 |x - x_0|^{s-\varepsilon}, \quad |x - x_0| < \frac{1}{\lambda^{l_0}}.$$

Thus  $H(f_1, x_0) \geq s - \varepsilon$  and hence  $H(f, x_0) \geq s - \varepsilon$ . Since  $\varepsilon > 0$  is arbitrary,  $H(f, x_0) \geq s$ .

Next, we consider the case  $n < s \leq n + 1$  for some  $n \in \mathbf{N}$ . In this case,  $f$  is  $n$ -times continuously differentiable on  $\mathbf{R}$  and we have

$$f^{(n)}(x) = \sum_{l=0}^{\infty} \frac{\alpha_l l^m}{\lambda^{l(s_l-n)}} \sin \left( \lambda^l x + \theta_l + \frac{n\pi}{2} \right).$$

Thus  $H(f^{(n)}, x_0) \geq s - n$  by an argument similar to the case where  $0 < s \leq 1$  and hence  $H(f, x_0) \geq s$  holds even for  $1 < s < \infty$ .

Finally, we consider the case  $s = \infty$ . In this case,  $f$  is obviously infinitely differentiable at  $x_0$  and hence  $H(f, x_0) = \infty$ .

(ii)  $H(g, x_0) \geq s$  follows from (i), if we put  $\alpha_l = 1$  for  $l \in \mathbf{Z}_+$  and  $m = 0$  in (i).

For  $\beta(g, x_0)$ , let us assume  $g$  locally belongs to  $\Gamma^\rho(x_0)$ . Let  $\psi$  be a function in  $\mathcal{S}_0(\mathbf{R})$  such that  $\hat{\psi}(\xi) = 0$  if  $|\xi - 1| \geq \frac{\lambda-1}{\lambda}$  and  $\hat{\psi}(1) = 2$ . Then there exist two constants  $M_3 \in (0, \infty)$  and  $\eta \in (0, 1]$  such that

$$\left| \int g(x) \frac{1}{a} \psi \left( \frac{x - x_0}{a} \right) dx \right| \leq M_3 a^\rho, \quad 0 < a \leq \eta. \quad (11)$$

Let  $j_0$  be a non-negative integer such that  $\frac{1}{\lambda^{j_0}} \leq \eta$ . For every  $j \geq j_0$ , we put  $a_j = \frac{1}{\lambda^j}$ . By (11), we have

$$\left| \int g(x) \lambda^j \psi(\lambda^j(x - x_0)) dx \right| \leq \frac{M_3}{\lambda^{j\rho}}, \quad j \geq j_0. \quad (12)$$

We estimate the left hand side of (12) as follows:

$$\left| \int g(x) \lambda^j \psi(\lambda^j(x - x_0)) dx \right| = \left| \int \sum_{l=0}^{\infty} \frac{1}{\lambda^{ls_l}} \sin(\lambda^{l-j}x + \lambda^l x_0 + \theta_l) \psi(x) dx \right|$$

$$\begin{aligned}
&= \left| \sum_{l=0}^{\infty} \frac{1}{\lambda^{ls_l}} \int \frac{e^{i(\lambda^{l-j}x + \lambda^l x_0 + \theta_l)} - e^{-i(\lambda^{l-j}x + \lambda^l x_0 + \theta_l)}}{2i} \psi(x) dx \right| \\
&= \left| \sum_{l=0}^{\infty} \frac{e^{i(\lambda^l x_0 + \theta_l)} \hat{\psi}(-\lambda^{l-j}) - e^{-i(\lambda^l x_0 + \theta_l)} \hat{\psi}(\lambda^{l-j})}{2i\lambda^{ls_l}} \right| \\
&= \frac{|\hat{\psi}(1)|}{2\lambda^{js_j}} \\
&= \frac{1}{\lambda^{js_j}}.
\end{aligned} \tag{13}$$

By (12) and (13),  $g \in \Gamma^\rho(x_0)$  implies  $\frac{1}{\lambda^{js_j}} \leq \frac{M_3}{\lambda^{j\rho}}$  for every  $j \geq j_0$  and hence  $\rho \leq \liminf_{j \rightarrow \infty} s_j = s \leq H(g, x_0)$ . Therefore  $\beta(g, x_0) \leq s \leq H(g, x_0)$ . Since  $H(g, x_0) \leq \beta(g, x_0)$  is trivial, we have  $H(g, x_0) = \beta(g, x_0) = s$ .  $\blacksquare$

**Theorem 2.** Let  $s$  be a function from  $\mathbf{R}$  to  $[0, \infty]$ , which is the lower limit of a sequence of continuous functions and let  $\{s_l\}_{l \in \mathbf{Z}_+}$  be a sequence of continuous functions satisfying part (i), (ii) and (iii) of Lemma 1.

Suppose  $\lambda > 1$  and  $\{\theta_l\}_{l \in \mathbf{Z}_+} \subset \mathbf{R}$  are chosen arbitrary. If we define a continuous function  $f$  by

$$f(x) = \sum_{l=0}^{\infty} \frac{1}{\lambda^{ls_l(x)}} \sin(\lambda^l x + \theta_l),$$

then we have

$$H(f, x_0) = \beta(f, x_0) = s(x_0)$$

at each point  $x_0$  in  $\mathbf{R}$ .

**Proof.** First, we consider the case  $n \leq s(x_0) < n+1$  for some  $n \in \mathbf{Z}_+$ . Using the Taylor expansion we have

$$\begin{aligned}
\frac{1}{\lambda^{ls_l(x)}} &= \frac{1}{\lambda^{ls_l(x_0)}} + \sum_{j=1}^n \frac{1}{j!} \frac{d^j}{dx^j} \frac{1}{\lambda^{ls_l(x)}} \Big|_{x=x_0} (x - x_0)^j \\
&\quad + \frac{1}{(n+1)!} \frac{d^{n+1}}{dx^{n+1}} \frac{1}{\lambda^{ls_l(x)}} \Big|_{x=\xi_l} (x - x_0)^{n+1},
\end{aligned} \tag{14}$$

where  $\xi_l \in (\min(x, x_0), \max(x, x_0))$ . It goes without saying that if  $n = 0$  the second term in the right hand side of (14) does not appear. By (14), we can write

$$f(x) = \sum_{l=0}^{\infty} \frac{1}{\lambda^{ls_l(x)}} \sin(\lambda^l x + \theta_l) = f_1(x) + f_2(x) + f_3(x), \tag{15}$$

$$f_1(x) = \sum_{l=0}^{\infty} \frac{1}{\lambda^{ls_l(x_0)}} \sin(\lambda^l x + \theta_l), \quad (16)$$

$$f_2(x) = \sum_{l=0}^{\infty} \sum_{j=1}^n \frac{1}{j!} \frac{d^j}{dx^j} \left. \frac{1}{\lambda^{ls_l(x)}} \right|_{x=x_0} \sin(\lambda^l x + \theta_l) (x - x_0)^j \quad (17)$$

and

$$f_3(x) = \frac{1}{(n+1)!} \sum_{l=0}^{\infty} \frac{d^{n+1}}{dx^{n+1}} \left. \frac{1}{\lambda^{ls_l(x)}} \right|_{x=\xi_l} \sin(\lambda^l x + \theta_l) (x - x_0)^{n+1}, \quad (18)$$

where  $\xi_l \in (\min(x, x_0), \max(x, x_0))$ .

By part (ii) of Lemma 2,  $H(f_1, x_0) = \beta(f_1, x_0) = s(x_0)$  follows at once.  $f_2$  does not appear if  $n = 0$ , and if  $n \geq 1$  we have

$$f_2(x) = \sum_{l=0}^{\infty} \sum_{j=1}^n \sum_{k=1}^j \sum_{(*)_j} \frac{1}{j!} \frac{(-\log \lambda)^{kl^k} \alpha_{j,i_1,\dots,i_k} s_l^{(i_1)}(x_0) \dots s_l^{(i_k)}(x_0)}{\lambda^{ls_l(x_0)}} \sin(\lambda^l x + \theta_l) (x - x_0)^j, \quad (19)$$

where  $\sum_{(*)_j}$  mean the summation under the condition  $i_1 + \dots + i_k = j$  with  $i_1 \leq \dots \leq i_k$  and  $\{\alpha_{j,i_1,\dots,i_k}\}$  are positive integers satisfying  $\sum_{(*)_j} \alpha_{j,i_1,\dots,i_k} \leq (k+1)^j$ . By (19), part (iii) of Lemma 1 and part (i) of Lemma 2, we can deduce that  $H(f_2, x_0) \geq s(x_0) + 1$ . For  $f_3$ , we have

$$f_3(x) = \frac{1}{(n+1)!} \sum_{l=0}^{\infty} \sum_{k=1}^{n+1} \sum_{(*)_{n+1}} \frac{(-\log \lambda)^{kl^k} \alpha_{n+1,i_1,\dots,i_k} s_l^{(i_1)}(\xi_l) \dots s_l^{(i_k)}(\xi_l)}{\lambda^{ls_l(\xi_l)}} \sin(\lambda^l x + \theta_l) (x - x_0)^{n+1}, \quad (20)$$

where  $\sum_{(*)_{n+1}}$  mean the summation under the condition  $i_1 + \dots + i_k = n+1$  with  $i_1 \leq \dots \leq i_k$  and  $\{\alpha_{n+1,i_1,\dots,i_k}\}$  are positive integers satisfying  $\sum_{(*)_{n+1}} \alpha_{n+1,i_1,\dots,i_k} \leq (k+1)^{n+1}$ . By (20) and part (iii) of Lemma 1, we can deduce that  $H(f_3, x_0) \geq n+1$ . By the estimates for  $f_1$ ,  $f_2$  and  $f_3$ , and (15), we can conclude that  $H(f, x_0) = \beta(f, x_0) = s(x_0)$ .

Next, we consider the case  $s(x_0) = \infty$ . Let  $n$  be a positive integer and let  $f = f_1 + f_2 + f_3$ , where  $f_1$ ,  $f_2$  and  $f_3$  are defined by (16), (17) and (18), respectively. But in this case, we have  $H(f_1, x_0) = H(f_2, x_0) = \infty$  and  $H(f_3, x_0) \geq n+1$  by part (iii) of Lemma 1 and part (i) of Lemma 2, since  $\liminf_{l \rightarrow \infty} s_l(x_0) = \infty$ . By the estimates for  $f_1$ ,  $f_2$  and  $f_3$ , and (15), we have  $H(f, x_0) \geq n+1$ . Since  $n$  is arbitrary, we can conclude that  $H(f, x_0) = \beta(f, x_0) = s(x_0)$  even for  $s(x_0) = \infty$ .  $\blacksquare$

In the case where  $s$  is a continuous function, we have the following result.

**Theorem 3.** *Let  $s$  be a continuous function from  $\mathbf{R}$  to  $(0, \infty)$  such that*

$$s(x_0) < H(s, x_0)$$

*at each point  $x_0$  in  $\mathbf{R}$ . Suppose  $\lambda > 1$  and  $\{\theta_l\}_{l \in \mathbf{Z}_+} \subset \mathbf{R}$  are chosen arbitrary. If we define a continuous function  $f$  by*

$$f(x) = \sum_{l=0}^{\infty} \frac{1}{\lambda^{ls(x)}} \sin(\lambda^l x + \theta_l),$$

*then we have*

$$H(f, x_0) = \beta(f, x_0) = s(x_0)$$

*at each point  $x_0$  in  $\mathbf{R}$ .*

**Proof.** Let  $x_0 \in \mathbf{R}$  be fixed arbitrary and let  $x$  be a real number such that  $|x - x_0| < 1$ . Then we have

$$\begin{aligned} f(x) &= \sum_{l=0}^{\infty} \frac{1}{\lambda^{ls(x_0)}} \sin(\lambda^l x + \theta_l) + \sum_{l=0}^{\infty} \left( \frac{1}{\lambda^{ls(x)}} - \frac{1}{\lambda^{ls(x_0)}} \right) \sin(\lambda^l x + \theta_l) \\ &= f_1(x) + f_2(x). \end{aligned} \tag{21}$$

By part (ii) of Lemma 2,  $H(f_1, x_0) = \beta(f_1, x_0) = s(x_0)$  follows at once. Let  $\varepsilon$  be a positive number such that  $s(x_0) + \varepsilon < H(s, x_0)$  and  $s(x_0) + \varepsilon \notin \mathbf{N}$ . Then  $s \in C^{s(x_0)+\varepsilon}(x_0)$  and there exist a polynomial  $P$  of degree at most  $[s(x_0) + \varepsilon]$ , two constants  $C \in (0, \infty)$  and  $\delta \in (0, 1)$  such that

$$s(x) = s(x_0) + P(x - x_0) + Q(x - x_0)$$

and

$$|Q(x - x_0)| \leq C|x - x_0|^{s(x_0)+\varepsilon}, \quad |x - x_0| \leq \delta.$$

To estimate  $f_2$ , using the mean value theorem, we write

$$\frac{1}{\lambda^{ls(x)}} - \frac{1}{\lambda^{ls(x_0)}} = \frac{(-\log \lambda)l(s(x) - s(x_0))}{\lambda^{l\tau_l}},$$

where  $\tau_l \in [\min(s(x), s(x_0)), \max(s(x), s(x_0))]$ . Then we have

$$\left| f_2(x) - \left( (-\log \lambda) \sum_{l=0}^{\infty} \frac{l}{\lambda^{l\tau_l}} \sin(\lambda^l x + \theta_l) \right) P(x - x_0) \right|$$

$$\begin{aligned}
&= (\log \lambda) \left| \sum_{l=0}^{\infty} \frac{l}{\lambda^{l\tau}} \sin(\lambda^l x + \theta_l) \right| |Q(x - x_0)| \\
&\leq C(\log \lambda) \sum_{l=0}^{\infty} \frac{l}{\lambda^{l\tau}} |x - x_0|^{s(x_0) + \varepsilon}.
\end{aligned}$$

Hence  $H(f_2, x_0) \geq s(x_0) + \varepsilon$ . By the estimates for  $f_1$  and  $f_2$ , and (21), we can conclude that  $H(f, x_0) = \beta(f, x_0) = s(x_0)$ . ■

**Corollary 1.** *Each point in  $\mathbf{R}$  is a cusp singularity of the Weierstrass functions.*

**Proof.** Let  $\mathcal{W}_c$  and  $\mathcal{W}_s$  be the Weierstrass functions (for the definitions of  $\mathcal{W}_c$  and  $\mathcal{W}_s$ , see (1) and (2)). If we put  $\lambda = b$ ,  $s(x) = \frac{\log(\frac{1}{a})}{\log b}$  and  $\theta_l = \frac{\pi}{2}$  for  $l \in \mathbf{Z}_+$  or  $\theta_l = 0$  for  $l \in \mathbf{Z}_+$ , then we have  $H(\mathcal{W}_c, x) = \beta(\mathcal{W}_c, x) = \frac{\log(\frac{1}{a})}{\log b} = H(\mathcal{W}_s, x) = \beta(\mathcal{W}_s, x)$  at each point  $x$  in  $\mathbf{R}$  from Theorem 3. ■

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