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BSPFA Combined with One Measurable Cardinal

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Abstract

We consider consequences of BSPFA (Bounded Semi-Proper Forcing Axiom) combined with an existence of a measurable cardinal. The large cardinal assures existences of relevant semiproper preorders via Chang's Conjecture-type arguments.

Introduction

In [T], a new combinatorial principle θ_{AC} is introduced. We recall its definition.

Definition. ([T]) θ_{AC} holds, if for every one-to-one list $\mathbf{r} = \langle r_i \mid i < \omega_1 \rangle$ in ${}^{\omega}2$ and every $S \subseteq \omega_1$, there exist ordinals $\gamma > \beta > \alpha \geq \omega_1$ and an increasing continuous decomposition $\gamma = \bigcup\{N_\nu \mid \nu < \omega_1\}$ of the ordinal γ into countable sets such that for all $\nu < \omega_1$, $N_\nu \cap \omega_1 \in S$ if and only if the following holds, where $i = \mathrm{o.t.}(N_\nu \cap \alpha)$, $j = \mathrm{o.t.}(N_\nu \cap \beta)$ and $k = \mathrm{o.t.}(N_\nu)$,

$$\Delta(r_i, r_j) = \mathrm{Max}\,\{\Delta(r_i, r_j), \Delta(r_i, r_k), \Delta(r_j, r_k)\}.$$

The notation $\Delta(r, r')$ stands for the least $n < \omega$ such that $r(n) \neq r'(n)$ for $r, r' \in {}^{\omega}2$ with $r \neq r'$. We also recall.

Definition. *BMM (Bounded Martin's Maximum)* holds, if for any $A \in H^V_{\omega_2^V}$ and any Σ_0-formula φ, if $\Vdash_P$ "$\exists y\, \varphi(y, A)$ in $H^{V[\dot{G}]}_{\omega_2^{V[\dot{G}]}}$" holds for some preorder P which preserves every stationary subset of ω_1, then we already have $\exists y\, \varphi(y, A)$ in $H^V_{\omega_2^V}$.

We may formulate a weaker forcing axiom by restricting the class of preorders to the semiproper ones.

Definition. *BSPFA (Bounded Semi-Proper Forcing Axiom)* holds, if for any $A \in H^V_{\omega_2^V}$ and any Σ_0-formula φ, if $\Vdash_P$ "$\exists y\, \varphi(y, A)$ in $H^{V[\dot{G}]}_{\omega_2^{V[\dot{G}]}}$" holds for some preorder P which is semiproper, then we already have $\exists y\, \varphi(y, A)$ in $H^V_{\omega_2^V}$.

In [T], it is shown

Theorem. ([T]) *(1) BMM implies* θ_{AC},

(2) θ_{AC} *implies* $2^\omega = 2^{\omega_1} = \omega_2$.

In this note, we consider θ^*_{AC} which is somewhat stronger than θ_{AC} of [T] and show

(3) If BSFPA holds and there exists a measurable cardinal, then θ^*_{AC} holds,

(4) θ^*_{AC} implies both θ_{AC} and CB (Complete Bounding).

While θ_{AC} of [T] demands existences of α, β and γ with $\omega_1 \leq \alpha < \beta < \gamma$, our θ^*_{AC} further demands $\alpha = \omega_1$. The consistency strength of the assumption in (1) is not well-known. A proper class of Woodin cardinals suffices (p. 867 in [W]). However they say it is unknown whether BMM implies $0^{\#}$ or not.

On the other hand, if we have a type of reflecting cardinal (which itself is very much weaker than Mahlo) and a measurable cardinal above it (and so lots of measurable must exist below it), then we get the consistency of the assumption in (3) via a revised countable support iteration (say, see [M2]).

§ 1. Basics with The One-to-one Lists in The Cantor Space

1.1 Definitin. A *one-to-one list* $\mathbf{r} = \langle r_i \mid i < \omega_1 \rangle$ in ${}^\omega 2$ means that for all $i < \omega_1$, $r_i : \omega \longrightarrow 2$ and for all $i, j < \omega_1$, if $i \neq j$, then $r_i \neq r_j$. In this case, we denote $\Delta(r_i, r_j) = \mathrm{Min}\ \{n < \omega \mid r_i(n) \neq r_j(n)\}$. More generally, we consider a one-to-one list $\mathbf{r} = \langle r_i \mid i \in T \rangle$ on a stationary set $T \subseteq \omega_1$ in ${}^\omega 2$. For a countable set X of ordinals, o.t.(X) denotes the order type of X. Hence o.t.$(X) < \omega_1$. For any ordinals $\alpha < \beta$, if o.t.$(X \cap \alpha) <$ o.t.$(X \cap \beta) < \omega_1$, then we denote $\Delta^{\mathbf{r}}_X(\alpha, \beta) = \Delta(r_{\text{o.t.}(X\cap\alpha)}, r_{\text{o.t.}(X\cap\beta)})$. We usually simply write $\Delta_X(\alpha, \beta)$ instead of $\Delta^{\mathbf{r}}_X(\alpha, \beta)$. For any ordinals α, β and γ, if o.t.$(X \cap \alpha) <$ o.t.$(X \cap \beta) <$ o.t.$(X \cap \gamma) < \omega_1$, then we denote $\mathrm{Max}\ \Delta_X(\alpha, \beta, \gamma) = \mathrm{Max}\ \{\Delta_X(\alpha, \beta), \Delta_X(\alpha, \gamma), \Delta_X(\beta, \gamma)\}$.

1.2 Lemma. Let $\mathbf{r} = \langle r_i \mid i < \omega_1 \rangle$ be a one-to-one list in ${}^\omega 2$. Then there exists $n < \omega$ such that both $\{i < \omega_1 \mid r_i(n) = 0\}$ and $\{i < \omega_1 \mid r_i(n) = 1\}$ are stationary.

Proof. Suppose not. For each $n < \omega$, there is a club C_n and ϵ_n such that for all $i \in C_n$, $r_i(n) = \epsilon_n$. Let $C = \bigcap\{C_n \mid n < \omega\}$. Then C is a club and for all $i \in C$ and all $n < \omega$, we have $r_i(n) = \epsilon_n$. Hence $\{r_i \mid i \in C\}$ has one element. This is a contradiction.

□

1.3 Lemma. Let $\mathbf{r} = \langle r_i \mid i \in T \rangle$ be a one-to-one list on a stationary set T in ${}^\omega 2$. Then there exist $m < \omega$ and $s \in {}^m 2$ such that both $\{i \in T \mid r_i \lceil m = s$ and $r_i(m) = 0\}$ and $\{i \in T \mid r_i \lceil m = s$ and $r_i(m) = 1\}$ are stationary.

Proof. Suppose not. For each $m < \omega$ and $s \in {}^m 2$, there exist a club C_{ms} and ϵ_{ms} such that for all $i \in C_{ms} \cap T$, we have if $r_i \lceil m = s$, then $r_i(m) = \epsilon_{ms}$. Let $C = \bigcap\{C_{ms} \mid m < \omega, s \in {}^m 2\}$. Then C is a club and for all $m < \omega$, all $s \in {}^m 2$ and all $i \in C \cap T$, we have if $r_i \lceil m = s$, then $r_i(m) = \epsilon_{ms}$. In particular, $r_i(m) = \epsilon_{m r_i \lceil m}$. Hence for $i, j \in C \cap T$, we may show $r_i \lceil m = r_j \lceil m$ for all $m < \omega$ by induction on m. Hence $\{r_i \mid i \in C \cap T\}$ has one element. This is a contradiction.

□

1.4 Lemma. Let $\mathbf{r} = \langle r_i \mid i < \omega_1 \rangle$ be a one-to-one list in ${}^\omega 2$. For any stationary S and any $n < \omega$, there exist m with $n \leq m < \omega$ and $s \in {}^m 2$ such that both $\{i \in S \mid r_i \lceil m = s$ and $r_i(m) = 0\}$ and $\{i \in S \mid r_i \lceil m = s$ and $r_i(m) = 1\}$ are stationary.

Proof. Let S and n be as given. Since $\{r_i \lceil n \mid i \in S\}$ is finite, S gets partitioned into finitely many cells according to $r_i \lceil n$. But S is stationary. Hence one of them is stationary. So there is $t \in {}^n 2$ such that $T = \{i \in S \mid r_i \lceil n = t\}$ is stationary. Now may apply lemma 1.3 to a one-to-one list $\langle r_i \lceil [n, \omega) \mid i \in T \rangle$ (somewhat abusive). Hence there exist m with $n \leq m < \omega$ and $u \in {}^{[n,m)}2$ such that both $\{i \in S \mid r_i \lceil n = t, r_i \lceil [n, m) = u, r_i(m) = 0\}$ and $\{i \in S \mid r_i \lceil n = t, r_i \lceil [n, m) = u, r_i(m) = 1\}$ are stationary.

□

1.5 Lemma. Let $\mathbf{r} = \langle r_i \mid i < \omega_1 \rangle$ be a one-to-one list in ${}^\omega 2$. For any $n < \omega$, there exists a club $C_{\mathbf{r}n}$ such that for any $i \in C_{\mathbf{r}n}$ there is m with $n \leq m < \omega$ such that both $\{j \in \omega_1 \mid r_j \lceil m = r_i \lceil m, r_j(m) = 0\}$ and $\{j \in \omega_1 \mid r_j \lceil m = r_i \lceil m, r_j(m) = 1\}$ are stationary.

Proof. Suppose not. For any club C, there is $i \in C$ such that for any m with $n \leq m < \omega$, there is η such that $\{j \in \omega_1 \mid r_j \lceil m = r_i \lceil m, r_j(m) = \eta\}$ is not stationary. Let $S = \{i < \omega_1 \mid$ for all m with $n \leq m < \omega$, there is η such that $\{j \in \omega_1 \mid r_j \lceil m = r_i \lceil m, r_j(m) = \eta\}$ is not stationary $\}$. Then S is stationary. By lemma 1.4, we have m with $n \leq m < \omega$ and $s \in {}^m 2$ such that both $S^0 = \{i \in S \mid r_i \lceil m = s, r_i(m) = 0\}$ and $S^1 = \{i \in S \mid r_i \lceil m = s, r_i(m) = 1\}$ are stationary. Pick any $i \in S^0 (\neq \emptyset)$. Then $r_i \lceil m = s$ and $i \in S$. Hence there is η such that $\{j \in \omega_1 \mid r_j \lceil m = s, r_j(m) = \eta\}$ is not stationary. Since S^0 is stationary, we have $\eta = 1$. Similary, since S^1 is stationary, we have $\eta = 0$. This is a contradiction.

□

1.6 Lemma. Let $\mathbf{r} = \langle r_i \mid i < \omega_1 \rangle$ be a one-to-one list in ${}^\omega 2$. Then there exists a club $C_{\mathbf{r}}$ such that for any $i \in C_{\mathbf{r}}$ and any $n < \omega$, there is m with $n \leq m < \omega$ such that both $\{j \in \omega_1 \mid r_j \lceil m = r_i \lceil m, r_j(m) = 0\}$ and $\{j \in \omega_1 \mid r_j \lceil m = r_i \lceil m, r_j(m) = 1\}$ are stationary.

Proof. Let $C_{\mathbf{r}} = \bigcap\{C_{\mathbf{r}n} \mid n < \omega\}$. Then this $C_{\mathbf{r}}$ works.

□

1.7 Lemma. Let $\mathbf{r} = \langle r_i \mid i < \omega_1 \rangle$ be a one-to-one list in ${}^{\omega}2$. Then there exists a club $C_{\mathbf{r}}$ such that for any $i \in C_{\mathbf{r}}$ and any $n < \omega$, we have $\{j < \omega_1 \mid \Delta(r_i, r_j) \geq n\}$ is stationary.

Proof. The $C_{\mathbf{r}}$ above works.

□

1.8 Lemma. Let $\mathbf{r} = \langle r_i \mid i < \omega_1 \rangle$ be a one-to-one list in ${}^{\omega}2$. Then there exist $n_{\mathbf{r}} < \omega$ and a club $C_{\mathbf{r}}$ such that

- Both $\{j < \omega_1 \mid r_j(n_{\mathbf{r}}) = 0\}$ and $\{j < \omega_1 \mid r_j(n_{\mathbf{r}}) = 1\}$ are stationary.

  And so

- For any $i < \omega_1$, $\{j < \omega_1 \mid \Delta(r_i, r_j) \leq n_{\mathbf{r}}\}$ is stationary,

  While

- For any $i \in C_{\mathbf{r}}$ and any $n < \omega$, $\{j < \omega_1 \mid \Delta(r_i, r_j) > n\}$ is stationary.

Proof. Let $n = n_{\mathbf{r}} < \omega$ be any number such that both $\{j < \omega_1 \mid r_j(n) = 0\}$ and $\{j < \omega_1 \mid r_j(n) = 1\}$ are stationary. Let $C_{\mathbf{r}}$ be as in above. These $n_{\mathbf{r}}$ and $C_{\mathbf{r}}$ work.

□

§ 2. Basics with Semiproper Preorders

2.1 Notation. Let λ be a regular cardinal. We write $N \prec H_\lambda$, if the structure $(N, \in)$ is an elementary substructure of $(H_\lambda, \in)$. For N and M, we denote $M \supseteq_{\text{end}} N$, if $M \supseteq N$ and $M \cap \omega_1 = N \cap \omega_1$. We write $\langle X_i \mid i < \omega_1 \rangle \nearrow X$, if $\langle X_i \mid i < \omega_1 \rangle$ is a sequence of continuously increasing countable subsets of X and $\bigcup\{X_i \mid i < \omega_1\} = X$.

2.2 Definition. Let κ be a regular uncountable cardinal and $S \subseteq [\kappa]^\omega$. We say S is *semiproper*, if there exists a club $C \subseteq [H_{(2^\kappa)^+}]^\omega$ such that for any $N \prec H_{(2^\kappa)^+}$ with $N \in C$, there is a countable $M \prec H_{(2^\kappa)^+}$ such that $M \supseteq_{\text{end}} N$ and $M \cap \kappa \in S$.

2.3 Lemma. Let κ be a regular uncountable cardinal, S, $T \subseteq [\kappa]^\omega$ be semiproper and disjoint. Then for any $B \subseteq \omega_1$, there is a semiproper p.o. set $P = P(S, T, B)$ such that in V^P, there is $\langle X_i \mid i < \omega_1 \rangle \nearrow \kappa$ such that for all $i < \omega_1$,

- If $i \in B$, then $X_i \in S$,
- If $i \notin B$, then $X_i \in T$,

  Hence

- $i \in B$ if and only if $X_i \in S$.

  Proof. Let $p \in P$, if $p = \langle X_i^p \mid i \leq \alpha^p \rangle$ such that

- p is continuously increasing and the X_i^p are countable subsets of κ with $\alpha^p < \omega_1$,

  For $i \leq \alpha^p$, we have

- If $i \in B$, then $X_i^p \in S$,
- If $i \notin B$, then $X_i^p \in T$.

For $p, q \in P$, let $q \leq p$, if $q \supseteq p$.

We show that this P works in a series of claims.

Claim 1. For any $p \in P$ and any $\xi \in \kappa$, there is X such that $\xi \in X$, $q = p \cup \{(\alpha^p + 1, X)\} \in P$ and $q \leq p$.

Proof. According to $\alpha^p + 1 \in B$ or not, we have two cases.

Case 1. $\alpha^p + 1 \in B$: Since S is semiproper, there is a countable $M \prec H_{(2^\kappa)^+}$ such that $p, \xi \in M$ and $M \cap \kappa \in S$. Let $X = M \cap \kappa$. Then this X works.

Case 2. $\alpha^p + 1 \notin B$: Since T is semiproper, there is a countable $M \prec H_{(2^\kappa)^+}$ such that $p, \xi \in M$ and $M \cap \kappa \in T$. Let $X = M \cap \kappa$. Then this X works.

□

Claim 2. For $i < \omega_1$ and $\xi \in \kappa$, $D(i, \xi) = \{q \in P \mid i \leq \alpha^q, \xi \in X^q_{\alpha^q}\}$ is open dense in P.

Proof. By induction on i for all ξ. By claim 1, it remains to deal with limit i. We show this by contradiction. Suppose for any $q \leq p$, $\alpha^q < i$. It suffices to derive a contradiction. Let $\langle i_n \mid n < \omega \rangle$ be increasing such that $i_0 = \alpha^p$ and $\sup\{i_n \mid n < \omega\} = i$. According to $i \in B$ or not, we have two cases.

Case 1. $i \in B$: Let $M \prec H_{(2^\kappa)^+}$ be such that $i, p, \xi \in M$ and $M \cap \kappa \in S$. Let $\langle \xi_n \mid n < \omega \rangle$ enumerate $M \cap \kappa$. By induction we have $\langle p_n \mid n < \omega \rangle$ so that $p_0 = p$, $p_n \in P \cap M$, $i_n \leq \alpha^{p_{n+1}} < i$ and $\xi_n \in X^{p_{n+1}}_{\alpha^{p_{n+1}}}$. Let $q = \bigcup\{p_n \mid n < \omega\} \cup \{(i, M \cap \kappa)\}$. Then $q \in P$ and $q \leq p$ with $\alpha^q = i$. This is a contradiction.

Case 2. $i \notin B$: Similarly to case 1, let $M \prec H_{(2^\kappa)^+}$ be such that $i, p, \xi \in M$ and $M \cap \kappa \in T$. Let $\langle \xi_n \mid n < \omega \rangle$ enumerate $M \cap \kappa$. By induction we have $\langle p_n \mid n < \omega \rangle$ so that $p_0 = p$, $p_n \in P \cap M$, $i_n \leq \alpha^{p_{n+1}} < i$ and $\xi_n \in X^{p_{n+1}}_{\alpha^{p_{n+1}}}$. Let $q = \bigcup\{p_n \mid n < \omega\} \cup \{(i, M \cap \kappa)\}$. Then $q \in P$ and $q \leq p$ with $\alpha^q = i$. This is a contradiction.

□

Claim 3. P is semiproper.

Proof. Let $P \in N \prec H_{(2^\kappa)^+}$ with $N \in C(S) \cap C(T)$, where $C(S)$ and $C(T)$ are clubs in $[H_{(2^\kappa)^+}]^\omega$ associated with semiproper S and T respectively. Let $p \in P \cap N$. We want to find $q \leq p$ which is (P, N)-semi-generic. According to $N \cap \omega_1 \in B$ or not, we have two cases.

Case 1. $N \cap \omega_1 \in B$: Since $N \in C(S)$, we may take a countable $M \prec H_{(2^\kappa)^+}$ such that $M \supseteq_{\text{end}} N$ and $M \cap \kappa \in S$. Let $\langle p_n \mid n < \omega \rangle$ be a (P, M)-generic sequence with $p_0 = p$. Let $q = \bigcup\{p_n \mid n < \omega\} \cup \{(M \cap \omega_1, M \cap \kappa)\}$. Then by claim 2, we know that $q \in P$ and so $q \leq p$. By construction, q is (P, M)-generic and so (P, N)-semi-generic.

Case 2. $N \cap \omega_1 \notin B$: Similarly to case 1, take a countable $M \prec H_{(2^\kappa)^+}$ such that $M \supseteq_{\text{end}} N$ and $M \cap \kappa \in T$. Let $\langle p_n \mid n < \omega \rangle$ be a (P, M)-generic sequence with $p_0 = p$. Let $q = \bigcup\{p_n \mid n < \omega\} \cup \{(M \cap \omega_1, M \cap \kappa)\}$. Then by claim 2, we know that $q \in P$ and so $q \leq p$. By construction, q is (P, M)-generic and so (P, N)-semi-generic.

□

Claim 4. Let G be any P-generic filter over V and let $\langle X_i \mid i < \omega_1 \rangle = \bigcup G$. Then $\langle X_i \mid i < \omega_1 \rangle \nearrow \kappa$ and for $i < \omega_1$, we have

- If $i \in B$, then $X_i \in S$,
- If $i \notin B$, then $X_i \in T$.

Proof. By construction of P and claim 2. Notice that $|\kappa| = \omega_1$ holds in the extension $V[G]$.

□

This completes the proof of lemma.

2.4 Lemma. Let κ be a regular uncountable cardinal and $S \subseteq [\kappa]^\omega$ be semiproper. Then there is a semiproper p.o. set $P = P(S)$ such that in V^P, there is $\langle X_i \mid i < \omega_1 \rangle \nearrow \kappa$ such that for all $i < \omega_1$, $X_i \in S$.

Proof. The proof is entirely similar to and simpler than lemma 2.3.

□

§ 3. First Use of A Measurable Cardinal and BSPFA

We prepare a lemma with a measurable cardinal which is by now well-known with stronger statements.

3.1 Lemma. Let κ be a measurable cardinal with a normal measure D on κ. Let N be a countable elementary substructure of $H_{(2^\kappa)^+}$ with $D \in N$.

(1) For any $\eta \in \kappa$ and any $s \in \bigcap(N \cap D)$ such that $\sup(N \cap \kappa)$, $\eta < s$, we may form a countable elementary substructure M of $H_{(2^\kappa)^+}$ such that $N \cup \{s\} \subset M$ and $M \cap s = N \cap s = N \cap \kappa$.

(2) There is a continuously increasing countable elementary substructures $\langle N_i \mid i < \omega_1 \rangle$ of $H_{(2^\kappa)^+}$ such that $N_0 = N$ and $\langle$ o.t.$(N_i \cap \kappa) \mid i < \omega_1 \rangle$ is a strictly increasing continuous sequence of countable ordinals.

(3) For any stationary $S \subseteq \omega_1$, there is a countable elementary substructure M of $H_{(2^\kappa)^+}$ such that $N \subseteq_{\text{end}} M$ and o.t.$(M \cap \kappa) \in S$.

Proof. For (1): Let $M = \{f(s) \mid f \in N\}$. Then this M works.

For (2): Construct $\langle N_i \mid i < \omega_1 \rangle$ by recursion on i. At the successor stages, apply (1). At the limit stages, just take a union.

For (3): Immediate by (2).

□

3.2 Lemma. Let κ be a measurable cardinal and $\mathbf{r} = \langle r_i \mid i < \omega_1 \rangle$ be a one-to-one list in ${}^\omega 2$. For any countable $N \prec H_{(2^\kappa)^+}$ with $\mathbf{r}, \kappa \in N$ and any $n < \omega$, there exists a countable $M \prec H_{(2^\kappa)^+}$ such that $M \supseteq_{\text{end}} N$ and $\Delta_M(\omega_1, \kappa) \geq n$. Namely, $S(\mathbf{r}, \kappa, n) = \{X \in [\kappa]^\omega \mid \Delta_X(\omega_1, \kappa) \geq n\}$ is semiproper.

Proof. Since $\mathbf{r} \in N$, we may assume $C_\mathbf{r} \in N$ and so $\delta = N \cap \omega_1 \in C_\mathbf{r}$. Therefore $S = \{j < \omega_1 \mid \Delta(r_\delta, r_j) \geq n\}$ is stationary. Since κ is measurable and $\kappa \in N$, we may take a countable $M \prec H_{(2^\kappa)^+}$ such that $M \supseteq_{\text{end}} N$ and $j = \text{o.t.}(M \cap \kappa) \in S$. Hence $\Delta_M(\omega_1, \kappa) = \Delta(r_{\text{o.t.}(M \cap \omega_1)}, r_{\text{o.t.}(M \cap \kappa)}) = \Delta(r_\delta, r_j) \geq n$.

□

3.3 Lemma. Let κ be a measurable cardinal and $\mathbf{r} = \langle r_i \mid i < \omega_1 \rangle$ be a one-to-one list in ${}^\omega 2$. For any $n < \omega$, there exists a semiproper p.o. set P such that in V^P, there exists $\langle \dot{X}_i \mid i < \omega_1 \rangle \nearrow \kappa$ such that for all $i < \omega_1$, $\Delta_{\dot{X}_i}(\omega_1, \kappa) \geq n$.

Proof. Apply lemma 2.4 to $S(\mathbf{r}, \kappa, n)$.

□

3.4 Lemma. (BSPFA) Let a measurable cardinal exist and $\mathbf{r} = \langle r_i \mid i < \omega_1 \rangle$ be a one-to-one list in ${}^\omega 2$. For any $n < \omega$, there exists β with $\omega_1 < \beta < \omega_2$ and $\langle X_i \mid i < \omega_1 \rangle \nearrow \beta$ such that for all $i < \omega_1$, $\Delta_{X_i}(\omega_1, \beta) \geq n$.

Proof. Apply BSPFA to lemma 3.3.

□

§ 4. Modifications and Summary

4.1 Lemma. Let $n < \omega$, $\omega_1 < \beta < \omega_2$ and $\langle X_i \mid i < \omega_1 \rangle \nearrow \beta$ be such that for any $i < \omega_1$, $\Delta_{X_i}(\omega_1, \beta) \geq n$. Then we have a continuously increasing $\langle N_i \mid i < \omega_1 \rangle$ such that

- For all $i < \omega_1$, $N_i \prec H_{\omega_2}$ and N_i is countable,
- $\beta \in N_0$, $\bigcup\{N_i \mid i < \omega_1\} \supset \omega_1$ and so $\bigcup\{N_i \mid i < \omega_1\} \supset \beta$,
- For all $i < \omega_1$, $\Delta_{N_i}(\omega_1, \beta) \geq n$.

Proof. Let $\langle N_i \mid i < \omega_1 \rangle$ be any continuously increasing sequence of countable $N_i \prec H_{\omega_2}$ such that $\bigcup\{N_i \mid i < \omega_1\} \supset \omega_1$ and $\beta \in N_0$. Then since $\beta < \omega_2$, we have $\bigcup\{N_i \cap \beta \mid i < \omega_1\} = \beta$ and so $C = \{i < \omega_1 : X_i = N_i \cap \beta\}$ is a club. By reenumerating $\{N_i \mid i \in C\}$, we are done.

□

4.2 Lemma. (BSPFA) Let a measurable cardinal exist and $\mathbf{r} = \langle r_i \mid i < \omega_1 \rangle$ be a one-to-one list in ${}^{\omega}2$. Then there exist $n_{\mathbf{r}} < \omega$, a club $C_{\mathbf{r}}$, $\beta_{\mathbf{r}}$ with $\omega_1 < \beta_{\mathbf{r}} < \omega_2$ and $\langle N_i^{\mathbf{r}} \mid i < \omega_1 \rangle$ continuously increasing such that

- Both $\{j < \omega_1 \mid r_j(n_{\mathbf{r}}) = 0\}$ and $\{j < \omega_1 \mid r_j(n_{\mathbf{r}}) = 1\}$ are stationary,
- For any $i < \omega_1$, $\{j < \omega_1 \mid \Delta(r_i, r_j) \leq n_{\mathbf{r}}\}$ is stationary,
- For any $i \in C_{\mathbf{r}}$ and any $n < \omega$, $\{j < \omega_1 \mid \Delta(r_i, r_j) > n\}$ is stationary,
- For any $i < \omega_1$, $N_i^{\mathbf{r}} \prec H_{\omega_2}$ and $N_i^{\mathbf{r}}$ is countable,
- $\beta_{\mathbf{r}} \in N_0^{\mathbf{r}}$, $\bigcup\{N_i^{\mathbf{r}} \mid i < \omega_1\} \supset \omega_1$ and so $\bigcup\{N_i^{\mathbf{r}} \cap \beta_{\mathbf{r}} \mid i < \omega_1\} = \beta_{\mathbf{r}}$,
- For any $i < \omega_1$, $\Delta^{\mathbf{r}}_{N_i^{\mathbf{r}}}(\omega_1, \beta_{\mathbf{r}}) \geq n_{\mathbf{r}} + 1$.

Proof. Combine lemma 1.8, lemma 3.4 and lemma 4.1.

□

§ 5. Second Use of The Same Measurable Cardinal and BSPFA

5.1 Definition. Let θ^*_{AC} denote the following statement. For any $\mathbf{r}$ one-to-one list in ${}^{\omega}2$ and any $B \subseteq \omega_1$, there exist β and γ with $\omega_1 < \beta < \gamma < \omega_2$ and $\langle X_i \mid i < \omega_1 \rangle \nearrow \gamma$ such that for any $i < \omega_1$, $i \in B$ if and only if $\Delta_{X_i}(\omega_1, \beta) = \mathrm{Max}\ \Delta_{X_i}(\omega_1, \beta, \gamma)$.

It is clear that θ^*_{AC} implies θ_{AC} of [T].

5.2 Theorem. (BSPFA) If there exists a measurable cardinal, then θ^*_{AC} holds.

We show this in a series of lemmas.

5.3 Lemma. Let κ be a measurable cardinal and $\mathbf{r}$ be a one-to-one list in ${}^{\omega}2$. For any β with $\omega_1 < \beta < \kappa$, any countable $N \prec H_{(2^\kappa)^+}$ with $\mathbf{r}, \beta, \kappa \in N$, there exists a countable $M \prec H_{(2^\kappa)^+}$ such that $M \supseteq_{\mathrm{end}} N$ and $\Delta_M(\omega_1, \beta) = \mathrm{Min}\ \Delta_M(\omega_1, \beta, \kappa)$.

Proof. Since $\mathbf{r} \in N$, we may assume $C_{\mathbf{r}} \in N$ and so $N \cap \omega_1 \in C_{\mathbf{r}}$. Hence for all $n < \omega$, we have $\{j < \omega_1 \mid \Delta(r_{N \cap \omega_1}, r_j) \geq n\}$ is stationary. Since $\omega_1 < \beta$, we may calculate $\Delta_N(\omega_1, \beta) = n$. Since κ is a measurable cardinal, we may choose a countable $M \prec H_{(2^\kappa)^+}$ such that $M \cap \beta = N \cap \beta$, if $j = \mathrm{o.t.}(M \cap \kappa)$, then $\Delta(r_{N \cap \omega_1}, r_j) \geq n + 1$. Since $\Delta_M(\omega_1, \beta) = \Delta_N(\omega_1, \beta) = n < \Delta(r_{N \cap \omega_1}, r_{\mathrm{o.t.}(M \cap \kappa)}) = \Delta_M(\omega_1, \kappa)$, we have $\Delta_M(\omega_1, \beta) = \mathrm{Min}\ \Delta_M(\omega_1, \beta, \kappa)$.

□

5.4 Lemma. (BSPFA) Let κ be a measurable cardinal and $\mathbf{r}$ be a one-to-one list in ${}^{\omega}2$. For any countable $N \prec H_{(2^\kappa)^+}$ with $\mathbf{r}, \kappa \in N$, there exists a countable $M \prec H_{(2^\kappa)^+}$ such that $M \supseteq_{\mathrm{end}} N$ and $\Delta_M(\omega_1, \beta_{\mathbf{r}}) = \mathrm{Max}\ \Delta_M(\omega_1, \beta_{\mathbf{r}}, \kappa)$.

Proof. Let $\eta = r_{N\cap\omega_1}(n_{\mathbf{r}})$. Let $\overline{\eta} \in \{0,1\}$ and $\eta \neq \overline{\eta}$. Since $\{j < \omega_1 \mid r_j(n_{\mathbf{r}}) = \overline{\eta}\}$ is stationary, we may choose a countable $M \prec H_{(2^\kappa)^+}$ such that $M \supseteq_{\text{end}} N$ and $r_{\text{o.t.}(M\cap\kappa)}(n_{\mathbf{r}}) = \overline{\eta}$. Hence $\Delta_M(\omega_1, \kappa) \leq n_{\mathbf{r}}$. On the other hand, since we may assume $\langle N_i^{\mathbf{r}} \mid i < \omega_1\rangle \in N$, if $\delta = N\cap\omega_1$, then we have $N_\delta^{\mathbf{r}} \subseteq_{\text{end}} N \cap H_{\omega_2}$. Since $\beta_{\mathbf{r}} \in N_0^{\mathbf{r}}$, we conclude $N_\delta^{\mathbf{r}} \cap \beta_{\mathbf{r}} = N \cap \beta_{\mathbf{r}} = M \cap \beta_{\mathbf{r}}$ holds. So $\Delta_M(\omega_1, \beta_{\mathbf{r}}) = \Delta_N(\omega_1, \beta_{\mathbf{r}}) = \Delta_{N_\delta^{\mathbf{r}}}(\omega_1, \beta_{\mathbf{r}}) \geq n_{\mathbf{r}}+1$. Therefore, $\Delta_M(\omega_1, \beta_{\mathbf{r}}) = \text{Max } \Delta_M(\omega_1, \beta_{\mathbf{r}}, \kappa)$.

□

5.5 Lemma. (BSPFA) Let κ be a measurable cardinal and $\mathbf{r}$ be a one-to-one list in ${}^\omega 2$. Let $S = \{X \in [\kappa]^\omega \mid \Delta_X(\omega_1, \beta_{\mathbf{r}}) = \text{Max } \Delta_X(\omega_1, \beta_{\mathbf{r}}, \kappa)\}$ and $T = \{X \in [\kappa]^\omega \mid \Delta_X(\omega_1, \beta_{\mathbf{r}}) = \text{Min } \Delta_X(\omega_1, \beta_{\mathbf{r}}, \kappa)\}$. Then both S and T are semiproper and disjoint.

Proof. By lemma 5.4 and lemma 5.3.

□

5.6 Lemma. (BSPFA) Let κ be a measurable cardinal. Let $\mathbf{r} = \langle r_i \mid i < \omega_1\rangle$ be a one-to-one list in ${}^\omega 2$ and $B \subseteq \omega_1$. Then there exists a semiproper p.o. set P such that in V^P, there is $\langle Y_i \mid i < \omega_1\rangle \nearrow \kappa$ such that for any $i < \omega_1$, $i \in B$ if and only if $\Delta_{Y_i}(\omega_1, \beta_{\mathbf{r}}) = \text{Max } \Delta_{Y_i}(\omega_1, \beta_{\mathbf{r}}, \kappa)$.

Proof. By lemma 5.5 and lemma 2.3.

□

Proof of theorem 5.2. Apply BSPFA to the p.o. set in lemma 5.6.

□

§ 6. θ^*_{AC} implies CB

6.1 Definition. *CB* (*complete bounding*) stands for the following. For any $f : \omega_1 \longrightarrow \omega_1$, there exist $\omega_1 < \gamma < \omega_2$, a club C and $\langle X_i \mid i < \omega_1\rangle \nearrow \gamma$ such that for all $i \in C$, $f(i) < \text{o.t.}(X_i)$.

6.2 Theorem. θ^*_{AC} implies CB.

Proof. We have two claims.

Claim 1. If for any one-to-one list $\mathbf{r}$ in ${}^\omega 2$, there exist $\omega_1 < \beta < \omega_2$ and $\langle X_i \mid i < \omega_1\rangle \nearrow \beta$ such that $\Delta_{X_i}(\omega_1, \beta) > 0$, then CB holds.

Proof. Let $f : \omega_1 \longrightarrow \omega_1$. We may assume that for all $i < \omega_1$, $i < f(i)$ and f is strictly increasing. Take a continuously increasing $\langle N_i \mid i < \omega_1\rangle$ such that each N_i is countable, $N_i \prec H_{\omega_2}$ and $N_i \in N_{i+1}$ and $f \in N_0$. Notice that $N_i \cap \omega_1 < f(N_i \cap \omega_1) < N_{i+1} \cap \omega_1$. It is easy to construct $\mathbf{r}$ so that $r_{N_i\cap\omega_1}(0) = 1$, for ξ with $N_i \cap \omega_1 < \xi \leq f(N_i \cap \omega_1)$, we have $r_\xi(0) = 0$. By assumption get $\omega_1 < \beta < \omega_2$ and $\langle X_i \mid i < \omega_1\rangle$ such that for all $i < \omega_1$, we have $\Delta_{X_i}(\omega_1, \beta) > 0$. Let $C = \{i < \omega_1 \mid N_i \cap \omega_1 = i = X_i \cap \omega_1, \omega_1 \in X_i\}$. Then for $i \in C$, since $\Delta_{X_i}(\omega_1, \beta) = \Delta(r_i, r_{\text{o.t.}(X_i)}) > 0$, we have $f(i) = f(N_i \cap \omega_1) < \text{o.t.}(X_i)$.

□

Claim 2. If θ^*_{AC} holds, then for any one-to-one list $\mathbf{r}$ in ${}^\omega 2$, there exist $\omega_1 < \beta < \omega_2$ and $\langle X_i \mid i < \omega_1\rangle \nearrow \beta$ such that $\Delta_{X_i}(\omega_1, \beta) > 0$.

Proof. By θ^*_{AC} for $B = \omega_1$, there exist $\omega_1 < \beta < \gamma < \omega_2$ and $\langle Y_i \mid i < \omega_1\rangle \nearrow \gamma$ such that for all $i < \omega_1$, we have $\Delta_{Y_i}(\omega_1, \beta) = \text{Max } \Delta_{Y_i}(\omega_1, \beta, \gamma)$. In particular, $\Delta_{Y_i}(\omega_1, \beta) > 0$. Let $X_i = Y_i \cap \beta$. Then these β and $\langle X_i \mid i < \omega_1\rangle$ work.

§ 7. Additional Observations

Now we make a few observations. We may consider to directly force our θ^*_{AC}. Namely, we may add the following to [D].

Theorem. ([D], [M]) The following are equiconsistent.

- Con(There exists a regular cardinal ρ such that $\{\kappa < \rho \mid \kappa$ is a measurable cardinal$\}$ is cofinal in ρ).
- Con(θ^*_{AC}),
- Con(CB).

□

Hence θ_{AC} of [T] accordingly has a large cardinal upper-bound.

Next, similarly to Con(PFA$^+$ + $\neg$ CB) (which we got from S. Todorcevic), we may show via ω_1-many Cohen reals **r**,

Theorem. ([M]) Con(PFA$^+$ and $\neg\theta_{\mathrm{AC}}$) and so Con(PFA$^+$ and $\neg$ BMM).

□

Lastly, starting with a Souslin tree in the ground model and preserving it ([M1]), we have

Theorem. ([M]) Con(There exists a Souslin tree and θ^*_{AC}) and so Con($\neg$ MA and θ^*_{AC})

□

Among others concerning the large cardinal strength of BMM, we may ask

Question. Does θ_{AC} of [T] imply any large cardinal, say, CB ?

More modestly,

Question. Does BMM imply the Weak Chang's Conjecture ?

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