

On the Pseudo-set

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Let X be a certain set and we shall call its element x , a *point* of set X . Taking X and an object M into consideration, we shall call M a *pseudo-set* with respect to X . For all points x in X and M one of the following three fundamental relations holds :

$$(1) \quad \begin{cases} x \bar{\in} M, \\ x \notin M, \\ x \parallel M. \end{cases}$$

In the following pages, let us consider that these three fundamental relations and set X are fixed, and we shall call M , shortly, a pseudo-set. When we talk about a point, we mean a point of the set X .

Definition of equality and inequality of pseudo-sets: Two pseudo-sets L and M are called equal, when they satisfy the following conditions. We indicate them by $L=M$:

$$(2) \quad \begin{cases} \text{if } x \bar{\in} L, \text{ then } x \bar{\in} M, \\ \text{if } x \notin L, \text{ then } x \notin M, \\ \text{if } x \parallel L, \text{ then } x \parallel M, \text{ and vice versa.} \end{cases}$$

By this definition, it is clear that the equality is reflexive, symmetric and transitive. Let for example $L=M$, $M=P$ and $x \bar{\in} L$, then $x \bar{\in} M$ for $L=M$, and hence $x \bar{\in} P$ for $M=P$.

Definition of the sum of two pseudo-sets: If there be an object S satisfying the following properties (3), then it is clear that S is a pseudo-set, and we shall call S a sum of two pseudo-sets L and M , and denote it by $S=L+M$;

$$(3) \quad \begin{cases} x \notin S \text{ when } x \bar{\in} L, x \bar{\in} M, \\ x \parallel S \text{ when } x \bar{\in} L, x \notin M, \\ x \bar{\in} S \text{ when } x \bar{\in} L, x \parallel M, \\ x \parallel S \text{ when } x \notin L, x \bar{\in} M, \\ x \bar{\in} S \text{ when } x \notin L, x \notin M, \\ x \notin S \text{ when } x \notin L, x \parallel M, \\ x \bar{\in} S \text{ when } x \parallel L, x \bar{\in} M, \\ x \notin S \text{ when } x \parallel L, x \notin M, \\ x \parallel S \text{ when } x \parallel L, x \parallel M. \end{cases}$$

These relations are shown in the accompanying table.

From this definition we obtain the following theorem :

Among three pseudo-sets M_1 , M_2 and M_3 , we have

$$\begin{aligned} M_1 + M_2 &= M_2 + M_1, \\ (M_1 + M_2) + M_3 &= M_1 + (M_2 + M_3), \end{aligned}$$

i. e. the summation of pseudo-sets is commutative and associative.

For let $x \in M_1$, $x \in M_2$, $x \in M_3$ for example, then $x \in (M_1 + M_2) + M_3$ for $x \parallel (M_1 + M_2)$, $x \in M_3$, and on the other hand $x \in M_1 + (M_2 + M_3)$ for $x \in M_1$, $x \in (M_2 + M_3)$.

Now let M be a pseudo-set and $\underline{M}$ be an object such as

$$\begin{aligned} x \in \underline{M} &\text{ when } x \in M, \\ x \in \underline{M} &\text{ when } x \in M, \\ x \parallel \underline{M} &\text{ when } x \parallel M, \end{aligned}$$

then, it is clear that $\underline{M}$ is a pseudo-set. We can call $\underline{M}$ a pseudo-anti-set of M . If all points x have the relation " $x \parallel M$ " with respect to M , then we can call M a pseudo-zero-set, and we will in general denote it by N . By these definitions, we have

$$\begin{aligned} M + N &= M, \\ M + \underline{M} &= N. \end{aligned}$$

Moreover, it is clear that M is a pseudo-anti-set of $\underline{M}$.

Definition of product of pseudo-sets: if there is an object P which has the following properties (4), then it is a pseudo-set and we shall call P a product of two pseudo-sets L and M , and denote it by $P = LM$:

$$(4) \quad \begin{cases} x \in P \text{ when } x \in L, x \in M, \\ x \in P \text{ when } x \in L, x \in M, \\ x \parallel P \text{ when } x \in L, x \parallel M, \\ x \in P \text{ when } x \in L, x \in M, \\ x \in P \text{ when } x \in L, x \in M, \\ x \parallel P \text{ when } x \in L, x \parallel M, \\ x \parallel P \text{ when } x \in L, x \parallel M, \\ x \parallel P \text{ when } x \parallel L, x \in M, \\ x \parallel P \text{ when } x \parallel L, x \in M, \\ x \parallel P \text{ when } x \parallel L, x \parallel M. \end{cases}$$

These relations are shown by the accompanying table.

From this definition, we obtain the following theorem :

Between three pseudo-sets M_1 , M_2 and M_3 , we have

		M		
		$\in$	$\in$	$\parallel$
L	$\in$	$\in$	$\parallel$	$\in$
	$\in$	$\parallel$	$\in$	$\in$
	$\parallel$	$\in$	$\in$	$\parallel$

		M		
		$\bar{\epsilon}$	ϵ	$\parallel$
L	$\bar{\epsilon}$	$\bar{\epsilon}$	ϵ	$\parallel$
	ϵ	ϵ	$\bar{\epsilon}$	$\parallel$
	$\parallel$	$\parallel$	$\parallel$	$\parallel$

$$M_1 M_2 = M_2 M_1,$$

$$(M_1 M_2) M_3 = M_1 (M_2 M_3),$$

i. e. the multiplication of the pseudo-sets is commutative and associative.

For let $x \bar{\epsilon} M_1$, $x \epsilon M_2$, $x \epsilon M_3$ for example, then $x \bar{\epsilon} (M_1 M_2) M_3$ for $x \epsilon (M_1 M_2)$, $x \epsilon M_3$, and on the other hand $x \bar{\epsilon} M_1 (M_2 M_3)$ for $x \bar{\epsilon} M_1$,

$x \bar{\epsilon} (M_2 M_3)$.

These two processes, the summation and the multiplication, are also distributive :

$$M_1(M_2 + M_3) = M_1 M_2 + M_1 M_3.$$

For, let $x \bar{\epsilon} M_1$, $x \epsilon M_2$, $x \bar{\epsilon} M_3$ for example, then $x \parallel M_1(M_2 + M_3)$ for $x \bar{\epsilon} M_1$, $x \parallel (M_2 + M_3)$; on the other hand $x \parallel M_1 M_2 + M_1 M_3$ for $x \epsilon M_1 M_2$, $x \bar{\epsilon} M_1 M_3$.

If all points x have the relation " $x \bar{\epsilon} M$ " with respect to M , we call M a pseudo-unit-set, and denote it by E . From this definition we have

$$\underline{E} \underline{E} = E,$$

$$\underline{M} \underline{E} = \underline{M}.$$

Because, all x have the relation " $x \epsilon \underline{E}$ " with respect to $\underline{E}$; therefore $x \epsilon \underline{M} \underline{E}$, $x \bar{\epsilon} \underline{M} \underline{E}$, $x \parallel \underline{M} \underline{E}$ provided $x \bar{\epsilon} M$, $x \epsilon M$, $x \parallel M$ respectively.

From the results which we have obtained, we may conclude the following proposition with respect to a system of pseudo-sets $\mathfrak{M}$:

If there exists a pseudo-zero-set in $\mathfrak{M}$, and if the sum and the product of any two pseudo-sets of $\mathfrak{M}$ and the pseudo-anti-set of any pseudo-sets of $\mathfrak{M}$ exist, then the system $\mathfrak{M}$ forms a ring, and if this $\mathfrak{M}$ contains also pseudo-unit-set, then $\mathfrak{M}$ forms a proper ring.

Example of pseudo-set : Let $\mathfrak{F}$ be a system of all one-valued functions $f(x)$ which are defined in the interval $0 < x < 1$, such as

$$f(x_i) = a_i \text{ (integer), for } i = 1, 2, 3, \dots, n,$$

where $0 < x_1, x_2, \dots, x_n < 1$, and let X be the set of these points $x_1, x_2, \dots, x_n$. When we define $x_i \bar{\epsilon} f(x)$, $x_i \epsilon f(x)$ and $x_i \parallel f(x)$, so that $a_i \equiv 1 \pmod{3}$, $a_i \equiv 2 \pmod{3}$ and $a_i \equiv 0 \pmod{3}$ respectively, then it is clear, by the definition of $f(x)$, that these $f(x)$ form pseudo-sets with respect to these three fundamental relations and to the set X . Therefore when we consider $\mathfrak{F}$ as a system of these pseudo-sets, two functions $f_1(x)$ and $f_2(x)$ of $\mathfrak{F}$ such as

$$f_1(x_i) = a_i^{(1)}, f_2(x_i) = a_i^{(2)} \quad (i = 1, 2, 3, \dots, n),$$

$$a_i^{(1)} \equiv a_i^{(2)} \pmod{3} \text{ for all } i = 1, 2, 3, \dots, n,$$

are considered equal: $f_1(x) = f_2(x)$.

Let f_1 and f_2 be any two functions of $\mathfrak{F}$. Then it is clear that $\mathfrak{F}$ contains functions $f_3(x)$, $f_4(x)$ as follows :

$$f_3(x_i) \equiv a_i^{(1)} + a_i^{(2)} \pmod{3}, \text{ for all } i=1, 2, 3, \dots, n,$$

$$f_4(x_i) \equiv a_i^{(1)} \times a_i^{(2)} \pmod{3}, \text{ for all } i=1, 2, 3, \dots, n,$$

and these functions $f_3(x)$, $f_4(x)$ are respectively the sum and the product of two pseudo-sets $f_1(x)$ and $f_2(x)$. By this and the definition of $\mathfrak{F}$, it forms a ring and a proper ring, for it contains a pseudo-zero-set, a pseudo-anti-set and the pseudo-unit-set.

At the conclusion we notice that we may give another definition to the sum and the product of pseudo-sets, although they are not fertile.

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