

A Central Extension of a Formal Loop Group

Takashi HASHIMOTO¹ and Ryuichi SAWAE²

1. Department of Mathematics, Faculty of Science, Hiroshima University,
Higashi-Hiroshima, 724, Japan

e-mail: r3104@math.sci.hiroshima-u.ac.jp

2. Department of Management and Information Science, Shikoku University,
Tokushima, 771-11, Japan

e-mail: sawae@keiei.shikoku-u.ac.jp

0. Introduction

In this article, we prove that there is an elegant relation between the conformal factor and a group 2-cocycle on the formal loop group with values in $SU(1, N+1)$, and show that the trivial central extension of the Hauser group acts transitively on the space of formal solutions of the Einstein-Maxwell field equations with N abelian gauge fields. The corresponding 2-cocycle on the Lie algebra of the formal loop group is the one which describes an affine Lie algebra [K]. This relation was first found by [BM].

Now we derive the equations, which are our starting point, from the stationary axisymmetric Einstein-Maxwell field equations with N abelian gauge potentials.

Let $ds^2 = g_{\mu\nu} dx^\mu \otimes dx^\nu$ be a metric on $\mathbb{R}^{1+3}$ and $\mathbf{A} = \mathbf{A}_\mu dx^\mu$ an abelian gauge potential with values in $\mathbb{R}^N$. Then the Einstein-Maxwell field equations with N abelian gauge fields are given by

$$R_{\mu\nu} = 8\pi T_{\mu\nu}, \quad \nabla_\kappa \mathbf{F}^{\mu\kappa} = 0 \quad (\mu, \nu = 0, 1, 2, 3),$$

where $R_{\mu\nu}$ is the Ricci curvature and

$$\begin{aligned} \mathbf{F}_{\mu\nu} &= \partial_\mu \mathbf{A}_\nu - \partial_\nu \mathbf{A}_\mu, \\ T_{\mu\nu} &= \frac{1}{4\pi} (\mathbf{F}_{\mu\kappa} {}^t \mathbf{F}_\nu{}^\kappa - \frac{1}{4} g_{\mu\nu} \mathbf{F}_{\kappa\iota} {}^t \mathbf{F}^{\kappa\iota}). \end{aligned}$$

We adopt the coordinates $(x^0, x^1, x^2, x^3) = (x^0, \phi, z, \rho)$ with x^0 being time and (ϕ, z, ρ) the cylindrical coordinates of $\mathbb{R}^3$. Stationary axisymmetric space-times amount to the assumption that a metric is of the form

$$g = \begin{pmatrix} h_{00} & h_{01} & & \\ h_{10} & h_{11} & & \\ & & -\lambda & 0 \\ & & 0 & -\lambda \end{pmatrix}$$

$$\det h = -\rho^2,$$

where $\lambda > 0$, $h_{01} = h_{10}$ and $h = (h_{ij})$. The field λ is called the conformal factor.

For abelian gauge potentials, we fix the gauge so as to $\mathbf{A}_2 = \mathbf{A}_3 = 0$. Since we assume that the fields are stationary and axisymmetric, the functions h_{ij} 's, λ and $\mathbf{A}_i$'s depend only on z and ρ . Further, we fix the gauge as follows :

$$h|_{(z,\rho)=(0,0)} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \mathbf{A}|_{(z,\rho)=(0,0)} = 0. \quad (0.1)$$

Introducing the Ernst potentials $u \in \mathbf{R}, v \in \mathbf{C}^N$ constructed from h and $\mathbf{A}$ by the standard method (cf. [DO][E]), we obtain

Proposition 0.1. *The stationary axisymmetric Einstein-Maxwell field equations with N abelian gauge fields are equivalent to the following equations :*

$$f(d * du + \rho^{-1} d\rho \wedge * du) = (du - 2v^* dv) \wedge * du \quad (0.2)$$

$$f(d * dv + \rho^{-1} d\rho \wedge * dv) = (du - 2v^* dv) \wedge * dv \quad (0.3)$$

$$\begin{aligned} \frac{\partial_z \lambda}{\lambda} = & -\frac{\partial_z f}{2f} + \frac{\rho}{2f^2} (\partial_z f \partial_\rho f) \\ & - \frac{\rho}{2f^2} (\partial_\rho u - \partial_\rho f - 2v^* \partial_\rho v) (\partial_z u - \partial_z f - 2v^* \partial_z v) \\ & + \frac{\rho}{f} (\partial_z v^* \partial_\rho v + \partial_z v^* \partial_\rho v) \end{aligned} \quad (0.4)$$

$$\begin{aligned} \frac{\partial_\rho \lambda}{\lambda} = & -\frac{\partial_\rho f}{2f} + \frac{\rho}{4f^2} \{(\partial_\rho f)^2 - (\partial_z f)^2\} \\ & + \frac{\rho}{4f^2} \{(\partial_z u - \partial_z f - 2v^* \partial_z v)^2 - (\partial_\rho u - \partial_\rho f - 2v^* \partial_\rho v)^2\} \\ & - \frac{\rho}{f} (\partial_z v^* \partial_z v - \partial_\rho v^* \partial_\rho v), \end{aligned} \quad (0.5)$$

where $v^* = {}^t \bar{v}$, $|v|^2 = v^* v$, $f = \text{Re } u - |v|^2$ and $*$ is the Hodge operator given by $*dz = d\rho$, $*d\rho = -dz$.

The first two equations are called the Ernst equations.

Corresponding to the gauge fixing (0.1), we shall consider the solutions under the conditions

$$u|_{(z,\rho)=(0,0)} = 1 \quad \text{and} \quad v|_{(z,\rho)=(0,0)} = 0. \quad (0.6)$$

It is essential to introduce the function $\tau = f^{1/2} \lambda$ and we shall consider τ , instead of λ , throughout this article.

1. Ernst Equation

Let θ be Cartan involution of $GL(N+2, \mathbb{C})$ defined by $g \mapsto g^{*-1}$ and G a subgroup of $GL(N+2, \mathbb{C})$ defined by

$$\{g \in GL(N+2, \mathbb{C}); g^* J g = J, \det g = 1\},$$

where $J = \begin{pmatrix} & i \\ & 1_N \\ -i & \end{pmatrix}$ and 1_N denotes the $N \times N$ identity matrix. Note that G is isomorphic to $SU(1, N+1)$. Let K be the subgroup of G such that each element of K is fixed by θ .

We fix subgroups A and N of G as follows :

$$A = \left\{ \begin{pmatrix} a & & \\ & 1_N & \\ & & 1/a \end{pmatrix}; a > 0 \right\}$$

$$N = \left\{ \begin{pmatrix} 1 & & & \\ v & & 1_N & \\ x + i|v|^2/2 & & iv^* & 1 \end{pmatrix}; x \in \mathbb{R}, v \in \mathbb{C}^N \right\},$$

where $|v|^2 = v^* v$. Then we have $G = KAN$ (Iwasawa decomposition).

Let R be a ring of formal power series in z and ρ over $\mathbb{C}$ i.e. $R = \mathbb{C}[[z, \rho]]$. We extend the complex conjugation $*$ of $\mathbb{C}$ to a conjugation of R by defining $\bar{z} = z, \bar{\rho} = \rho$. Let G_R be a subgroup of $GL(N+2, R)$ defined by

$$\{g \in GL(N+2, R); g^* J g = J, \det g = 1\}.$$

Then, corresponding to $G = KAN$, G_R decomposes as $G_R = K_R A_R N_R$, where K_R , A_R and N_R denote subgroups of G_R consisting of matrices with values in K , A and N respectively, each of whose components is an element of R .

Now we parametrize an element of $A_R N_R$ as follows :

$$P = \begin{pmatrix} f^{1/2} & 0 & 0 \\ \sqrt{2}v & 1_N & 0 \\ (\psi + i|v|^2)/f^{1/2} & \sqrt{2}iv^*/f^{1/2} & f^{-1/2} \end{pmatrix}, \quad (1.1)$$

where f and v are the same ones as in (0.2) and (0.3), and $\psi = \text{Im } u$.

The following fact is well known.

Proposition 1.1. *Under the parametrization of (1.1), we put $M = P^* P$. Then the Ernst equations (0.2) and (0.3) are equivalent to the following equation:*

$$d(\rho * dM M^{-1}) = 0. \quad (1.2)$$

Moreover the function τ is a solution of (0.4) and (0.5) if and only if it is a solution of the following equations :

$$\tau^{-1} \partial_z \tau = \frac{\rho}{4} \text{tr}(\partial_z M M^{-1} \partial_\rho M M^{-1}) \quad (1.3)$$

$$\tau^{-1} \partial_\rho \tau = \frac{\rho}{8} \text{tr}((\partial_\rho M M^{-1})^2 - (\partial_z M M^{-1})^2). \quad (1.4)$$

The integrability of τ follows easily from (1.3) and (1.4). Equation (1.2) is also called the Ernst equation. We shall consider the solutions satisfying

$$P|_{(z,\rho)=(0,0)} = 1,$$

which corresponds to the gauge fixing condition (0.6).

It is also known that the equation (1.2) can be rewritten as the integrability condition of a 1-form with values in $\mathfrak{g}$ each of whose component is an element of $\mathbb{C}(z, \rho) \otimes_{\mathbb{C}} \mathbb{C}[[t]]$, where $\mathbb{C}(z, \rho)$ is the quotient field of $R = \mathbb{C}[[z, \rho]]$ and t an indeterminate called "spectral parameter". Namely, let $\mathcal{A}$ and $\mathcal{I}$ be 1-forms defined by

$$\mathcal{A} = \frac{1}{2}(dPP^{-1} - (dPP)^*) \quad \mathcal{I} = \frac{1}{2}(dPP^{-1} + (dPP)^*)$$

for any $P \in A_R N_R$, and put

$$\Omega_P = \mathcal{A} + \left(\frac{1-t^2}{1+t^2} - \frac{2t}{1+t^2} * \right) \mathcal{I},$$

where $*$ is the Hodge operator given by $*dz = d\rho$, $*d\rho = -dz$. We extend the canonical exterior derivative d on $\mathbb{C}(z, \rho)$ to that on $\mathbb{C}(z, \rho) \otimes_{\mathbb{C}} \mathbb{C}[[t]]$ by defining

$$dt = \frac{t}{(1+t^2)\rho} ((1-t^2)d\rho + 2tdz). \quad (1.5)$$

Note then that $d^2 t = 0$. Now we have

Proposition 1.2. Ω_P satisfies the integrability condition, i.e.,

$$d\Omega_P - \Omega_P \wedge \Omega_P = 0 \quad (1.6)$$

if and only if P is a solution of (1.2).

It follows from Proposition 1.2 that if P is a solution of the Ernst equation, then there exists a *potential* $p = \sum_{n \geq 0} p_n t^n$ such that each entry of p_n is an element of $\mathbb{C}(z, \rho)$ and

$$dp = \Omega_P \cdot p \quad \text{and} \quad p_0 = P. \quad (1.7)$$

2. Hauser Group

We introduce formal loop algebras and formal loop groups, following [T].

Put $F_0 = R = \mathbb{C}[[z, \rho]]$ and $F_n = \rho^{|n|}R$ for a nonzero integer n . We introduce a topology in R by declaring that $\{F_n\}_{n \geq 0}$ forms a fundamental neighborhoods system of 0. Note that $F_m F_n \subset F_{m+n}$ for $m, n \geq 0$.

Then we define a formal loop algebra $\mathcal{Fgl}$ by

$$\mathcal{Fgl} = \left\{ X = \sum_{n \in \mathbb{Z}} X_n t^n; X_n \in \mathfrak{gl}(N+2, F_n) \right\}. \quad (2.1)$$

Let $*$ be an anti-involution of $\mathcal{Fgl}$ defined by

$$X^* = \sum_{n \in \mathbb{Z}} X_n^* (-1/t)^n$$

for $X = \sum_{n \in \mathbb{Z}} X_n t^n$. This is well-defined by the definition of our filtration $\{F_n\}_{n \in \mathbb{Z}}$.

We define a formal loop group $\mathcal{FG}_0$, following [T], by

$$\mathcal{FG}_0 = \left\{ g = \sum_{n \in \mathbb{Z}} g_n t^n \in \mathcal{Fgl}; g^* J g = J, \det g = 1, g_0|_{(z, \rho)=(0,0)} = 1 \right\} \quad (2.2)$$

and its subgroups by

$$\mathcal{FK} = \left\{ k = \sum_{n \in \mathbb{Z}} k_n t^n \in \mathcal{FG}_0; \theta^{(\infty)} k = k \right\} \quad (2.3)$$

$$\mathcal{FP} = \left\{ p = \sum_{n \in \mathbb{Z}} p_n t^n \in \mathcal{FG}_0; p_0 \in A_R N_R, p_n = 0 \text{ if } n < 0 \right\}. \quad (2.4)$$

Since $\mathcal{FG}_0$ is canonically embedded in $\mathcal{Fgl}$, we can define an involution $\theta^{(\infty)}$ of $\mathcal{FG}_0$ by

$$\theta^{(\infty)}(g) = (g^*)^{-1} \quad \text{for } g \in \mathcal{FG}_0,$$

which we call Cartan involution of $\mathcal{FGL}$.

Then, using the Birkhoff decomposition ((3.17), [T]), we can decompose uniquely an element $g \in \mathcal{FG}$ as

$$g = kp \quad (k \in \mathcal{FK}, p \in \mathcal{FP}). \quad (2.5)$$

Let s be another indeterminate. Define an infinite dimensional group $\mathcal{G}^{(\infty)}$, which we call Hauser group, by

$$\mathcal{G}^{(\infty)} = \left\{ g = \sum_{n \geq 0} g_n s^n \in GL(N+2, \mathbb{C}[[s]]); g^* J g = J, \det g = 1, g_0 = 1 \right\},$$

where $\mathbb{C}[[s]]$ is a ring of formal power series in s over $\mathbb{C}$ and $g^* = \sum g_n^* s^n$.

Let j be a homomorphism of $GL(N+2, \mathbb{C}[[s]])$ into $\mathcal{FGL}$ given by

$$j : g = \sum_{n \geq 0} g_n s^n \mapsto j(g) = \sum_{n \geq 0} g_n \left(\rho \left(\frac{1}{t} - t \right) + 2z \right)^n.$$

Then it is easy to see that j is injective and that the image of $\mathcal{G}^{(\infty)}$ by j is in $\mathcal{FG}_0$. We denote by $\mathcal{FH}$ the image of $\mathcal{G}^{(\infty)}$ by j . The following equations characterize the elements of $\mathcal{FH}$ in $\mathcal{FG}$.

Lemma 2.1. *An element $g \in \mathcal{FG}$ belongs to $\mathcal{FH}$ if and only if g satisfies the following equations :*

$$\partial_t g = -\rho \left(\partial_z + \frac{1}{t} \partial_\rho \right) g \quad (2.6)$$

$$\partial_t g = -\frac{\rho}{2} \left(1 + \frac{1}{t^2} \right) \partial_z g. \quad (2.7)$$

This characterization will play an important role in the proof of our main theorem.

Definition. Let $\mathcal{FP}$ be as in (2.4). We define $\mathcal{SP}$ to be a subset of $\mathcal{FP}$ consisting of elements $p = \sum_{n \geq 0} p_n t^n$ which satisfy the following conditions :

$$dp = \Omega_{p_0} \cdot p \quad \text{and} \quad p_0|_{(z, \rho)=(0,0)} = 1. \quad (2.8)$$

We call $\mathcal{SP}$ the space of potentials.

It follows from (2.8) that p_0 is a solution of the Ernst equation (1.2) for $p = \sum_{n \geq 0} p_n t^n \in \mathcal{SP}$.

Theorem 2.2. *Let $p \in \mathcal{FP}$. Then $p \in \mathcal{SP}$ if and only if $p^* p \in \mathcal{FH}$.*

Let $p \in \mathcal{SP}$ and $g \in \mathcal{G}^{(\infty)}$. By (2.5) there exist $k \in \mathcal{FK}$ and $p_g \in \mathcal{FP}$ such that

$$p \cdot j(g) = k^{-1} \cdot p_g. \quad (2.9)$$

Then, it follows immediately from Theorem 2.2 that p_g is in $\mathcal{SP}$. Thus we can define an action of the Hauser group $\mathcal{G}^{(\infty)}$ on $\mathcal{SP}$ to the right by

$$\mathcal{SP} \times \mathcal{G}^{(\infty)} \longrightarrow \mathcal{SP} \quad (p, g) \mapsto p_g, \quad (2.10)$$

where p_g is given by (2.9).

From the fact that an element $g = \sum_{n \geq 0} g_n s^n \in \mathcal{G}^{(\infty)}$ such that $g^* = g$ and such that g_0 is positive definite decomposes as $g = h^* h$ for some $h \in \mathcal{G}^{(\infty)}$, we have

Corollary 2.3. *The action of $\mathcal{G}^{(\infty)}$ on $\mathcal{SP}$ given by (2.10) is transitive.*

Remark. As we mentioned in [S], our group $\mathcal{G}^{(\infty)}$ is too small to obtain all solutions of the Ernst equation (1.2) through the action (2.10).

3. 2-Cocycle on $\mathcal{FG}_0$

The formal loop algebra $\mathcal{Fgl}$ becomes a Lie algebra with Lie bracket $[X, Y] = XY - YX$. The map

$$\exp : \mathcal{Fgl} \longrightarrow \mathcal{FGL}$$

given by

$$\exp X = e^X = \sum_{n \geq 0} \frac{X^n}{n!} \quad (3.1)$$

is called the *formal exponential map*. Note that for any $g \in \mathcal{FG}_0$ we can find a unique element X in $\mathcal{Fgl}$ such that $g = e^X$, since the *logarithm* given by

$$\log(1 + A) = \sum_{n \geq 1} \frac{(-1)^{n-1}}{n} A^n \quad (3.2)$$

is well-defined and satisfies

$$e^{\log(1+A)} = 1 + A \quad (3.3)$$

for $A = \sum_{n \in \mathbb{Z}} a_n t^n \in \mathcal{Fgl}$ with $a_0 \in \mathfrak{gl}(N+2, \mathfrak{m})$, where $\mathfrak{m}$ is the maximal ideal of R .

For X, Y in $\mathcal{Fgl}$, let $c_n(X, Y)$ ($n = 1, 2, \dots$) be the elements in $\mathcal{Fgl}$ which are determined by

$$\exp vX \exp vY = \exp \sum_{n \geq 0} c_n(X, Y) v^n,$$

where v is an indeterminate. Furthermore c_n 's are uniquely determined by the following recursion formulas (see [V]) :

$$\begin{aligned} c_1(X, Y) &= X + Y \\ (n+1)c_{n+1}(X, Y) &= \frac{1}{2}[X - Y, c_n(X, Y)] \\ &+ \sum_{p \geq 1, 2p \leq n} K_{2p} \sum_{\substack{k_1, \dots, k_{2p} > 0 \\ k_1 + \dots + k_{2p} = n}} [c_{k_1}(X, Y), [\dots, [c_{k_{2p}}(X, Y), X + Y] \dots]] \quad (n \geq 1), \end{aligned}$$

where K_{2p} 's are determined by

$$\frac{x}{1 - e^{-x}} - \frac{1}{2}x = 1 + \sum_{p \geq 1} K_{2p} x^{2p}.$$

We set $C(X, Y) = \sum_{n \geq 1} c_n(X, Y)$. Then $C(X, Y)$ is a well-defined element of $\mathcal{Fgl}$ for X, Y such that $X_0, Y_0 \in \mathfrak{gl}(N+2, \mathfrak{m})$.

Lemma 3.1. For $n \geq 2$, there exists a $\mathcal{Fgl}$ -valued function $L_n(\cdot, \cdot)$ which satisfies

$$c_n(X, Y) = [X, L_n(X, Y)] + [Y, L_n(-Y, -X)]. \quad (3.4)$$

for $X, Y \in \mathcal{Fgl}$.

Note that L_n 's are not uniquely determined, however, we fix L_n 's so that there holds

$$L(X, vY) = \left(\frac{e^{-\text{ad}X} - 1 + \text{ad}X}{\text{ad}X(1 - e^{-\text{ad}X})} - \frac{1}{4} \right) vY + O(v^2), \quad (3.5)$$

where we put $L(X, Y) = \sum_{n \geq 2} L_n(X, Y)$ for $X, Y \in \mathcal{Fgl}$ such that $X_0, Y_0 \in \mathfrak{gl}(N+2, \mathfrak{m})$. Thus, we obtain

$$C(X, Y) = X + Y + [X, L(X, Y)] + [Y, L(-Y, -X)].$$

For a series $f = \sum_{n \in \mathbb{Z}} f_n t^n \in R[[t, t^{-1}]]$, we write

$$\text{Res}_t f = f_{-1} \in R.$$

Let $R_0 = \mathbb{R}[[z, \rho]] \subset R$, the formal power series in z and ρ over $\mathbb{R}$. We define a R_0 -valued 2-cocycle ω on $\mathcal{Fgl}$ by

$$\omega(X, Y) = \text{Res}_t \text{Re tr} X \partial_t Y$$

for $X, Y \in \mathcal{Fgl}$. Note that

$$\omega(X^*, Y^*) = -\omega(X, Y) \quad (3.6)$$

for $X, Y \in \mathcal{Fgl}$.

Now we introduce a group 2-cocycle on $\mathcal{FG}_0$, following [BM]. Note that, from (3.3), any element $g \in \mathcal{FG}_0$ can be uniquely written as $g = e^X$ for $X \in \mathcal{Fgl}$ with $X_0 \in \mathfrak{gl}(N+2, \mathfrak{m})$.

Definition. Let Ξ be a R_0 -valued function on $\mathcal{FG}_0 \times \mathcal{FG}_0$ defined by

$$\Xi(e^X, e^Y) = \omega(X, L(X, Y)) + \omega(Y, L(-Y, -X)).$$

Then Ξ defines a 2-cocycle on $\mathcal{FG}_0$, i.e. satisfies the cocycle condition :

$$\Xi(e^X, e^Y) + \Xi(e^X e^Y, e^Z) = \Xi(e^Y, e^Z) + \Xi(e^X, e^Y e^Z) \quad (3.7)$$

for $X, Y, Z \in \mathcal{Fgl}$.

4. Central Extension

For any $p \in \mathcal{SP}$, we can find an element $g \in \mathcal{FH}$ which sends the identity element $1 \in \mathcal{SP}$ to p by Corollary 2.2. Then we have $p = kg$ for some $k \in \mathcal{FK}$.

Proposition 4.1. *For $p = \sum_{n \geq 0} p_n t^n \in \mathcal{SP}$, let $g \in \mathcal{FH}$ and $k \in \mathcal{FK}$ be such that $p = kg$. Let τ be a solution of (1.3) and (1.4) corresponding to $P = p_0$. Then we have the following relations:*

$$\tau^{-1} \partial_z \tau = \partial_z \Xi(kg, g^{-1}) \quad (4.1)$$

$$\tau^{-1} \partial_\rho \tau = \partial_\rho \Xi(kg, g^{-1}). \quad (4.2)$$

Now we define a central extension of $\mathcal{FG}_0$ in terms of the cocycle Ξ .

Definition. Let $(\mathcal{FG}_0)^\sim$ be the set given by

$$(\mathcal{FG}_0)^\sim = \{(g, e^\mu); g \in \mathcal{FG}_0, \mu \in R_0\}.$$

Define a product of any two elements of $(\mathcal{FG}_0)^\sim$ by

$$(g_1, e^{\mu_1}) \cdot (g_2, e^{\mu_2}) = (g_1 g_2, e^{\mu_1 + \mu_2 + \Xi(g_1, g_2)}) \quad (4.3)$$

for $(g_1, e^{\mu_1}), (g_2, e^{\mu_2}) \in (\mathcal{FG}_0)^\sim$. Since Ξ satisfies the cocycle condition (3.7), $(\mathcal{FG}_0)^\sim$ forms a group with group multiplication given by (4.3). Namely, $(\mathcal{FG}_0)^\sim$ is a *central extension* of $\mathcal{FG}_0$.

Let $\tilde{\theta}^{(\infty)}$ be an involution of $(\mathcal{FG}_0)^\sim$ given by

$$\tilde{\theta}^{(\infty)}(g, e^\mu) = (\theta^{(\infty)}(g), e^{-\mu}).$$

If we denote by $(\mathcal{FK})^\sim$ the subgroup of $(\mathcal{FG}_0)^\sim$ consisting of elements which are fixed by $\tilde{\theta}^{(\infty)}$, then we have

$$(\mathcal{FK})^\sim = \{(k, 1) \in (\mathcal{FG}_0)^\sim; k \in \mathcal{FK}\}.$$

Let $(\mathcal{FP})^\sim$ be a subgroup of $(\mathcal{FG}_0)^\sim$ given by

$$(\mathcal{FP})^\sim = \{(p, e^\mu) \in (\mathcal{FG}_0)^\sim; p \in \mathcal{FP}, \mu \in R_0\}.$$

It follows immediately from the decomposition (2.5) of $\mathcal{FG}$ that $(\mathcal{FG}_0)^\sim$ has a unique decomposition :

$$(\mathcal{FG}_0)^\sim = (\mathcal{FK})^\sim \cdot (\mathcal{FP})^\sim. \quad (4.4)$$

Furthermore, we put

$$(\mathcal{FH})^\sim = \{(g, e^\gamma) \in (\mathcal{FG}_0)^\sim; g \in \mathcal{FH}, \gamma \in \mathbb{R}\}.$$

It follows from Lemma 3.2, [HS2] that $\mathcal{FH}$ can be regarded as a subgroup of $(\mathcal{FH})^\sim$ by

$$\mathcal{FH} \longrightarrow (\mathcal{FH})^\sim, \quad g \longmapsto (g, 1).$$

Let $(\mathcal{SP})^\sim$ be the subset of $(\mathcal{FP})^\sim$ given by

$$(\mathcal{SP})^\sim = \left\{ (p, e^\mu) \in (\mathcal{FP})^\sim; p = \sum_{n \geq 0} p_n t^n \in \mathcal{SP}, \right. \\ \left. \tau = e^{-\mu} \text{ satisfies (1.3) and (1.4) with } P = p_0 \right\}. \quad (4.5)$$

We call $(\mathcal{SP})^\sim$ the space of potentials with conformal factor.

Proposition 4.2. For $p \in \mathcal{SP}$, let $k \in \mathcal{FK}$ and $g \in \mathcal{FH}$ be as above, i.e. $p = kg$. Then we have

$$\Xi(p^*, p) = 2\Xi(kg, g^{-1}). \quad (4.6)$$

Therefore, any element of $(\mathcal{SP})^\sim$ can be written as $(p, e^{-\frac{1}{2}\Xi(p^*, p) + \gamma})$ for $p \in \mathcal{SP}, \gamma \in \mathbb{R}$.

Define an action of $(\mathcal{FH})^\sim$ on the space of potentials with conformal factor $(\mathcal{SP})^\sim$ to the right through the decomposition (4.4) :

$$(\mathcal{SP})^\sim \times (\mathcal{FH})^\sim \longrightarrow (\mathcal{SP})^\sim, \quad ((p, e^\mu), (g, e^\gamma)) \longmapsto (p_g, e^\alpha). \quad (4.7)$$

Namely, we can find a unique element $(k, 1) \in (\mathcal{FK})^\sim$ and $(p_g, e^\alpha) \in (\mathcal{FP})^\sim$ such that

$$(p, e^\mu)(g, e^\gamma) = (k, 1)^{-1}(p_g, e^\alpha),$$

where k and p_g are the elements given in (2.9). Since we have

$$\tilde{\theta}^{(\infty)}((p, e^\mu)(g, e^\gamma))^{-1} \cdot (p, e^\mu)(g, e^\gamma) = (g^* p^* p g, e^{2(\mu + \gamma) + \Xi(p^*, p)})$$

and

$$\tilde{\theta}^{(\infty)}(p_g, e^\alpha)^{-1} \cdot (p_g, e^\alpha) = (p_g^* p_g, e^{2\alpha + \Xi(p_g^*, p_g)}),$$

we obtain

$$\begin{aligned} \alpha &= \mu + \gamma + \frac{1}{2}(\Xi(p^*, p) - \Xi(p_g^*, p_g)) \\ &= \gamma' - \frac{1}{2}\Xi(p_g^*, p_g) \end{aligned}$$

for some $\gamma' \in \mathbb{R}$, where we used Proposition 4.4. Thus (p_g, e^α) belongs to $(\mathcal{SP})^\sim$, i.e. the action (4.7) of $(\mathcal{FH})^\sim$ is well-defined.

Now we state our main theorem :

Theorem 4.3. The group $(\mathcal{FH})^\sim$ acts transitively on the space of potentials with conformal factor $(\mathcal{SP})^\sim$ by (4.7).

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