

A GENERALIZATION OF THE MORITA-MUMFORD CLASSES TO EXTENDED MAPPING CLASS GROUPS FOR SURFACES

NARIYA KAWAZUMI (河澄 響矢)

Department of Mathematics, Faculty of Sciences, (北大理)
Hokkaido University, Sapporo, 060 Japan

ABSTRACT. Let $\Sigma_{g,1}$ be an orientable compact surface of genus g with 1 boundary component, and $\Gamma_{g,1}$ the mapping class group of $\Sigma_{g,1}$. We define a bigraded series of cohomology classes $m_{i,j} \in H^{2i+j-2}(\Gamma_{g,1}; \wedge^j H_1(\Sigma_{g,1}; \mathbb{Z}))$, $2i+j-2 \geq 1$, $i, j \geq 0$. When $j = 0$, the class $m_{i+1,0}$ is the i -th Morita-Mumford class $[Mo][Mu]$. It is proved that $H^r(\Gamma_{g,1}; \wedge^s H_1(\Sigma_{g,1}; \mathbb{Q}))$ is generated by $m_{i,j}$'s for the case $r+s=2$ and the case $g \geq 5$ and $(r,s) = (1,3)$. Especially the Johnson homomorphism extended to the whole mapping class group by Morita [Mo3] has an implicit representation by the classes $m_{0,3}$ and $m_{0,2}m_{1,1}$ over $\mathbb{Q}$.

INTRODUCTION

Let $g \geq 2$, $r, n \geq 0$ be integers. Let $\Sigma_{g,r}^n$ denote a 2-dimensional compact oriented C^∞ manifold (i.e., compact oriented surface) of genus g with r boundary components and (ordered) n punctures. The group of path-components $\pi_0(\text{Diff}_+(\Sigma_{g,r}^n))$ is denoted by $\Gamma_{g,r}^n$ (or $\mathcal{M}_{g,r}^n$) and called the mapping class group of genus g with r boundary components and (ordered) n punctures. Here $\text{Diff}_+(\Sigma_{g,r}^n)$ denotes the topological group (endowed with C^∞ topology) consisting of all orientation preserving diffeomorphisms of $\Sigma_{g,r}^n$ which fix the boundary components and the punctures pointwise. When $n = 0$, we drop the indices: $\Sigma_{g,r} = \Sigma_{g,r}^0$, $\Gamma_{g,r} = \Gamma_{g,r}^0$ and similarly $\Sigma_g = \Sigma_{g,0}^0$, $\Gamma_g = \Gamma_{g,0}^0$. Throughout this paper we denote by $H_1(\Sigma_{g,r}^n)$ the first integral singular homology of the space $\Sigma_{g,r}^n$, on which the group $\Gamma_{g,s}^m$ act in an obvious way provided that $s \geq r$ and $m \geq n$.

By the *extended mapping class group* we mean the semi-direct product

$$\widetilde{\Gamma}_{g,r}^n := H_1(\Sigma_{g,1}) \rtimes \Gamma_{g,r}^n.$$

The purpose of the present paper is to define a bigraded series $\widetilde{m}_{i,j}$ of cohomology classes of the extended group $\widetilde{\Gamma}_{g,1}$, which is a generalization of the Morita-Mumford cohomology classes of the group Γ_g , and to investigate the ones of lower degree.

In §1 we prepare a theory of cohomology of pairs of groups, which is essential to the construction of the classes in the succeeding two sections. The E_2 -term of the

Lyndon-Hochschild-Serre spectral sequence of the group $\widetilde{\Gamma}_{g,1}$ with respect to the normal subgroup $H_1(\Sigma_{g,1})$ is given by

$$E_2^{p,q} = H^p(\Gamma_{g,1}; \bigwedge^q H^1(\Sigma_{g,1})).$$

So the classes $\widetilde{m}_{i,j}$ induce cohomology classes $m_{i,j}$ of the group $\Gamma_{g,1}$ with values in $\bigwedge^* H^1(\Sigma_{g,1})$. When $j = 0$, the class $m_{i+1,0}$ is the i -th Morita-Mumford class [Mo][Mu]. In §4, in order to see the non-triviality, we evaluate the classes $m_{0,2}$, $m_{1,1}$ and $m_{0,3}$ and prove that $H^r(\Gamma_{g,1}; \bigwedge^s H^1(\Sigma_{g,1}; \mathbb{Q}))$ is generated by $m_{i,j}$'s for the case $r + s = 2$ (Proposition 4.1, Theorem 4.3, Corollary 4.5) and the case $g \geq 5$ and $(r, s) = (1, 3)$ (Theorem 4.4). Especially the Johnson homomorphism extended to the whole mapping class group by Morita [Mo3] has an implicit representation by the classes $m_{0,3}$ and $m_{0,2}m_{1,1}$ over $\mathbb{Q}$.

The author would like to express his gratitude to Prof. S. Morita and Prof. A. Kohno for helpful discussions.

Contents.

- §1. Cohomology of Pairs of Groups.
- §2. Mapping Class Groups.
- §3. Construction of Cohomology Classes.
- §4. Evaluations.

1. Cohomology of Pairs of Groups.

In this section we define cohomology groups $H^*(G, H : M)$ of a pair of groups (G, H) in the most naive sense. Denote by $C^*(G; M)$ the normalized cochain complex of a group G with values in a G -module M .

Let G be a group, H a subgroup of G , and M a G -module. We denote by $H^*(G, H; M)$ the cohomology group of the kernel of the restriction map

$$\text{res} : C^*(G; M) \rightarrow C^*(H; M)$$

and call it *the cohomology group of the pair of groups (G, H) with values in the G -module M* . Since the restriction map res is surjective in the cochain level, we have a cohomology exact sequence

$$(1.1) \quad \cdots \rightarrow H^{q-1}(H; M) \rightarrow H^q(G, H; M) \rightarrow H^q(G; M) \rightarrow H^q(H; M) \rightarrow \cdots,$$

In a natural way the cup product

$$\cup : H^*(G; M') \otimes H^*(G, H; M'') \rightarrow H^*(G, H; M' \otimes M'')$$

is defined.

Let $K \triangleleft G$ be a normal subgroup satisfying the condition

$$(1.2) \quad HK = G.$$

Then we have the following Lyndon-Hochschild-Serre (LHS) spectral sequence [HS].

GENERALIZATION OF MORITA-MUMFORD CLASSES

Proposition 1.3. *There is a spectral sequence converging to $H^*(G, H; M)$ whose E_2 term is given by*

$$E_2^{p,q} = H^p(G/K; H^q(K, K \cap H; M)).$$

It should be remarked how the quotient group G/K acts on the cohomology group $H^*(K, K \cap H; M)$. Since K is a normal subgroup of G , the group H acts on the normalized complex $C^*(K, K \cap H; M)$ by

$$(h \cdot c)(x_1, \dots, x_n) := h(c(h^{-1}x_1h, \dots, h^{-1}x_nh)),$$

where $h \in H$, $c \in C^n(K, K \cap H; M)$ and $x_1, \dots, x_n \in K$. For any element $h \in K \cap H$ consider a homotopy map

$$\Phi = \Phi_h : C^n(K, K \cap H; M) \rightarrow C^{n-1}(K, K \cap H; M)$$

given by

$$(\Phi_h c)(x_1, \dots, x_{n-1}) := \sum_{j=0}^{n-1} (-1)^j c(x_1, \dots, x_j, h, h^{-1}x_{j+1}h, \dots, h^{-1}x_{n-1}h),$$

This map is well-defined and satisfies a homotopy equation

$$(d\Phi_h + \Phi_h d)c = h \cdot c - c \quad (\forall c \in C^*(K, K \cap H; M)).$$

Hence the subgroup $K \cap H$ acts on the cohomology group $H^*(K, K \cap H; M)$ trivially. From the condition (1.2) and the Second Isomorphism Theorem we have a natural isomorphism

$$G/K = H/K \cap H.$$

Thus the quotient group G/K acts on the cohomology group $H^*(K, K \cap H; M)$.

Let M , M_1 and M_2 be G/K -modules. Suppose

$$(1.4) \quad H^q(K, K \cap H; \mathbb{Z}) = \begin{cases} \mathbb{Z}, & \text{if } q = n, \\ 0, & \text{if } q > n. \end{cases}$$

Then the spectral sequence (1.3) induces a homomorphism

$$(1.5) \quad \pi_! : H^p(G, H; M) \rightarrow H^{p-n}(G/K; M),$$

which is called *the Gysin map* or *the fiber integral*. As usual we have

$$(1.6) \quad \pi_!(u \cup \pi^* v) = (\pi_! u) \cup v \in H^{p+q-n}(G/K; M_1 \otimes M_2),$$

for $u \in H^p(G, H; M_1)$ and $v \in H^q(G/K; M_2)$.

2. Mapping Class Groups.

From now on we consider mainly the mapping class groups $\Gamma_{g,1}$ and $\Gamma_{g,1}^1$. First we remark that the surface $\Sigma_{g,1}^1$ is obtained by glueing the surfaces $\Sigma_{g,1}$ and $\Sigma_{0,2}^1$ along the boundaries. So the diffeomorphism of $\Sigma_{g,1}$ is naturally extended to that of $\Sigma_{g,1}^1$. The infinite cyclic group $\mathbb{Z}$ acts on the surface $\Sigma_{0,2}^1$ by rotating the puncture and fixing the boundaries pointwise. Similarly this action is extended to that on $\Sigma_{g,1}^1$ in a natural way. Thus we obtain a natural homomorphism $\Gamma_{g,1} \times \mathbb{Z} \rightarrow \Gamma_{g,1}^1$, which is injective (see [I] §5). In the sequel we regard the group $\Gamma_{g,1} \times \mathbb{Z}$ as a subgroup of $\Gamma_{g,1}^1$ through the injection. Especially we may consider the cohomology group $H^*(\Gamma_{g,1}^1, \Gamma_{g,1} \times \mathbb{Z}; M)$ for an arbitrary $\Gamma_{g,1}^1$ -module M . By forgetting the puncture we obtain an extension

$$(2.1) \quad 1 \rightarrow \pi_1(\Sigma_{g,1}) \rightarrow \Gamma_{g,1}^1 \xrightarrow{\pi} \Gamma_{g,1} \rightarrow 1.$$

Next we prepare a cycle induced by the "fiber" $\pi_1(\Sigma_{g,1})$. Choose a usual symplectic generator system of the fundamental group $\pi_1(\Sigma_{g,1})$:

$$a_1, a_2, \dots, a_g, b_1, b_2, \dots, b_g.$$

The loop on the boundary induces an element of $\pi_1(\Sigma_{g,1})$

$$w := \prod_{i=1}^g [a_i b_i], \quad [a_i, b_i] = a_i b_i a_i^{-1} b_i^{-1}.$$

We identify the group $\mathbb{Z}$ with the subgroup generated by w in $\pi_1(\Sigma_{g,1})$, and consider the cohomology group of the pair $H^*(\pi_1(\Sigma_{g,1}), \mathbb{Z})$.

Following Meyer [Me], we construct a normalized bar 2-chain $[\Sigma_{g,1}, \partial]$ as follows. For $1 \leq j \leq 4g$ let $w_j = a_i^{\pm 1}, b_i^{\pm 1}$ be the j -th generator in the element w , and $\widetilde{w}_j := w_1 w_2 \cdots w_j = a_1 b_1 \cdots w_j$. Let $\widetilde{w}_0 = 1$. We define

$$(2.2) \quad [\Sigma_{g,1}, \partial] := \sum_{j=1}^{4g} [\widetilde{w}_{j-1} | w_j] - \sum_{i=1}^g ([a_i | a_i^{-1}] + [b_i | b_i^{-1}]) \in C_2(\pi_1(\Sigma_{g,1})).$$

Lemma 2.3. *For any trivial $\pi_1(\Sigma_{g,1})$ -module M , we have*

$$H^*(\pi_1(\Sigma_{g,1}), \mathbb{Z}; M) = \begin{cases} H \otimes M, & \text{if } * = 1, \\ M, & \text{if } * = 2, \\ 0, & \text{otherwise,} \end{cases}$$

where $H = H_1(\Sigma_{g,1}; \mathbb{Z}) \cong \mathbb{Z}^{2g}$. The evaluation

$$\langle \cdot, [\Sigma_{g,1}, \partial] \rangle: H^2(\pi_1(\Sigma_{g,1}), \mathbb{Z}; M) \rightarrow M$$

is a well-defined isomorphism.

The first half of the lemma follows from the exact sequence (1.1), and the second from straightforward calculations.

GENERALIZATION OF MORITA-MUMFORD CLASSES

Now let M be a $\Gamma_{g,1}$ -module. The condition (1.2) is satisfied for our case $G = \Gamma_{g,1}^1$, $H = \Gamma_{g,1} \times \mathbb{Z}$ and $K = \pi_1(\Sigma_{g,1})$. It follows from Proposition 1.3 there exists a spectral sequence converging to

$$H^*(\Gamma_{g,1}^1, \Gamma_{g,1} \times \mathbb{Z}; M),$$

whose E_2 term is given by

$$H^p(\Gamma_{g,1}; H^q(\pi_1(\Sigma_{g,1}), \mathbb{Z}; M)) = \begin{cases} H^p(\Gamma_{g,1}; H \otimes M), & \text{if } * = 1, \\ H^p(\Gamma_{g,1}; M), & \text{if } * = 2, \\ 0, & \text{otherwise.} \end{cases}$$

Hence it induces a Gysin exact sequence

$$\begin{aligned} \cdots \rightarrow H^{q-1}(\Gamma_{g,1}; M) \rightarrow H^{q+1}(\Gamma_{g,1}; H \otimes M) \\ \rightarrow H^{q+2}(\Gamma_{g,1}^1, \Gamma_{g,1} \times \mathbb{Z}; M) \xrightarrow{\pi_!} H^q(\Gamma_{g,1}; M) \rightarrow \cdots \end{aligned}$$

Here the homomorphism $\pi_!$ is the *fiber integral* introduced in (1.5).

The Gysin sequence splits as follows. The identity map $1_{\mathbb{Z}} : \mathbb{Z} \rightarrow \mathbb{Z}$ generates the cohomology group $H^1(\mathbb{Z}) \cong \mathbb{Z}$. Regard $1_{\mathbb{Z}}$ as an element of $H^1(\Gamma_{g,1} \times \mathbb{Z})$ through the natural projection $\Gamma_{g,1} \times \mathbb{Z} \rightarrow \mathbb{Z}$ and denote by θ the image of $1_{\mathbb{Z}}$ under the connecting homomorphism δ^* :

$$\theta := \delta^*(1_{\mathbb{Z}}) \in H^2(\Gamma_{g,1}^1, \Gamma_{g,1} \times \mathbb{Z}; \mathbb{Z}).$$

Since $\langle \theta, [\Sigma_{g,1}, \partial] \rangle = -1$, we have

$$(2.4) \quad \pi_! \theta = -1 \in H^0(\Gamma_{g,1}; \mathbb{Z}).$$

Thus, from the property (1.6) of the fiber integral $\pi_!$, the sequence splits. Consequently we have

Proposition 2.5. *For any $\Gamma_{g,1}$ -module M , we have an exact sequence*

$$0 \rightarrow H^{q+1}(\Gamma_{g,1}; H \otimes M) \rightarrow H^{q+2}(\Gamma_{g,1}^1, \Gamma_{g,1} \times \mathbb{Z}; M) \xrightarrow{\pi_!} H^q(\Gamma_{g,1}; M) \rightarrow 0,$$

which splits as follows:

$$H^{q+2}(\Gamma_{g,1}^1, \Gamma_{g,1} \times \mathbb{Z}; M) = H^{q+1}(\Gamma_{g,1}; H \otimes M) \oplus \theta \cup H^q(\Gamma_{g,1}; M).$$

On the other hand, taking the semi-direct product of the extension (2.1) and the $\Gamma_{g,1}$ -module $H_1(\Sigma_{g,1}; \mathbb{Z})$, we have an extension of groups

$$(2.6) \quad 1 \rightarrow \pi_1(\Sigma_{g,1}) \rightarrow \widetilde{\Gamma_{g,1}^1} \xrightarrow{\tilde{\pi}} \widetilde{\Gamma_{g,1}^1} \rightarrow 1.$$

In a similar way to the fiber integral $\pi_!$ we obtain the *fiber integral*

$$\tilde{\pi}_! : H^q(\widetilde{\Gamma_{g,1}^1}, \widetilde{\Gamma_{g,1}} \times \mathbb{Z}; \mathbb{Z}) \rightarrow H^{q-2}(\widetilde{\Gamma_{g,1}^1}; \mathbb{Z}).$$

3. Construction of Cohomology Classes.

For the rest we often abbreviate

$$H := H_1(\Sigma_{g,1}; \mathbb{Z}) = H^1(\Sigma_{g,1}; \mathbb{Z}).$$

The isomorphism on the right-hand side is the Poincaré duality, which is $\Gamma_{g,1}$ -equivariant. We remark this H plays a different role in the sequel from the subgroup H in the preceding sections.

Denote by $\cdot$ the intersection form on $H \cong H_1(\Sigma_g; \mathbb{Z})$.

Choose a simple curve l on $\Sigma_{g,1}^1$ connecting the puncture to a point on the boundary. Define a 2-cochain $\tilde{\omega}_l \in C^2(\widetilde{\Gamma_{g,1}^1}; \mathbb{Z})$ by

$$(3.1) \quad \tilde{\omega}_l(u_1\gamma_1, u_2\gamma_2) := \gamma_1(\gamma_2 l - l) \cdot u_1, \quad u_1, u_2 \in H, \gamma_1, \gamma_2 \in \Gamma_{g,1}^1,$$

and a 1-cochain $\omega_l \in C^1(\Gamma_{g,1}^1; H)$ by

$$(3.2) \quad \omega_l(\gamma) = \gamma l - l \in H, \quad \gamma \in \Gamma_{g,1}^1,$$

where we remark $\gamma l - l$ can be regarded as a closed curve on $\Sigma_{g,1}$. A straightforward computation shows the cochains $\tilde{\omega}_l$ and ω_l are cocycles. On the other hand, if $\gamma \in \Gamma_{g,1} \times \mathbb{Z}$, the curve $\gamma l - l$ is homotopic to a curve in the boundary $\partial\Sigma_{g,1}$. Hence $\gamma l - l = 0 \in H$. Thus we have

$$(3.3) \quad \tilde{\omega}_l \in Z^2(\widetilde{\Gamma_{g,1}^1}, \widetilde{\Gamma_{g,1} \times \mathbb{Z}}; \mathbb{Z}) \quad \text{and} \quad \omega_l \in Z^2(\Gamma_{g,1}^1, \Gamma_{g,1} \times \mathbb{Z}; H).$$

To study the dependence of the cohomology classes $[\tilde{\omega}_l]$ and $[\omega_l]$ on the choice of the curve l , choose another simple curve l' on $\Sigma_{g,1}^1$ connecting the puncture to the boundary. The cycle $v := l' - l$ on $\Sigma_{g,1}^1$ may be regarded as an element in H . So we have

$$(3.4) \quad \omega_{l'} - \omega_l = dv \in C^1(\Gamma_{g,1}^1; H).$$

When we define a 1-cochain $c_v \in C^1(\widetilde{\Gamma_{g,1}^1})$ by

$$c_v(u\gamma) := (\gamma v) \cdot u, \quad u \in H, \gamma \in \Gamma_{g,1}^1,$$

we have

$$(3.5) \quad \tilde{\omega}_{l'} - \tilde{\omega}_l = dc_v.$$

Let $e \in H^2(\Gamma_g^1; \mathbb{Z})$ be the Euler class of the central extension

$$1 \rightarrow \mathbb{Z} \rightarrow \Gamma_{g,1} \rightarrow \Gamma_g^1 \rightarrow 1.$$

The class e may be regarded as a cohomology class in $H^2(\Gamma_{g,1}^1, \Gamma_{g,1} \times \mathbb{Z}; \mathbb{Z})$ in an obvious way. From (3.4) and (3.5), if $i + j \geq 2$, the products

$$\begin{aligned} e^i [\tilde{\omega}_l]^j &\in H^{2i+2j}(\widetilde{\Gamma_{g,1}^1}, \widetilde{\Gamma_{g,1} \times \mathbb{Z}}; \mathbb{Z}) \quad \text{and} \\ e^i [\omega_l]^j &\in H^{2i+j}(\Gamma_{g,1}^1, \Gamma_{g,1} \times \mathbb{Z}; \bigwedge^j H) \end{aligned}$$

GENERALIZATION OF MORITA-MUMFORD CLASSES

are independent of the choice of the curve l . We denote them by $e^i \tilde{\omega}^j$ and $e^i \omega^j$ respectively.

Recall $H^p(\Gamma_{g,1}^1; \bigwedge^q H)$ is the $E_2^{p,q}$ -term of the LHS spectral sequence of $\widetilde{\Gamma_{g,1}}$ with respect to the normal subgroup H . Since we have

$$\tilde{\omega}_l(u_1, u_2 \gamma_2) = \omega_l(\gamma_2) \cdot u_1$$

for $\forall u_1, u_2 \in H$ and $\gamma_2 \in \Gamma_{g,1}^1$, the class $[\omega_l] \in H^1(\Gamma_{g,1}^1, \Gamma_{g,1} \times \mathbb{Z}; H)$ is equal to that induced by $\tilde{\omega}_l \in H^2(\widetilde{\Gamma_{g,1}^1}, \widetilde{\Gamma_{g,1}} \times \mathbb{Z}; \mathbb{Z})$. Now we can define the cohomology classes $\widetilde{m}_{i,j}$ and $m_{i,j}$. Consider two extensions of groups

$$(2.1) \quad 1 \rightarrow \pi_1(\Sigma_{g,1}) \rightarrow \Gamma_{g,1}^1 \xrightarrow{\pi} \Gamma_{g,1}^1 \rightarrow 1$$

$$(2.6) \quad 1 \rightarrow \pi_1(\Sigma_{g,1}) \rightarrow \widetilde{\Gamma_{g,1}^1} \xrightarrow{\tilde{\pi}} \widetilde{\Gamma_{g,1}^1} \rightarrow 1.$$

We define

$$(3.6) \quad \begin{aligned} m_{i,j} &:= \pi_!(e^i \omega^j) \in H^{2i+j-2}(\Gamma_{g,1}; \bigwedge^j H) \\ \widetilde{m}_{i,j} &:= \tilde{\pi}_!(e^i \tilde{\omega}^j) \in H^{2i+2j-2}(\widetilde{\Gamma_{g,1}}; \mathbb{Z}) \end{aligned}$$

for $i, j \in \mathbb{N}$. Here $\pi_!$ and $\tilde{\pi}_!$ are the fiber integrals introduced in the previous section. Clearly $m_{i+1,0}$ and $\widetilde{m}_{i+1,0}$ are equal to (the image of) the i -th Morita-Mumford (tautological) class $e_i (= \kappa_i) \in H^{2i}(\Gamma_g; \mathbb{Z})$ [Mo][Mu]:

$$(3.7) \quad m_{i+1,0} = \widetilde{m}_{i+1,0} = e_i \in H^{2i}(\Gamma_{g,1}; \mathbb{Z}).$$

Remark 3.8. Let $\mathcal{F}_{g-1}$ be the dressed moduli of pairs of compact Riemann surfaces of genus g and holomorphic line bundles of degree $g-1$ on the surfaces. The space $\mathcal{F}_{g-1}$ is aspherical and its π_1 is equal to $\widetilde{\Gamma_{g,1}}$. As is known, the Lie algebra of holomorphic differential operators "near S^1 " has an infinitesimal and transitive action on the dressed moduli $\mathcal{F}_{g-1}$ [ADKP]. The $\widetilde{m}_{i,j}$'s have their origins in the equivariant cohomology of $\mathcal{F}_{g-1}$ under this action [Ka1].

4. Evaluations.

The purpose of this section is to evaluate the classes $m_{2,0}$, $m_{1,1}$ and $m_{0,3}$ and to prove that $H^r(\Gamma_{g,1}; \bigwedge^s H_1(\Sigma_{g,1}; \mathbb{Q}))$ is generated by $m_{i,j}$'s for the case $r + s = 2$ and the case $g \geq 5$ and $(r, s) = (1, 3)$.

Denote by Ω the symplectic form on H induced by the cup product:

$$\Omega := \sum_{i=1}^g a_i \otimes b_i - b_i \otimes a_i \in \bigwedge^2 H,$$

where $\{a_i, b_i; 1 \leq i \leq g\}$ is (the homology classes induced by) a symplectic generator system of the fundamental group $\pi_1(\Sigma_{g,1})$ as in §2.

Proposition 4.1.

$$m_{0,2} = \pi_!(\omega^2) = 2\Omega \in H^0(\Gamma_{g,1}; \bigwedge^2 H).$$

Proof. It suffices to show that

$$\langle \omega^2, [\Sigma_{g,1}, \partial] \rangle = 2\Omega.$$

Here $[\Sigma_{g,1}, \partial]$ is a 2-chain introduced in (2.2). Since $\omega(\widetilde{w_{4i}}) = 0$, we have

$$\begin{aligned} \langle \omega^2, [\Sigma_{g,1}, \partial] \rangle &= \sum_{j=1}^{4g} \omega^2(\widetilde{w_{j-1}}, w_j) - \sum_{i=1}^g (\omega^2(a_i, a_i^{-1}) + \omega^2(b_i, b_i^{-1})) \\ &= \sum_{i=1}^g a_i \wedge b_i - (a_i + b_i) \wedge a_i - (a_i + b_i - a_i) \wedge b_i + a_i \wedge a_i + b_i \wedge b_i \\ &= \sum_{i=1}^g a_i \wedge b_i - b_i \wedge a_i = 2\Omega, \end{aligned}$$

as was to be shown. $\square$

Next we study the classes $m_{1,1}$ and $m_{0,3}$. In [Mo1] and [Mo2] Morita proved

$$(4.2) \quad H^1(\Gamma_{g,1}; H) = \mathbb{Z}, \quad \text{and} \quad H^1(\Gamma_{g,1}; \bigwedge^3 H) = \mathbb{Z}^2,$$

where we denote $H = H_1(\Sigma_{g,1}; \mathbb{Z})$ as before. Our results are

Theorem 4.3. *The class $m_{1,1}$ generates the group $H^1(\Gamma_{g,1}; H)$.*

Theorem 4.4. *If $g \geq 5$, the classes $m_{0,2}m_{1,1}$ and $m_{0,3}$ generate the group $H^1(\Gamma_{g,1}; \bigwedge^3 H \otimes \mathbb{Q})$.*

The rest of this section is devoted to the proof of the theorems. As was shown by Harer [H], if $g \geq 3$, we have $H^2(\Gamma_{g,1}; \mathbb{Q}) = \mathbb{Q}$ and the class $m_{2,0} = e_1$ generates it. Hence in the case $r + s = 2$ the groups $H^r(\Gamma_{g,1}; \bigwedge^s H \otimes \mathbb{Q})$ are generated by the classes $m_{i,j}$'s. Consequently

GENERALIZATION OF MORITA-MUMFORD CLASSES

Corollary 4.5. *If $g \geq 3$, the group $H^2(\widetilde{\Gamma_{g,1}}; \mathbb{Q})$ is isomorphic to $\mathbb{Q}^3$ and the classes $\widetilde{m_{0,2}}$, $\widetilde{m_{1,1}}$ and $\widetilde{m_{2,0}}$ form its free basis.*

The first half of the corollary has been already shown by Arbarello et. al. ([ADKP] §5).

To prove the theorems we endow the surface Σ_g with a Riemannian metric. Fix a sufficiently small positive real ϵ . Let $\varpi : ST\Sigma_g \rightarrow \Sigma_g$ be the unit tangent bundle of the surface Σ_g . Denote by D^2 the unit disk in $\mathbb{C}$: $D^2 := \{z \in \mathbb{C}; |z| \leq 1\}$. We define a disk bundle D_g over $ST\Sigma_g$ by

$$D_g := \{(v_1, x_2) \in ST\Sigma_g \times \Sigma_g; \text{dist}(\varpi(v_1), x_2) \leq \epsilon\},$$

The first projection induces its projection $p_1 : D_g \rightarrow ST\Sigma_g$. The disk bundle is trivial through the projection

$$ST\Sigma_g \times D^2 \rightarrow D_g, \quad (v, z) \mapsto (v, \text{Exp}_{\varpi(v)}(\epsilon z v)).$$

Here we use the (almost) complex structure induced by the given Riemannian metric.

Consider a $\Sigma_{g,1}$ -bundle

$$p_1 : Y_g := ST\Sigma_g \times \Sigma_g - \text{int } D_g \rightarrow ST\Sigma_g$$

induced by the first projection. The fundamental group $\pi_1(ST\Sigma_g)$ is embedded into the group $\Gamma_{g,1}$ through the classifying map ι of the bundle $p_1 : Y_g \rightarrow ST\Sigma_g$, and is identified with the kernel of the forgetting map $\Gamma_{g,1} \rightarrow \Gamma_g$:

$$1 \rightarrow \pi_1(ST\Sigma_g) \xrightarrow{\iota} \Gamma_{g,1} \rightarrow \Gamma_g \rightarrow 1.$$

Since the spaces Σ_g , $ST\Sigma_g$, D_g and Y_g are all aspherical, we drop the notations $\pi_1(\cdot)$ in the cohomology groups.

The identity map $1_H \in \text{Hom}(H, H)$ induces a cohomology class

$$1_H \in H^1(\Sigma_g; H) \cong \text{Hom}(H, H).$$

By abuse of notation we denote also by 1_H the pull-back $\varpi^*(1_H)$ through the projection $\varpi : ST\Sigma_g \rightarrow \Sigma_g$:

$$1_H = \varpi^*(1_H) \in H^1(ST\Sigma_g; H) \cong \text{Hom}(H, H).$$

In [Mo1] Morita proved the following theorem (see also [Mo2] p.81 1.4 ff).

Theorem 4.6 (Morita).

$$H^1(\Gamma_{g,1}; H) = \mathbb{Z}.$$

Furthermore a crossed homomorphism $k : \Gamma_{g,1} \rightarrow H$ represents a generator of the group $H^1(ST\Sigma_g; H)$ if and only if the restriction of k to $\pi_1(ST\Sigma_g)$ is equal to $\pm(2 - 2g)1_H$:

$$\iota^*(k) = \pm(2 - 2g)1_H \in H^1(ST\Sigma_g; H).$$

As for $\bigwedge^3 H = \bigwedge^3 H_1(\Sigma_{g,1}; H)$ he proved the following ([Mo3] Theorem 5.1, see also the proof of Corollary 5.7). Let k_0 be a generator of the group $H^1(\Gamma_{g,1}; H)$.

NARIYA KAWAZUMI

Theorem 4.7 (Morita). *If $g \geq 3$,*

$$H^1(\Gamma_{g,1}; \bigwedge^3 H) = \mathbb{Z} \oplus \mathbb{Z}.$$

The class $\Omega \wedge k_0$ and a class he named $2\tilde{k}$ form its free basis. Furthermore their restriction to $\pi_1(ST\Sigma_g)$ are given by

$$\iota^*(\Omega \wedge k_0) = \pm(2 - 2g)\Omega \wedge 1_H \in H^1(ST\Sigma_g; \bigwedge^3 H),$$

$$\iota^*(2\tilde{k}) = 2\Omega \wedge 1_H \in H^1(ST\Sigma_g; \bigwedge^3 H).$$

Therefore our theorems are reduced to

Assertion 4.8.

$$(1) \quad \iota^*(m_{1,1}) = -(2 - 2g)1_H \in H^1(ST\Sigma_g; H)$$

$$(2) \quad \iota^*(m_{0,3}) = -6\Omega \wedge 1_H \in H^1(ST\Sigma_g; \bigwedge^3 H)$$

In fact, (1) implies Theorem 4.3 by Theorem 4.6. So we have $m_{2,0}m_{1,1} = \pm 2\Omega \wedge k_0$. From Theorem 4.7 the class $m_{0,3}$ has a representation $m_{0,3} = a\Omega \wedge k_0 + b(2\tilde{k})$ for some integers a and b . Since $H^1(ST\Sigma_g; \bigwedge^3 H) = H \otimes \bigwedge^3 H$ is $\mathbb{Z}$ -free, we have

$$-6 = \pm a(2 - 2g) + 2b,$$

and so $b \equiv -3 \pmod{g-1}$, while $g-1 \geq 4$. Thus we have $b \neq 0$.

This completes the proof of Theorems 4.3 and 4.4 modulo Assertion 4.8.

Let M be a $\pi_1(ST\Sigma_g)$ -module. By excision we may consider the map

$$j^* : H^*(Y_g, \partial Y_g; M) \xrightarrow[\text{exc.}]{\cong} H^*(ST\Sigma_g \times \Sigma_g, D_g; M) \rightarrow H^*(ST\Sigma_g \times \Sigma_g; M).$$

The fiber integral $p_{1!} : H^*(Y_g, \partial Y_g; M) \rightarrow H^{*-2}(ST\Sigma_g; M)$ decomposes itself into

$$H^*(Y_g, \partial Y_g; M) \xrightarrow{j^*} H^*(ST\Sigma_g \times \Sigma_g, D_g; M) \xrightarrow{p_{1!}} H^{*-2}(ST\Sigma_g; M).$$

Here the latter fiber integral $p_{1!}$ is the usual one induced by the first projection $p_1 : ST\Sigma_g \times \Sigma_g \rightarrow ST\Sigma_g$. Thus we have

$$\iota^*m_{1,1} = p_{1!}j^*(e\omega) \quad \text{and} \quad \iota^*m_{0,3} = p_{1!}j^*(\omega^3).$$

Now we have

$$j^*(e) = p_2^*e' \in H^2(ST\Sigma_g \times \Sigma_g; \mathbb{Z})$$

$$j^*(\omega) = p_2^*1_H - p_1^*1_H \in H^1(ST\Sigma_g \times \Sigma_g; H),$$

where $p_2 : ST\Sigma_g \times \Sigma_g \rightarrow \Sigma_g$ is the second projection and

$$e' = e(T\Sigma_g) \in H^2(\Sigma_g; \mathbb{Z}).$$

GENERALIZATION OF MORITA-MUMFORD CLASSES

Since $e'1_H \in H^3(\Sigma_g; H) = 0$, we have

$$\begin{aligned} \iota^* m_{1,1} &= p_{1!} j^*(e\omega) = p_{1!}(p_2^* e')(p_2^* 1_H - p_1^* 1_H) \\ &= -(p_{1!} p_2^* e')1_H = -(2-2g)1_H. \end{aligned}$$

On the other hand, since $(1_H)^3 \in H^3(\Sigma_g; \bigwedge^3 H) = 0$ and $p_{1!} p_2^* 1_H \in H^{-1}(ST\Sigma_g; H) = 0$, we have

$$j^*(\omega^3) = (p_2^* 1_H - p_1^* 1_H)^3 = -3(p_2^*(1_H)^2)p_1^* 1_H + 3(p_2^* 1_H)p_1^*(1_H)^2$$

and

$$p_{1!} j^*(\omega^3) = -3(p_{1!} p_2^*(1_H)^2)1_H + 3(p_{1!} p_2^* 1_H)(1_H)^2 = -3 \langle (1_H)^2, [\Sigma_g] \rangle 1_H,$$

where we denote by $[\Sigma_g] \in H_2(\Sigma_g; \mathbb{Z})$ the fundamental class. From a similar calculation to Proposition 4.1 follows $\langle (1_H)^2, [\Sigma_g] \rangle = 2\Omega$. Therefore

$$\iota^* m_{0,3} = p_{1!} j^*(\omega^*) = -6\Omega \wedge 1_H.$$

This completes the proof of Assertion 4.8 and so those of Theorems 4.3 and 4.4.

Remark 4.9. The crossed homomorphism $\tilde{k} = \frac{1}{2} 2\tilde{k} : \Gamma_{g,1} \rightarrow \frac{1}{2} \bigwedge^3 H$ in (4.7) is the Johnson homomorphism extended to the whole mapping class group by Morita [Mo3]. Hence Theorem 4.4 implies the Johnson homomorphism $\tilde{k}$ is represented by $m_{0,3}$ and $m_{0,2}m_{1,1}$ over $\mathbb{Q}$. The author, however, doesn't know the explicit representation of $\tilde{k}$ by $m_{0,3}$ and $m_{0,2}m_{1,1}$.

REFERENCES

- [ADKP] E. Arbarello, C. DeContini, V.G. Kac, and C. Procesi, *Moduli spaces of curves and representation theory*, Commun. Math. Phys. **117** (1988), 1–36.
- [H] J.L. Harer, *The second homology group of the mapping class group of an orientable surface*, Invent. math. **72** (1983), 221–239.
- [HS] G. Hochschild and J.-P. Serre, *Cohomology of group extensions*, Trans. Amer. Math. Soc. **74** (1953), 110–134.
- [I] A. Ishida, *Master Thesis*, (in Japanese), Univ. of Tokyo (1994).
- [Ka] N. Kawazumi, *Homology of hyperelliptic mapping class groups for surfaces*, preprint. Hokkaido Univ. **262**.
- [Ka1] ———, *Moduli space and complex analytic Gel'fand-Fuks cohomology of Riemann surfaces, III*, in preparation.
- [Me] W. Meyer, *Die Signatur von Flächenbündeln*, Math. Ann. **201** (1973), 239–264.
- [Mo] S. Morita, *Characteristic classes of surface bundles*, Inventiones math. **90** (1987), 551–577.
- [Mo1] ———, *Families of Jacobian manifolds and characteristic classes of surface bundles, I*, Ann. Inst. Fourier **39** (1989), 777–810.
- [Mo2] ———, *Families of Jacobian manifolds and characteristic classes of surface bundles, II*, Math. Proc. Camb. Phil. Soc. **105** (1989), 79–101.
- [Mo3] ———, *The extension of Johnson's homomorphism from the Torelli group to the mapping class group*, Invent. math. **111** (1993), 197–224.
- [Mu] D. Mumford, *Towards an enumerative geometry of the moduli space of curves*, Arithmetic and Geometry., Progr. Math. **36** (1983), 271–328.