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CYCLIC VECTORS IN FOCK-TYPE SPACES

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1. Introduction

Let $\mathbb{D}$ be the open unit disk of the complex plane $\mathbb{C}$. We denote the polynomial ring by $\mathcal{C}$, and the space of all entire functions by $Hol(\mathbb{C})$. Let X be a complete semi-normed space of holomorphic functions on a domain Ω in $\mathbb{C}$. For a subset E of X, let $\overline{E}$ be the closure of E in X. A function f is said to be *cyclic* in X if $f\mathcal{C} \subset X$ and $\overline{f\mathcal{C}} = X$. In the Hardy spaces $H^p(\mathbb{D})$ $(0 < p < \infty)$, it is well known that a function is cyclic if and only if it is $H^p(\mathbb{D})$-outer (see [Gar]). Also in the Bergman spaces $L^p_a(\mathbb{D})$ $(0 < p < \infty)$, it is known that a function is cyclic if and only if it is $L^p_a(\mathbb{D})$-outer (see [HKZ]). Recently the author has characterized the cyclic vectors in the classical Fock space. The classical Fock space $L^2_a(\mathbb{C})$ is

$$L^2_a(\mathbb{C}) = \left\{ f \in Hol(\mathbb{C}) : \|f\|_{L^2_a(\mathbb{C})} = \left\{ \int_{\mathbb{C}} |f(z)|^2 d\mu(z) \right\}^{1/2} < \infty \right\}$$

where

$$d\mu(z) = e^{-\frac{|z|^2}{2}} dA(z)/2\pi$$

is the Gaussian measure on $\mathbb{C}$ and dA is the ordinary Lebesgue measure. In [Izu1], we have proved the following:

Theorem A . *Let $h(z) \in Hol(\mathbb{C})$. Then the following are equivalent:*

(i) *$f(z)$ is a nonvanishing function in $L^2_a(\mathbb{C})$.*
(ii) *$f(z) = e^{h(z)}$, $h(z) = \alpha z^2 + \beta z + \gamma$ for $\alpha, \beta, \gamma \in \mathbb{C}$, $|\alpha| < \frac{1}{4}$.*
(iii) *$f(z)$ is cyclic in $L^2_a(\mathbb{C})$.*

It is known that there are non-vanishing functions in $H^p(\mathbb{D})$ and $L^p_a(\mathbb{D})$ which are not cyclic in the respective spaces (see [Gar] and [HKZ]). In fact, Brown and Shields posed the following question [BS]:

Question B . *Let Ω be bounded region in $\mathbb{C}$. Does there exist a polynomially dense Banach space X of analytic functions in Ω with the two properties*

(i) $zX \subset X$

(ii) *for any* $\lambda \in \Omega$, *point evaluation functional for* λ *is bounded,*

in which a function $f(z)$ *is cyclic if and only if* $f(z) \neq 0$ *for all* $z \in \Omega$*?*

The above theorem is not the answer of this question. But it says that there exists a polynomially dense Banach space in which every non-vanishing function is cyclic.

In this paper, we consider the cyclic vectors in more generalized spaces.

Let $0 < p < \infty$, $s > 0$ and $\alpha > 0$. Let ϕ be a positive function on $[0, \infty)$. The space $L^p_a(\mathbb{C}, \phi)$ consists of those entire functions whose semi-norm

$$\|f\|_{L^p_a(\mathbb{C},\phi)} = \left\{ \frac{1}{2\pi} \int_{\mathbb{C}} |f(z)|^p e^{-p\phi(|z|)} dA(z) \right\}^{1/p}$$

is finite. This space is called Fock-type space. Throughout this paper, we put $\phi(|z|) = \frac{\alpha}{p}|z|^s$. We study the cyclic vectors in $L^p_a(\mathbb{C}, \phi)$.

This is a summary of the paper [Izu2].

2. Results

The following is our main result:

Theorem 1. *Let* f *be a function in* $L^p_a(\mathbb{C}, \phi)$ *satisfying* $fC \subset L^p_a(\mathbb{C}, \phi)$. *Then the following are equivalent:*

(i) $f(z)$ *is a non-vanishing function.*

(ii) $f(z) = e^{h(z)}$ *for* $h(z) = \sum_{k=0}^{[s]} a_k z^k$, $a_k \in \mathbb{C}$, *where* $[s]$ *is the largest integer with* $[s] \leq s$, *and in addition* $|a_s| < \frac{\alpha}{p}$ *if* s *is an integer.*

(iii) $f(z)$ *is cyclic in* $L^p_a(\mathbb{C}, \phi)$.

We know that every non-vanishing function in the classical Fock space $L^2_a(\mathbb{C})$ is cyclic. In our case, we notice that it is not valid for some positive numbers; that is, if s is not an integer or $s = 1, 2, 3, 4$, then $L^p_a(\mathbb{C}, \phi)$ has the same property as the one in $L^2_a(\mathbb{C})$, but if $s = 5, 6, 7, \cdots$, the situation is different. For example, although $f(z) = e^{\frac{\alpha}{p} z^s}$ is a non-vanishing function in $L^p_a(\mathbb{C}, \phi)$, the function $f(z)$ does not satisfy $fC \subset L^p_a(\mathbb{C}, \phi)$. Obviously this function $f(z)$ is not cyclic. But if we consider the non-vanishing functions just satisfying $fC \subset L^p_a(\mathbb{C}, \phi)$, then the situation is similar.

To prove Theorem 1, we introduce the space $\mathcal{F}^p_\phi$ which is studied in [MMO]. The space is

$$\mathcal{F}^p_\phi = \left\{ f \in Hol(\mathbb{C}) : \|f\|^p_{\mathcal{F}^p_\phi} = \int_{\mathbb{C}} |f(z)|^p e^{-p\phi(|z|)} \rho^{-1} \Delta\phi \, dA(z) < \infty \right\}$$

where $\Delta\phi$ is the Laplacian of ϕ and $\rho^{-1}\Delta\phi$ is a regular version of $\Delta\phi$. If $p = 2$, then $\mathcal{F}^2_\phi$ is a Hilbert space with inner product

$$\langle f, g \rangle_{\mathcal{F}^2_\phi} = \int_{\mathbb{C}} f(z)\overline{g(z)} e^{-2\phi(z)} \rho^{-1} \Delta\phi \, dA(z).$$

We denote the reproducing kernel of $\mathcal{F}^2_\phi$ by K_λ, $\lambda \in \mathbb{C}$. The following lemma is proved by Marco, Massaneda and Ortega-Cerdà in [MMO, Lemma 21].

Lemma 2. *There exists a positive number C such that for any $\lambda \in \mathbb{C}$*

$$C^{-1} e^{2\phi(\lambda)} \leq \|K_\lambda\|^2_{\mathcal{F}^2_\phi} \leq C e^{2\phi(\lambda)}.$$

In [CGH], Chen, Guo and Hou proved the following:

Lemma 3.

$$\lim_{|\lambda| \to \infty} \frac{\langle f, K_\lambda \rangle_{\mathcal{F}^2_\phi}}{\|K_\lambda\|_{\mathcal{F}^2_\phi}} = 0$$

for any $f \in \mathcal{F}^2_\phi$.

By Lemma 2 and 3, we get the following:

Lemma 4. *The following are equivalent:*

(i) *$f(z) \in L^p_a(\mathbb{C}, \phi)$ is a non-vanishing function satisfying $f\mathcal{C} \subset L^p_a(\mathbb{C}, \phi)$.*

(ii) *$f(z) = e^{h(z)}$ where $h(z) = \sum_{k=0}^{[s]} a_k z^k$ and in addition $|a_s| < \frac{\alpha}{p}$ if s is an integer.*

By Lemma 4, (i)$\Leftrightarrow$(ii) in Theorem 1 has been proved.

The following two lemmas are the generalizations of the results in [GW].

Lemma 5. *Let $f \in L^p_a(\mathbb{C}, \phi)$ with $f(z) = \sum_{n=0}^{\infty} c_n z^n$. Then we have the following:*

(i) *There exists a constant $C_1 > 0$, which depends on f, satisfying*

$$|c_n| \leq C_1 e^{\frac{2-s}{ps}} \left(\frac{s\alpha e}{pn + 2 - s} \right)^{\frac{n}{s}} \|f\|_{L^p_a(\mathbb{C}, \phi)}.$$

(ii) *For large n,*

$$\|z^n\|^p_{L^p_a(\mathbb{C},\phi)} = \frac{\alpha^{-\frac{pn+2}{s}}}{s}\Gamma\left(\frac{pn+2}{s}\right) \sim \frac{1}{s\alpha}\left(2\pi\frac{pn+2-s}{s}\right)^{\frac{1}{2}}\left(\frac{pn+2-s}{s\alpha e}\right)^{\frac{pn+2-s}{s}},$$

where Γ denotes the gamma function.

(iii) *There is a constant $C_2>0$, which depends on f, satisfying*

$$\|c_n z^n\|_{L^p_a(\mathbb{C},\phi)} \leq C_2\left(\frac{pn+2-s}{s}\right)^{\frac{1}{2p}}\left(\frac{pn+2-s}{s\alpha}\right)^{\frac{2-s}{ps}}\|f\|_{L^p_a(\mathbb{C},\phi)}.$$

Using Lemma 5, we get the following lemma:

Lemma 6. *The polynomial ring C is dense in $L^p_a(\mathbb{C},\phi)$.*

Finally we show (ii)$\Leftrightarrow$(iii) in Theorem 1. Since every cyclic vector is non-vanishing, (iii)$\Rightarrow$(ii) is trivial. The idea for proving the opposite direction is from [Izu1].

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