

On the existence of duck solutions in a four-dimensional dynamic economic model

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Abstract

We consider the existence of duck solutions in a two-region business cycle model where each of the regions is described as Goodwin's business cycle model and they are coupled by interregional trade. We show that there exist duck solutions in our model with monotonic investment functions, and present results from numerical experiments.

1 Preliminaries

1.1 Duck in $\mathbb{R}^3$

We describe some results of Benoît [1] by following Kakiuchi and Tchizawa [3]. Consider the following system of differential equations in $\mathbb{R}^3$:

$$\begin{cases} \dot{x} = f(x, y, z, \varepsilon), \\ \dot{y} = g(x, y, z, \varepsilon), \\ \varepsilon \dot{z} = h(x, y, z, \varepsilon), \end{cases} \quad (1.1)$$

where f , g , and h are defined on $\mathbb{R}^3 \times \mathbb{R}^1$ and ε is infinitesimally small. We assume that system (1.1) satisfies the following conditions.

(A1) f and g are of class $\mathcal{C}^1$, and h is of class $\mathcal{C}^2$.

(A2) The slow manifold $S_1 = \{(x, y, z) \in \mathbb{R}^3 | h(x, y, z, 0) = 0\}$ is a two-dimensional differentiable manifold and intersects the set $T_1 = \{(x, y, z) \in \mathbb{R}^3 | \partial h(x, y, z, 0) / \partial z = 0\}$ transversely so that the pli set $PL = \{(x, y, z) \in S_1 \cap T_1\}$ is a one-dimensional differentiable manifold.

(A3) Either the value of f or that of g is nonzero at any point of PL .

The following equation holds by differentiating $h(x, y, z, 0)$ with respect to t :

$$h_x(x, y, z, 0)f(x, y, z, 0) + h_y(x, y, z, 0)g(x, y, z, 0) + h_z(x, y, z, 0)\dot{z} = 0,$$

where $h_\alpha(x, y, z, 0) = \partial h(x, y, z, 0) / \partial \alpha$ ($\alpha = x, y, z$). (1.1) becomes the following:

$$\begin{cases} \dot{x} = f(x, y, z, 0), \\ \dot{y} = g(x, y, z, 0), \\ \dot{z} = -\{h_x(x, y, z, 0)f(x, y, z, 0) + h_y(x, y, z, 0)g(x, y, z, 0)\} / h_z(x, y, z, 0), \end{cases} \quad (1.2)$$

where $(x, y, z) \in S_1 \setminus PL$. To avoid degeneracy in (1.2), we consider the newly revised system:

$$\begin{cases} \dot{x} = -h_x(x, y, z, 0)f(x, y, z, 0), \\ \dot{y} = -h_x(x, y, z, 0)g(x, y, z, 0), \\ \dot{z} = h_x(x, y, z, 0)f(x, y, z, 0) + h_y(x, y, z, 0)g(x, y, z, 0). \end{cases} \quad (1.3)$$

Note that system (1.3) is well defined at any point of $\mathbb{R}^3$. Therefore, system (1.3) is well defined indeed at any point of PL .

Definition 1.1 A singular point of (1.3), which is contained in PL and satisfies

$$h_x(x, y, z, 0)f(x, y, z, 0) + h_y(x, y, z, 0)g(x, y, z, 0) = 0,$$

is called a *pseudo singular point*.

(A4) For any $(x, y, z) \in S_1$, either $h_x(x, y, z, 0) \neq 0$ or $h_y(x, y, z, 0) \neq 0$ holds.

Then the slow manifold S_1 can be expressed like as $y = \varphi(x, z)$ in the neighborhood of PL and we obtain the following system, which restricts system (1.3) on S_1 :

$$\begin{cases} \dot{x} = -h_x(x, \varphi(x, z), z, 0)f(x, \varphi(x, z), z, 0), \\ \dot{z} = h_x(x, \varphi(x, z), z, 0)f(x, \varphi(x, z), z, 0) + h_y(x, \varphi(x, z), z, 0)g(x, \varphi(x, z), z, 0). \end{cases} \quad (1.4)$$

(A5) All singular points of (1.4) are nondegenerate, that is, the linearization of (1.4) at a singular point has two nonzero eigenvalues. Note that all pseudo singular points are the singular points of (1.4).

Definition 1.2 Let λ_1, λ_2 be two eigenvalues of the linearization of (1.4) at a pseudo singular point. The pseudo singular point with real eigenvalues is called a *pseudo singular saddle point* if $\lambda_1 < 0 < \lambda_2$.

Benoît [1] finally obtained the following theorem (for the definition of a duck solution in (1.1), see e.g. [3]).

Theorem 1.3 If (1.1) has a pseudo singular saddle point, then there exists a duck solution in (1.1).

1.2 Duck in $\mathbb{R}^4$

In this subsection, we consider a slow-fast system in $\mathbb{R}^4$ with a two-dimensinal slow manifold. We reduce it to the system in $\mathbb{R}^2$ by following Tchizawa [5, 6] and provide the condition for the existence of a duck solution. Consider the following system of differential equations in $\mathbb{R}^4$:

$$\begin{cases} \varepsilon \dot{x}_1 = h_1(x_1, x_2, y_1, y_2, \varepsilon), \\ \varepsilon \dot{x}_2 = h_2(x_1, x_2, y_1, y_2, \varepsilon), \\ \dot{y}_1 = f_1(x_1, x_2, y_1, y_2, \varepsilon), \\ \dot{y}_2 = f_2(x_1, x_2, y_1, y_2, \varepsilon), \end{cases} \quad (1.5)$$

where f_1, f_2, h_1 , and h_2 are defined on $\mathbb{R}^4 \times \mathbb{R}^1$ and ε is infinitesimally small. In the following we use the notations $x = (x_1, x_2)^\top$, $y = (y_1, y_2)^\top$, $f = (f_1, f_2)^\top$, and $h = (h_1, h_2)^\top$. We assume that system (1.5) satisfies the following conditions.

(B1) f is of class $\mathbb{C}^1$ and h is of class $\mathbb{C}^2$.

(B2) The slow manifold $S_2 = \{(x, y) \in \mathbb{R}^4 | h(x, y, 0) = 0\}$ is a two-dimensional differentiable manifold and intersects the set $T_2 = \{(x, y) \in \mathbb{R}^4 | \det(\frac{\partial h}{\partial x}(x, y, 0)) = 0\}$ transversely so that the generalized pli set $GPL = \{(x, y) \in S_2 \cap T_2\}$ is a one-dimensional differentiable manifold.

(B3) Either the value of f_1 or that of f_2 is nonzero at any point of GPL .

(B4) $\text{rank}(\frac{\partial h}{\partial x}(x, y, 0)) = 2$ for any $(x, y) \in S_2 \setminus GPL$, $\text{rank}(\frac{\partial h}{\partial y}(x, y, 0)) = 2$ for any $(x, y) \in S_2$, and $\frac{\partial h_1}{\partial x_2}(x, y, 0) \neq 0$ or $\frac{\partial h_2}{\partial x_1}(x, y, 0) \neq 0$ for any $(x, y) \in GPL$.

From the last part of (B4) we see that the implicit function theorem guarantees the existence of a unique function $x_2 = \psi_2(x_1, y_1, y_2)$ (respectively, $x_1 = \psi_1(x_2, y_1, y_2)$) such that $h_1(x_1, \psi_2(x_1, y_1, y_2), y_1, y_2, 0) = 0$ (respectively, $h_2(\psi_1(x_2, y_1, y_2), x_2, y_1, y_2, 0) = 0$). By using the relation $x_2 = \psi_2(x_1, y_1, y_2)$ and h_2 instead of h_1 to avoid redundancy, (1.5) can be reduced the following slow-fast system in $\mathbb{R}^3$ under the condition that $\dot{x}_1$ and $\dot{x}_2$ are limited, that is, $\varepsilon|\dot{x}_1 - \dot{x}_2|$ tends to 0 as ε tends to 0:

$$\begin{cases} \dot{y}_1 = f_1(x_1, \psi_2(x_1, y_1, y_2), y_1, y_2, \varepsilon), \\ \dot{y}_2 = f_2(x_1, \psi_2(x_1, y_1, y_2), y_1, y_2, \varepsilon), \\ \varepsilon \dot{x}_1 = h_2(x_1, \psi_2(x_1, y_1, y_2), y_1, y_2, \varepsilon). \end{cases} \quad (1.6)$$

Similarly, we can get the following system:

$$\begin{cases} \dot{y}_1 = f_1(\psi_1(x_2, y_1, y_2), x_2, y_1, y_2, \varepsilon), \\ \dot{y}_2 = f_2(\psi_1(x_2, y_1, y_2), x_2, y_1, y_2, \varepsilon), \\ \varepsilon \dot{x}_2 = h_1(\psi_1(x_2, y_1, y_2), x_2, y_1, y_2, \varepsilon). \end{cases} \quad (1.7)$$

Definition 1.4 If there exist duck solutions in both (1.6) and (1.7) at the common pseudo singular point, they are called *duck solutions* in (1.5). If there exists a duck solution in either of them, it is called a *partial duck solution* in (1.5).

From Theorem 1.3 we have the following corollary.

Corollary 1.5 *If either (1.6) or (1.7) has a pseudo singular saddle point, then there exists a partial duck solution in (1.5). If both (1.6) and (1.7) have a common pseudo singular saddle point, then there exist duck solutions in (1.5).*

By differentiating $h(x, y, 0)$ with respect to t , we have

$$\frac{\partial h}{\partial x}(x, y, 0)\dot{x} + \frac{\partial h}{\partial y}(x, y, 0)\dot{y} = 0, \quad (1.8)$$

where $\dot{x} = (\dot{x}_1, \dot{x}_2)^\top$ and $\dot{y} = (\dot{y}_1, \dot{y}_2)^\top$. By using the relation $\dot{y} = f(x, y, 0)$, (1.8) becomes

$$\dot{x} = - \left[\frac{\partial h}{\partial x}(x, y, 0) \right]^{-1} \frac{\partial h}{\partial y}(x, y, 0) f(x, y, 0).$$

By applying the second part of (B4), y is uniquely described like as $y = \varphi(x)$ and we have

$$\dot{x} = - \left[\frac{\partial h}{\partial x}(x, \varphi(x), 0) \right]^{-1} \frac{\partial h}{\partial y}(x, \varphi(x), 0) f(x, \varphi(x), 0). \quad (1.9)$$

To avoid degeneracy in (1.9), we consider the following system:

$$\dot{x} = -\det \left(\frac{\partial h}{\partial x}(x, \varphi(x), 0) \right) \left[\frac{\partial h}{\partial x}(x, \varphi(x), 0) \right]^{-1} \frac{\partial h}{\partial y}(x, \varphi(x), 0) f(x, \varphi(x), 0). \quad (1.10)$$

(B5) All singular points of (1.10) are nondegenerate.

Definition 1.6 A singular point of (1.10) is called a *generalized pseudo singular point*.

Definition 1.7 Let λ_1, λ_2 be two eigenvalues of the linearization of (1.10) at a generalized pseudo singular point. The pseudo singular point with real eigenvalues is called a *generalized pseudo singular saddle point* if $\lambda_1 < 0 < \lambda_2$.

By applying Benoît's criterion, Tchizawa [5, 6] finally obtained the following theorem.

Theorem 1.8 *If (1.5) has a generalized pseudo singular saddle point, then there exists a partial duck solution in (1.5).*

2 Economic models

2.1 Goodwin's business cycle model

The Goodwin model consists of a national income identity $y(t)$, a consumption function $c(t)$, and an investment function $\dot{k}(t)$:

$$\begin{aligned} y(t) &= c(t) + \dot{k}(t) - \varepsilon \dot{y}(t), \\ c(t) &= \alpha y(t) + \beta(t), \\ \dot{k}(t + \theta) &= \varphi(\dot{y}(t)) + l(t + \theta), \end{aligned} \quad (2.1)$$

where $k(t)$ denotes capital stock, ε (> 0) a constant expressing a lag in the multiplier process, α ($0 < \alpha < 1$) the marginal propensity to consume, $\beta(t)$ an autonomous consumption, $\varphi(\dot{y}(t))$ the induced investment function as shown in Figure 1, $l(t)$ is the autonomous investment, and θ the lag between the decision to invest and the corresponding outlays, respectively. Goodwin finally obtained the following second-order differential equation (see [2] for details):

$$\varepsilon \theta \ddot{z} + [\varepsilon + (1 - \alpha)\theta] \dot{z} - \varphi(\dot{z}) + (1 - \alpha)z = 0, \quad (2.2)$$

where z is the deviations from the equilibrium income. Using graphical integration method, Goodwin showed that (2.2) has a unique limit cycle. Viewing recent progress in information and production technologies, we may take ε and θ to be small. As ε is the parameter depending on the speed of information propagation, we can consider the situation where ε tends to 0. On the other hand, as θ concerns production process, we would not take θ to be small comparable to ε . Hence we shall henceforth assume

$$0 < \varepsilon \ll \theta \ll 1.$$

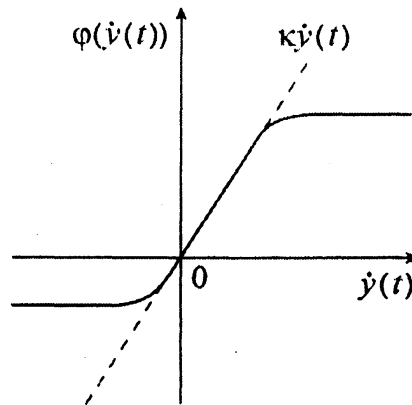


Figure 1 The induced investment function.

2.2 Two-region business cycle model

Now we present a two-region business cycle model which is a natural extension of the Goodwin model obtained by introducing interregional trade. More precisely, the model consists of the following equations:

$$\begin{aligned} y_i(t) &= c_i(t) + \dot{k}_i(t) - \varepsilon_i \dot{y}_i(t) + e_i(t) - m_i(t), \\ c_i(t) &= \alpha_i y_i(t) + \beta_i(t), \\ \dot{k}_i(t + \theta_i) &= \varphi_i(\dot{y}_i(t)) + l_i(t + \theta_i), \end{aligned} \quad (2.3)$$

where the subscript i ($i = 1, 2$) denotes the region i , $e_i(t)$ the export of the region i , and $m_i(t)$ the import of the region i , respectively. For simplicity, we put $\varepsilon_1 = \varepsilon_2 = \varepsilon$ and $\theta_1 = \theta_2 = \theta$. As to the export and import terms, we put

$$e_i(t + \theta) = m_j(t + \theta) = a_j y_j(t) + b_j \varphi_j(\dot{y}_j(t)),$$

where the subscript j ($j = 1, 2$) denotes the region different from the region i , and $a_i \geq 0$ and $b_i > 0$ are constants.

By the same transformation as that in the Goodwin model, we have the following second-order equation:

$$\varepsilon \theta \ddot{z}_i + [\varepsilon + (1 - \alpha_i) \theta] \dot{z}_i - (1 - b_i) \varphi_i(\dot{z}_i) - b_j \varphi_j(\dot{z}_j) + (1 - \alpha_i + a_i) z_i - a_j z_j = 0.$$

Setting new variables, $x_i = \dot{z}_i$ ($i = 1, 2$), we obtain the following system:

$$\begin{cases} \varepsilon \dot{x}_1 = -\frac{1 - \alpha + a_1}{\theta} z_1 + \frac{a_2}{\theta} z_2 - \left(\frac{\varepsilon}{\theta} + 1 - \alpha\right) x_1 + \frac{1 - b_1}{\theta} \varphi_1(x_1) + \frac{b_2}{\theta} \varphi_2(x_2), \\ \varepsilon \dot{x}_2 = \frac{a_1}{\theta} z_1 - \frac{1 - \alpha + a_2}{\theta} z_2 + \frac{b_1}{\theta} \varphi_1(x_1) - \left(\frac{\varepsilon}{\theta} + 1 - \alpha\right) x_2 + \frac{1 - b_2}{\theta} \varphi_2(x_2), \\ \dot{z}_1 = x_1, \\ \dot{z}_2 = x_2. \end{cases} \quad (2.4)$$

System (2.4) is the specific case of system (1.10) when we consider the situation where ε tends to 0, so we can apply Tchizawa's result to (2.4) in order to investigate the

existence of a duck solution. It can be shown that there does not exist a duck solution in the Goodwin model as far as the induced investment function φ is the type of the function as shown in Figure 1 (see [4]). Tchizawa *et al* [7] considered the Goodwin-like business cycle model and showed that there exists the condition on the economic parameters under which a duck solution exists when we use a cubic polynomial as the function φ . In the next section, we prove that there exist duck solutions in (2.4) even though we use a monotone increasing function with upper and lower limits as the investment function.

3 Duck solutions in the two-region model

By following the procedure described in Section 1.2, we obtain the following system in $\mathbb{R}^2$, which corresponds to (1.10):

$$\begin{cases} \dot{x}_1 = \frac{1}{\theta^2} \left[\left(-(1 - \alpha_2)\theta + (1 - b_2) \frac{d\varphi_2(x_2)}{dx_2} \right) (1 - \alpha_1 + a_1) + a_1 b_2 \frac{d\varphi_2(x_2)}{dx_2} \right] x_1 \\ \quad - \frac{1}{\theta^2} \left[\left(-(1 - \alpha_2)\theta + (1 - b_2) \frac{d\varphi_2(x_2)}{dx_2} \right) a_2 + b_2 (1 - \alpha_2 + a_2) \frac{d\varphi_2(x_2)}{dx_2} \right] x_2, \\ \dot{x}_2 = -\frac{1}{\theta^2} \left[b_1 (1 - \alpha_1 + a_1) \frac{d\varphi_1(x_1)}{dx_1} + \left(-(1 - \alpha_1)\theta + (1 - b_1) \frac{d\varphi_1(x_1)}{dx_1} \right) a_1 \right] x_1 \\ \quad + \frac{1}{\theta^2} \left[a_2 b_1 \frac{d\varphi_1(x_1)}{dx_1} + \left(-(1 - \alpha_1)\theta + (1 - b_1) \frac{d\varphi_1(x_1)}{dx_1} \right) (1 - \alpha_2 + a_2) \right] x_2. \end{cases} \quad (3.1)$$

In what follows, we put $\alpha_1 = \alpha_2 = \alpha$ and $\varphi_i(x_i) = \tanh x_i$ ($i = 1, 2$) for the sake of the specific calculation of the generalized pseudo singular points. Note that the hyperbolic tangent is a typical example of the function as shown in Figure 1. Then the generalized pseudo singular points, that is, the singular points of (3.1) are determined by the following system:

$$\begin{cases} \left[\left(-(1 - \alpha)\theta + \frac{4(1 - b_2)}{(\exp(x_2) + \exp(-x_2))^2} \right) (1 - \alpha + a_1) \right. \\ \quad \left. + \frac{4a_1 b_2}{(\exp(x_2) + \exp(-x_2))^2} \right] x_1 \\ - \left[\left(-(1 - \alpha)\theta + \frac{4(1 - b_2)}{(\exp(x_2) + \exp(-x_2))^2} \right) a_2 \right. \\ \quad \left. + \frac{4b_2(1 - \alpha + a_2)}{(\exp(x_2) + \exp(-x_2))^2} \right] x_2 = 0, \\ - \left[\frac{4b_1(1 - \alpha + a_1)}{(\exp(x_1) + \exp(-x_1))^2} \right. \\ \quad \left. + \left(-(1 - \alpha)\theta + \frac{4(1 - b_1)}{(\exp(x_1) + \exp(-x_1))^2} \right) a_1 \right] x_1 \\ + \left[\frac{4a_2 b_1}{(\exp(x_1) + \exp(-x_1))^2} \right. \\ \quad \left. + \left(-(1 - \alpha)\theta + \frac{4(1 - b_1)}{(\exp(x_1) + \exp(-x_1))^2} \right) (1 - \alpha + a_2) \right] x_2 = 0. \end{cases} \quad (3.2)$$

In the case $x_1 = -x_2 (\neq 0)$, (3.2) can be reduced to the equation:

$$\begin{cases} \left((1-\alpha)(1-\alpha+a_1+a_2)\theta - \frac{4(1-\alpha+a_1+a_2)}{(\exp(x_1) + \exp(-x_1))^2} \right) x_1 = 0, \\ \left((1-\alpha)(1-\alpha-a_1-a_2)\theta - \frac{4(1-\alpha-a_1-a_2)}{(\exp(x_1) + \exp(-x_1))^2} \right) x_1 = 0. \end{cases}$$

Therefore the generalized pseudo singular points satisfy the following equation:

$$(1-\alpha)\theta = \frac{4}{(\exp(x_1) + \exp(-x_1))^2}.$$

Putting $Y = \sqrt{\frac{4}{(1-\alpha)\theta}}$ and $Z = \exp(x_1)$, we obtain

$$Z = \frac{Y \pm \sqrt{Y^2 - 4}}{2}.$$

From $0 < \alpha < 1$ and $\theta \ll 1$, we have $Y^2 - 4 > 0$. Then we get the following two generalized pseudo singular points:

$$P_1 = (X, -X), P_2 = (-X, X),$$

where

$$X = \log \frac{Y + \sqrt{Y^2 - 4}}{2} > \log \frac{2+0}{2} = 0.$$

Next we investigate the eigenvalues of the linearization of (3.1) at these generalized pseudo singular points. The matrix we consider is as follows:

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

where

$$\begin{aligned} A &= (1-\alpha)(1-\alpha+a_1)\theta - \frac{4[(1-\alpha)(1-b_2)+a_1]}{(\exp(X) + \exp(-X))^2}, \\ B &= -\frac{8(1-\alpha+a_1+a_2)(\exp(X) - \exp(-X))X}{(\exp(X) + \exp(-X))^3} - (1-\alpha)\theta a_2 + \frac{(1-\alpha)b_2+a_2}{(\exp(X) + \exp(-X))^2}, \\ C &= -\frac{8(1-\alpha+a_1+a_2)(\exp(X) + \exp(-X))X}{(\exp(X) + \exp(-X))^3} - (1-\alpha)\theta a_1 + \frac{(1-\alpha)b_1+a_1}{(\exp(X) + \exp(-X))^2}, \\ D &= (1-\alpha)(1-\alpha+a_2)\theta - \frac{4[(1-\alpha)(1-b_1)+a_2]}{(\exp(X) + \exp(-X))^2}. \end{aligned}$$

The characteristic equation is $\lambda^2 - (A+D)\lambda + AD - BC = 0$ and we have two eigenvalues

$$\lambda_1, \lambda_2 = \frac{(A+D) \pm \sqrt{(A+D)^2 - 4(AD - BC)}}{2}.$$

In a general economic condition, we can prove

$$\lambda_1 \lambda_2 = AD - BC < -2(1-\alpha)^3(1-\alpha+a_1+a_2)\theta^2 X(2X \tanh X - b_1 - b_2) < 0.$$

Therefore, we have two generalized pseudo singular saddle points and the following theorem is established by Theorem 1.8.

Theorem 3.1 *If $\alpha_1 = \alpha_2 = \alpha$ and $\varphi_i(x_i) = \tanh x_i$ ($i = 1, 2$), then there exist partial duck solutions in (2.4).*

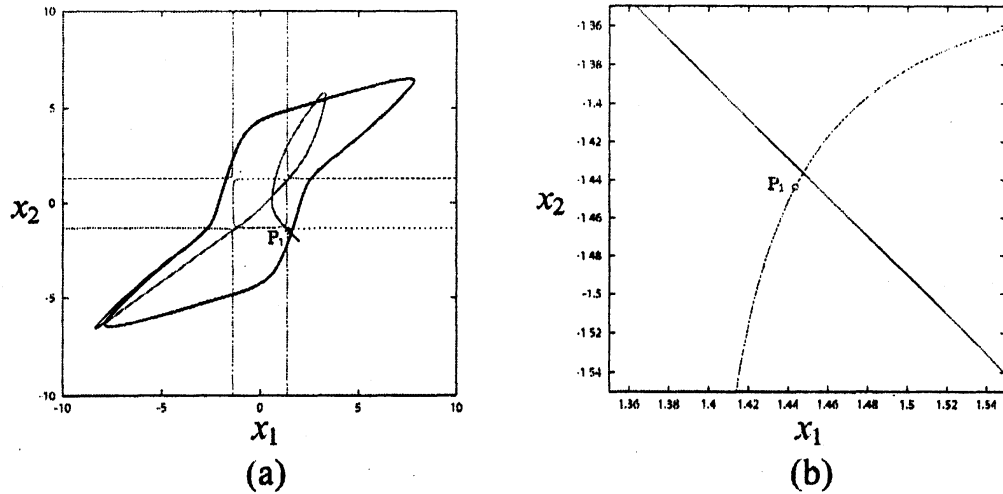


Figure 2 The solution of (2.4) and the generalized pseudo singular point P_1 . (a) Projection onto the (x_1, x_2) plane. The dotted lines are *GPL*. (b) Enlarged view of (a) in the neighborhood of P_1 .

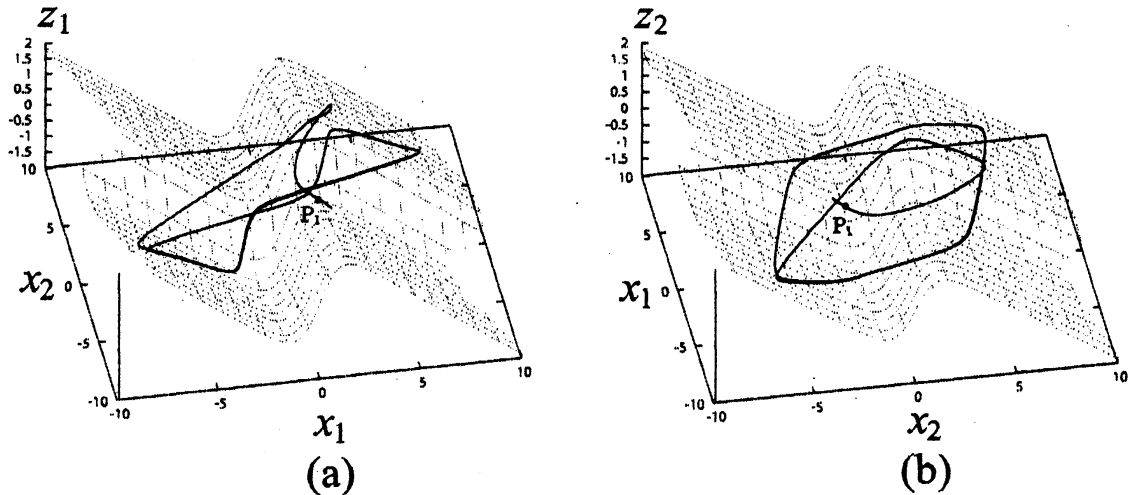


Figure 3 The solution and the slow manifold of (2.4), and P_1 . (a) Projection onto the (x_1, x_2, z_1) space. (b) Projection onto the (x_2, x_1, z_2) space.

4 Numerical example

We illustrate our results with numerical examples. The parameters values are as follows:

$$\alpha = 0.6, \theta = 0.5, \varepsilon = 0.003, a_1 = 0.1, a_2 = 0.2, b_1 = 0.1, b_2 = 0.2,$$

and then we obtain $P_1 = (1.44364, -1.44364)$ and $P_2 = (-1.44364, 1.44364)$, the eigenvalues 0.82036 and -0.358036 , and the corresponding eigenvectors $(0.686459, -0.727169)$ and $(0.678577, 0.73453)$, respectively. Hence we have two generalized pseudo singular saddle points in (2.4).

Finally, we present the results of numerical simulation of (2.4). The results shown in Figures 2-5 are calculated by using the fourth-order Runge-Kutta method. In (x_1, x_2) plane, after the solution passes near P_1 , it jumps upward and then converges to a limit cycle. In the following we focus on studying the behavior of the solution around P_1 . Next

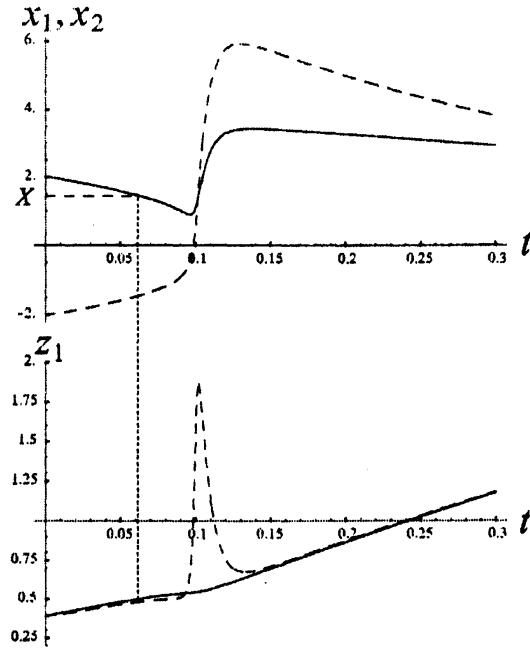


Figure 4 (Top panel) $x_1(t)$ (solid line) and $x_2(t)$ (dashed line) of (2.4). (Bottom panel) $z_1(t)$ of (2.4) (solid line) and z_1 coordinate of the slow manifold projected onto (x_1, x_2, z_1) space when x_1 and x_2 are the components of the solution to (2.4) (dashed line). The dotted line indicates the position when the solution passes near P_1 .

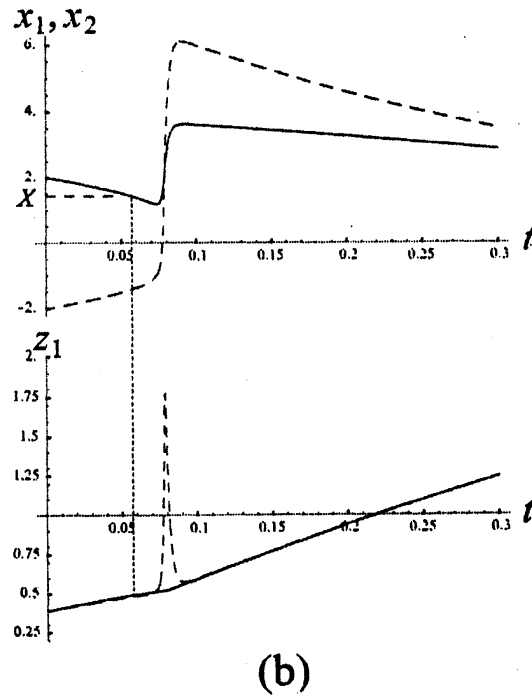
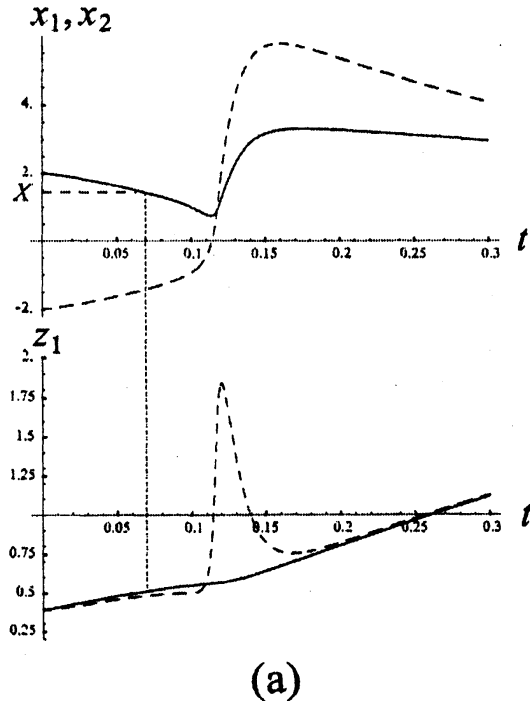


Figure 5 (Top panels) $x_1(t)$ (solid line) and $x_2(t)$ (dashed line) of (2.4) under the condition that the position and distance between the solution and P_1 are the same as those in Fig. 2(b). (Bottom panels) $z_1(t)$ of (2.4) (solid line) and z_1 coordinate of the slow manifold projected onto (x_1, x_2, z_1) space when x_1 and x_2 are the components of the solution to (2.4) (dashed line); (a) $\varepsilon = 0.005$; (b) $\varepsilon = 0.001$.

we observe this trajectory by giving three-dimensional views. Two three-dimensional projections of the solution, the slow manifold of (2.4), and P_1 are depicted in Figure 3. One can see that, after passing near P_1 , the solution moves along the slow manifold for a short distance and then turns the direction. Figure 4 indicates that the time during which the solution stays on the slow manifold is much larger than the order of ε . As shown in Figures 4 and 5, even though ε decreases, the solution moves close to the slow manifold during about 0.02 independent of the value of ε . These observed phenomena demonstrate the properties of a duck solution. In our examples, duck solutions occur at the moment of the transition from the state in out of phase in the business cycles to their synchronization.

5 Concluding remarks

In this paper, we have shown that there exist duck solutions in the two-region model even though we use monotonic investment functions. Notice that the Goodwin model never has duck solutions unless we have an artificial setting for the induced investment function as shown in [7].

In our numerical experiments, we adopted the hyperbolic tangent. It still remains a question whether our model exhibits a duck phenomenon under more general investment functions.

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