

A quandle cocycle invariant with non-commutative flows for a handlebody-knot

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Abstract

This is a summary of the construction of the quandle cocycle invariant obtained in the joint work with Iwakiri, Jang and Oshiro [7]. Iwakiri and the author [6] introduced a notion of a flow, and defined a quandle cocycle invariant for handlebody-knots. The quandle cocycle invariant given in this article is defined by using “non-commutative” flows.

1 A G -family of quandles

A quandle [8, 9] is a non-empty set X with a binary operation $*$: $X \times X \rightarrow X$ satisfying

- $x * x = x$ ($x \in X$),
- $*x : X \rightarrow X$ is bijective ($x \in X$),
- $(x * y) * z = (x * z) * (y * z)$ ($x, y, z \in X$).

An Alexander quandle $(M, *)$ is a Λ -module M with the binary operation defined by $x * y = tx + (1 - t)y$, where $\Lambda := \mathbb{Z}[t, t^{-1}]$. A conjugation quandle $(G, *)$ is a group G with the binary operation defined by $x * y = y^{-1}xy$.

A G -family of quandles is a non-empty set X with a family of binary operations $*^g : X \times X \rightarrow X$ ($g \in G$) satisfying

- $x *^g x = x$ ($x \in X, g \in G$),
- $x *^{gh} y = (x *^g y) *^h y, x *^e y = x$ ($x, y \in X, g, h \in G$),

$$\bullet (x *^g y) *^h z = (x *^h z) *^{h^{-1}gh} (y *^h z) \quad (x, y, z \in X, g, h \in G).$$

Proposition 1. Let G be a group, and $(X, \{*\^g\}_{g \in G})$ a G -family of quandles.

- (1) For any $g \in G$, $(X, *\^g)$ is a quandle.
- (2) We define $*$: $(X \times G) \times (X \times G) \rightarrow X \times G$ by

$$(x, g) * (y, h) = (x *^h y, h^{-1}gh).$$

Then $(X \times G, *)$ is a quandle

We call the quandle $(X \times G, *)$ given in Proposition 1 the *associated quandle* of X .

Proposition 2. Let R be a ring, G a group, and X a right $R[G]$ -module. We define a binary operation $*^g : X \times X \rightarrow X$ by $x *^g y = xg + y(e - g)$. Then X is a G -family of quandles.

Let X be a G -family of quandles, and Q the associated quandle of X . The *associated group* of X , denoted by $\text{As}(X)$, is defined by

$$\text{As}(X) = \left\langle q \in Q \left| \begin{array}{l} q_1 * q_2 = q_2^{-1} q_1 q_2 \quad (q_1, q_2 \in Q), \\ (x, gh) = (x, g)(x, h) \quad (x \in X, g, h \in G) \end{array} \right. \right\rangle.$$

An X -set Y is a set equipped with a right action of the associated group $\text{As}(X)$. We denote by $y * q$ the image of an element $y \in Y$ by the action $q \in \text{As}(X)$. We also denote $y * (x, g)$ by $y *^g x$. Any singleton set $\{y\}$ is an X -set with the trivial action, which is a trivial X -set. The set X is also an X -set with the action defined by $y * (x, g) = y *^g x$ for $y \in X, (x, g) \in Q$.

2 A handlebody-link

A *handlebody-link* is a disjoint union of handlebodies embedded in the 3-sphere S^3 . Two handlebody-links are *equivalent* if there is an orientation-preserving self-homeomorphism of S^3 which sends one to the other. A *spatial graph* is a finite graph embedded in S^3 . Two spatial graphs are *equivalent* if there is an orientation-preserving self-homeomorphism of S^3 which sends one to the other. When a handlebody-link H is a regular neighborhood of a spatial graph K , we say that H is represented by K . In this article,

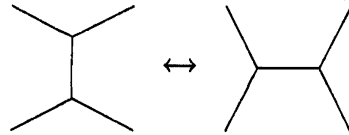


Figure 1:

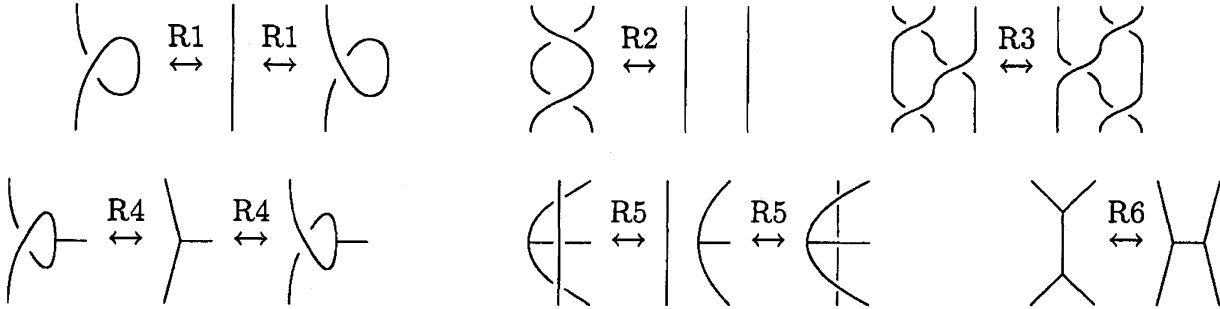


Figure 2:

a trivalent graph may contain circle components. Then any handlebody-link can be represented by some spatial trivalent graph. A *diagram* of a handlebody-link is a diagram of a spatial trivalent graph which represents the handlebody-link. An *IH-move* is a local spatial move on spatial trivalent graphs as described in Figure 1.

Theorem 3 ([5]). For spatial trivalent graphs K_1 and K_2 , the following are equivalent.

- K_1 and K_2 represent an equivalent handlebody-link.
- K_1 and K_2 are related by a finite sequence of IH-moves.
- Diagrams of K_1 and K_2 are related by a finite sequence of the moves depicted in Figure 2.

3 A coloring with G -family of quandles

Let D be a diagram of a handlebody-link H . Putting an orientation to each edge in D , we obtain a diagram D of an oriented spatial trivalent graph. We may represent an orientation of an edge by a normal orientation, which is obtained by rotating a usual orientation $\pi/2$ counterclockwise on the diagram.

For an arc incident to a vertex ω , we define $\epsilon(\alpha; \omega) \in \{1, -1\}$ by

$$\epsilon(\alpha; \omega) = \begin{cases} 1 & \text{the orientation of the arc } \alpha \text{ points to the vertex } \omega, \\ -1 & \text{otherwise.} \end{cases}$$

We denote by $\mathcal{A}(D)$ (resp. $\mathcal{R}(D)$) the set of arcs (resp. complementary regions) of D . Let D be a diagram of an oriented spatial trivalent graph. Let X be a G -family of quandles, Y an X -set, and Q be the associated quandle of X . Let p_X and p_G be the projections from Q to X and G , respectively. An X_Y -coloring of D is a map $C : \mathcal{A}(D) \cup \mathcal{R}(D) \rightarrow Q \cup Y$ satisfying the following conditions (see Figures 3, 4).

- $C(\mathcal{A}(D)) \subset Q$, $C(\mathcal{R}(D)) \subset Y$.
- Let χ_3 be the over-arc at a crossing χ . Let χ_1, χ_2 be the under-arc at the crossing χ such that the normal orientation of χ_3 points from χ_1 to χ_2 . Then

$$C(\chi_2) = C(\chi_1) * C(\chi_3).$$

- Let $\omega_1, \omega_2, \omega_3$ be the arcs incident to a vertex ω . Then

$$\begin{aligned} (p_X \circ C)(\omega_1) &= (p_X \circ C)(\omega_2) = (p_X \circ C)(\omega_3), \\ (p_G \circ C)(\omega_1)^{\epsilon(\omega_1; \omega)} (p_G \circ C)(\omega_2)^{\epsilon(\omega_2; \omega)} (p_G \circ C)(\omega_3)^{\epsilon(\omega_3; \omega)} &= e. \end{aligned}$$

- For any arc $\alpha \in \mathcal{A}(D)$, we have

$$C(\alpha_1) * C(\alpha) = C(\alpha_2),$$

where α_1, α_2 are the regions facing the arc α so that the normal orientation of α points from α_1 to α_2 .

We denote by $\text{Col}_X(D)_Y$ the set of X_Y -colorings of D .

For two diagrams D and E which locally differ, we denote by $\mathcal{A}(D, E)$ (resp. $\mathcal{R}(D, E)$) the set of arcs (resp. regions) that D and E share.

Lemma 4. Let X be a G -family of quandles, and Y an X -set. Let D be a diagram of an oriented spatial trivalent graph. Let E be a diagram obtained by applying one of the R1–R6 moves to the diagram D once, where we choose orientations for E which agree with those for D on $\mathcal{A}(D, E)$. For $C \in \text{Col}_X(D)_Y$, there is a unique X_Y -coloring $C_{D,E} \in \text{Col}_X(E)_Y$ such that $C|_{\mathcal{A}(D,E)} = C_{D,E}|_{\mathcal{A}(D,E)}$ and $C|_{\mathcal{R}(D,E)} = C_{D,E}|_{\mathcal{R}(D,E)}$.

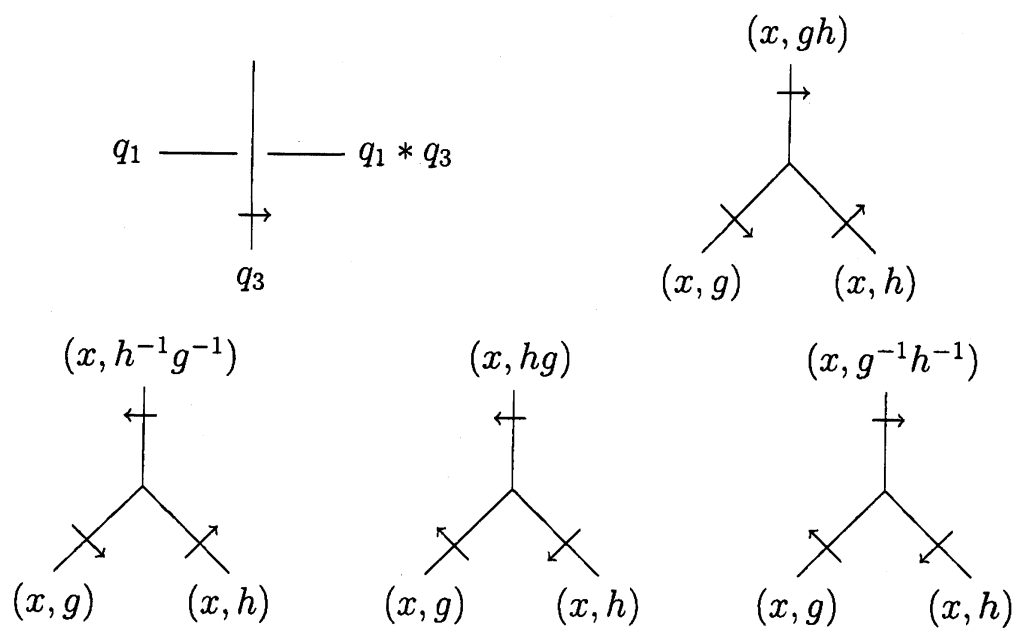


Figure 3:

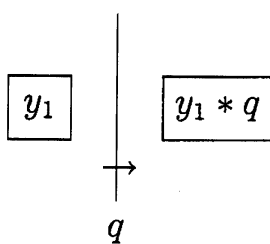


Figure 4:

4 A homology

Let X be a G -family of quandles, Y an X -set, and Q the associated quandle of X . Let $B_n(X)_Y$ be the free abelian group generated by the elements of $Y \times Q^n$ if $n \geq 0$, and let $B_n(X)_Y = 0$ otherwise. We put

$$((y, q_1, \dots, q_i) * q, q_{i+1}, \dots, q_n) := (y * q, q_1 * q, \dots, q_i * q, q_{i+1}, \dots, q_n)$$

for $y \in Y$ and $q, q_1, \dots, q_n \in Q$. We define a boundary homomorphism $\partial_n : B_n(X)_Y \rightarrow B_{n-1}(X)_Y$ by

$$\begin{aligned} \partial_n(y, q_1, \dots, q_n) &= \sum_{i=1}^n (-1)^i (y, q_1, \dots, q_{i-1}, q_{i+1}, \dots, q_n) \\ &\quad - \sum_{i=1}^n (-1)^i ((y, q_1, \dots, q_{i-1}) * q_i, q_{i+1}, \dots, q_n) \end{aligned}$$

for $n > 0$, and $\partial_n = 0$ otherwise. Then $B_*(X)_Y = (B_n(X)_Y, \partial_n)$ is a chain complex (see [1, 2, 3, 4]).

Let $D_n(X)_Y$ be the subgroup of $B_n(X)_Y$ generated by the elements of

$$\bigcup_{i=1}^{n-1} \left\{ (y, q_1, \dots, q_{i-1}, (x, g), (x, h), q_{i+2}, \dots, q_n) \mid \begin{array}{l} y \in Y, x \in X, g, h \in G \\ q_1, \dots, q_n \in Q \end{array} \right\}$$

and

$$\bigcup_{i=1}^n \left\{ \begin{array}{l} (y, q_1, \dots, q_{i-1}, (x, gh), q_{i+1}, \dots, q_n) \\ -(y, q_1, \dots, q_{i-1}, (x, g), q_{i+1}, \dots, q_n) \\ -((y, q_1, \dots, q_{i-1}) * (x, g), (x, h), q_{i+1}, \dots, q_n) \end{array} \mid \begin{array}{l} y \in Y, x \in X, \\ g, h \in G, \\ q_1, \dots, q_n \in Q \end{array} \right\}.$$

Lemma 5. For $n \in \mathbb{Z}$, we have $\partial_n(D_n(X)_Y) \subset D_{n-1}(X)_Y$. Thus $D_*(X)_Y = (D_n(X)_Y, \partial_n)$ is a subcomplex of $B_*(X)_Y$.

We put $C_n(X)_Y = B_n(X)_Y / D_n(X)_Y$. Then $C_*(X)_Y = (C_n(X)_Y, \partial_n)$ is a chain complex. For an abelian group A , we define the cochain complex $C^*(X; A)_Y = \text{Hom}(C_*(X)_Y, A)$. We denote by $H_n(X)_Y$ the n th homology group of $C_*(X)_Y$.

5 A cocycle invariant

Let D be a diagram of an oriented spatial trivalent graph. For an X_Y -coloring $C \in \text{Col}_X(D)_Y$, we define the weight $w(\chi; C) \in C_2(X)_Y$ at a crossing χ of D as follows. Let χ_1, χ_2 and χ_3 be respectively the under-arcs and the over-arc at a crossing χ such that the normal orientation of χ_3 points from χ_1 to χ_2 . Let R_χ be the region facing χ_1 and χ_3 such that the normal orientations χ_1 and χ_3 point from R_χ to the opposite regions with respect to χ_1 and χ_3 , respectively. Then we define

$$w(\chi; C) = \epsilon(\chi)(C(R_\chi), C(\chi_1), C(\chi_3)),$$

where $\epsilon(\chi) \in \{1, -1\}$ is the sign of a crossing χ . We define a chain $W(D; C) \in C_2(X)_Y$ by

$$W(D; C) = \sum_{\chi} w(\chi; C),$$

where χ runs over all crossings of D .

Lemma 6. The chain $W(D; C)$ is a 2-cycle of $C_*(X)_Y$. Further, for cohomologous 2-cocycles θ, θ' of $C^*(X; A)_Y$, we have $\theta(W(D; C)) = \theta'(W(D; C))$.

Lemma 7. Let D be a diagram of an oriented spatial trivalent graph. Let E be a diagram obtained by applying one of the R1–R6 moves to the diagram D once, where we choose orientations for E which agree with those for D on $\mathcal{A}(D, E)$. For $C \in \text{Col}_X(D)_Y$ and $C_{D,E} \in \text{Col}_X(E)_Y$ such that $C|_{\mathcal{A}(D,E)} = C_{D,E}|_{\mathcal{A}(D,E)}$ and $C|_{\mathcal{R}(D,E)} = C_{D,E}|_{\mathcal{R}(D,E)}$, we have $[W(D; C)] = [W(E; C_{D,E})] \in H_2(X)_Y$.

We denote by G_H (resp. G_K) the fundamental group of the exterior of a handlebody-link H (resp. a spatial graph K). When H is represented by K , G_H and G_K are identical. Let D be a diagram of an oriented spatial trivalent graph K . By the definition of an X_Y -coloring C of D , the map $p_G \circ C|_{\mathcal{A}(D)}$ represents a homomorphism from G_K to G , which we denote by $\rho_C \in \text{Hom}(G_K, G)$. For $\rho \in \text{Hom}(G_K, G)$, we define

$$\text{Col}_X(D; \rho)_Y = \{C \in \text{Col}_X(D)_Y \mid \rho_C = \rho\}.$$

For a 2-cocycle θ of $C^*(X; A)_Y$, we define

$$\begin{aligned}\mathcal{H}(D) &:= \{[W(D; C)] \in H_2(X)_Y \mid C \in \text{Col}_X(D)_Y\}, \\ \Phi_\theta(D) &:= \{\theta(W(D; C)) \in A \mid C \in \text{Col}_X(D)_Y\}, \\ \mathcal{H}(D; \rho) &:= \{[W(D; C)] \in H_2(X)_Y \mid C \in \text{Col}_X(D; \rho)_Y\}, \\ \Phi_\theta(D; \rho) &:= \{\theta(W(D; C)) \in A \mid C \in \text{Col}_X(D; \rho)_Y\}\end{aligned}$$

as multisets.

Lemma 8. Let D be a diagram of an oriented spatial trivalent graph K . For $\rho, \rho' \in \text{Hom}(G_K, G)$ such that ρ and ρ' are conjugate, we have

$$\mathcal{H}(D; \rho) = \mathcal{H}(D; \rho') \quad \Phi_\theta(D; \rho) = \Phi_\theta(D; \rho').$$

We denote by $\text{Conj}(G_K, G)$ the set of conjugacy classes of homomorphisms from G_K to G . By Lemma 8, $\mathcal{H}(D; \rho)$ and $\Phi_\theta(D; \rho)$ are well-defined for $\rho \in \text{Conj}(G_K, G)$.

Lemma 9. Let D be a diagram of an oriented spatial trivalent graph K . Let E be a diagram obtained from D by reversing the orientation of an edge e . For $\rho \in \text{Hom}(G_K, G)$, we have

$$\begin{aligned}\mathcal{H}(D) &= \mathcal{H}(E), & \mathcal{H}(D; \rho) &= \mathcal{H}(E; \rho), \\ \Phi_\theta(D) &= \Phi_\theta(E), & \Phi_\theta(D; \rho) &= \Phi_\theta(E; \rho).\end{aligned}$$

By Lemma 9, $\mathcal{H}(D)$, $\Phi_\theta(D)$, $\mathcal{H}(D; \rho)$ and $\Phi_\theta(D; \rho)$ are well-defined for a diagram D of an unoriented spatial trivalent graph, which is a diagram of a handlebody-link. For a diagram D of a handlebody-link H , we define

$$\begin{aligned}\mathcal{H}^{\text{hom}}(D) &:= \{\mathcal{H}(D; \rho) \mid \rho \in \text{Hom}(G_H, G)\}, \\ \Phi_\theta^{\text{hom}}(D) &:= \{\Phi_\theta(D; \rho) \mid \rho \in \text{Hom}(G_H, G)\}, \\ \mathcal{H}^{\text{conj}}(D) &:= \{\mathcal{H}(D; \rho) \mid \rho \in \text{Conj}(G_H, G)\}, \\ \Phi_\theta^{\text{conj}}(D) &:= \{\Phi_\theta(D; \rho) \mid \rho \in \text{Conj}(G_H, G)\}\end{aligned}$$

as “multisets of multisets.” We remark that, for X_Y -colorings C and $C_{D,E}$ in Lemma 7, we have $\rho_C = \rho_{C_{D,E}}$. Then, by Lemmas 6–9, we have the following theorem.

Theorem 10. Let X be a G -family of quandles, Y an X -set. Let θ be a 2-cocycle of $C^*(X; A)_Y$. Let H be a handlebody-link represented by a diagram D . Then the followings are invariants of a handlebody-link H .

$$\mathcal{D}(H), \quad \Phi_\theta(D), \quad \mathcal{H}^{\text{hom}}(D), \quad \Phi_\theta^{\text{hom}}(D), \quad \mathcal{H}^{\text{conj}}(D), \quad \Phi_\theta^{\text{conj}}(D).$$

References

- [1] J. S. Carter, D. Jelsovsky, S. Kamada, L. Langford and M. Saito, *Quandle cohomology and state-sum invariants of knotted curves and surfaces*, Trans. Amer. Math. Soc. **355** (2003) 3947–3989.
- [2] J. S. Carter, D. Jelsovsky, S. Kamada and M. Saito, *Quandle homology groups, their Betti numbers, and virtual knots*, J. Pure Appl. Algebra **157** (2001) 135–155.
- [3] R. Fenn, C. Rourke and B. Sanderson, *Trunks and classifying spaces*, Appl. Categ. Structures **3** (1995), 321–356.
- [4] R. Fenn, C. Rourke and B. Sanderson, *The rack space*, Trans. Amer. Math. Soc. **359** (2007), 701–740.
- [5] A. Ishii, *Moves and invariants for knotted handlebodies*, Algebr. Geom. Topol. **8** (2008), 1403–1418.
- [6] A. Ishii and M. Iwakiri, *Quandle cocycle invariants for spatial graphs and knotted handlebodies*, to appear in Canad. J. Math.
- [7] A. Ishii, M. Iwakiri, Y. Jang and K. Oshiro, *A G -family of quandles and handlebody-knots*, preprint.
- [8] D. Joyce, *A classifying invariant of knots, the knot quandle*, J. Pure Appl. Alg. **23** (1982) 37–65.
- [9] S. V. Matveev, *Distributive groupoids in knot theory*, Mat. Sb. (N.S.) **119(161)** (1982) 78–88.