

Rotation number for the one-dimensional Schrödinger operator with periodic singular potentials

首都大学東京大学院 理工学研究科 数理情報科学専攻 新國裕昭 (Hiroaki Niikuni)

Department of Mathematics and Information Sciences,
Tokyo Metropolitan University

1. Introduction and main result

In this article, we survey the results in [13, 14, 15]. In those papers, we study the one-dimensional Schrödinger operator with singular potentials. In order to explain the motivation of our study, we describe its background. Such operators play an important role in solid state physics (see [10]) and have been studied in numerous works [1, 2, 5, 6, 8, 11, 16, 17]. In 1931, Kronig and Penney introduced the Hamiltonians which is formally expressed as

$$L_1 = -\frac{d^2}{dx^2} + \beta \sum_{l=-\infty}^{\infty} \delta(x - 2\pi l) \quad \text{in } L^2(\mathbb{R}),$$

where $\delta(x)$ is the Dirac delta function at the origin and $\beta \in \mathbb{R} \setminus \{0\}$. The precise definition of L_1 is given through the boundary conditions on the lattice $2\pi\mathbb{Z}$ as follows.

$$(L_1 y)(x) = -\frac{d^2}{dx^2} y(x), \quad x \in \mathbb{R} \setminus 2\pi\mathbb{Z},$$

$$\text{Dom}(L_1) = \left\{ y \in H^2(\mathbb{R} \setminus 2\pi\mathbb{Z}) \left| \begin{array}{c} \begin{pmatrix} y(x+0) \\ y'(x+0) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \beta & 1 \end{pmatrix} \begin{pmatrix} y(x-0) \\ y'(x-0) \end{pmatrix} \\ \text{for } x \in 2\pi\mathbb{Z} \end{array} \right. \right\},$$

where $H^2(D)$ denotes the Sobolev space of order 2 on an open set $D \subset \mathbb{R}$. This operator is the Hamiltonian for an electron in a one-dimensional crystal and is called Kronig-Penney Hamiltonian. The Dirac delta function is the most typical point interaction. The δ -interaction was widely generalized. In [5, 6], Gesztesy, Holden, and Kirsch inspired a new class of point interactions. They studied the operator in $L^2(\mathbb{R})$ of the form

$$(L_2 y)(x) = -\frac{d^2}{dx^2} y(x), \quad x \in \mathbb{R} \setminus 2\pi\mathbb{Z},$$

$$\text{Dom}(L_2) = \left\{ y \in H^2(\mathbb{R} \setminus 2\pi\mathbb{Z}) \left| \begin{array}{c} \begin{pmatrix} y(x+0) \\ y'(x+0) \end{pmatrix} = \begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix} \begin{pmatrix} y(x-0) \\ y'(x-0) \end{pmatrix} \\ \text{for } x \in 2\pi\mathbb{Z} \end{array} \right. \right\}.$$

This operator has the formal expression

$$L_2 = -\frac{d^2}{dx^2} + \beta \sum_{l=-\infty}^{\infty} \delta'(x - 2\pi l) \quad \text{in } L^2(\mathbb{R}).$$

In [16], Šeba found that the domain of any self-adjoint extension of $(-d^2/dx^2)|_{C_0^\infty(\mathbb{R} \setminus \{0\})}$ in $L^2(\mathbb{R})$ of coupled type is expressed as

$$\left\{ y \in H^2(\mathbb{R} \setminus \{0\}) \left| \begin{pmatrix} y(+0) \\ y'(+0) \end{pmatrix} = cA \begin{pmatrix} y(-0) \\ y'(-0) \end{pmatrix} \right. \right\}$$

with $A \in SL(2, \mathbb{R})$, $c \in \mathbb{C}$, and $|c| = 1$, where $SL(2, \mathbb{R})$ denotes the special linear group (see also [2] and [1, Section K.1.4]). In [8], Hughes gave the Floquet-Bloch decomposition of the Schrödinger operator in $L^2(\mathbb{R})$ with generalized point interaction on a lattice $2\pi\mathbb{Z}$ defined as

$$(L_3 y)(x) = -\frac{d^2}{dx^2} y(x), \quad x \in \mathbb{R} \setminus 2\pi\mathbb{Z},$$

$$\text{Dom}(L_3) = \left\{ y \in H^2(\mathbb{R} \setminus 2\pi\mathbb{Z}) \left| \begin{array}{c} \begin{pmatrix} y(x+0) \\ y'(x+0) \end{pmatrix} = cA \begin{pmatrix} y(x-0) \\ y'(x-0) \end{pmatrix} \\ \text{for } x \in 2\pi\mathbb{Z} \end{array} \right. \right\}.$$

These backgrounds motivate us to study the spectra of the one-dimensional Schrödinger operators with periodic generalized point interactions.

To define the operators, we introduce notations. We fix $n \in \mathbb{N} = \{1, 2, 3, \dots\}$. Let $0 = \kappa_0 < \kappa_1 < \dots < \kappa_n = 2\pi$ be a partition of the interval $(0, 2\pi)$. We put $\Gamma_j = \{\kappa_j\} + 2\pi\mathbb{Z}$ for $j = 1, 2, \dots, n$, and $\Gamma = \Gamma_1 \cup \Gamma_2 \cup \dots \cup \Gamma_n$. For $\{\theta_j\}_{j=1}^n \subset \mathbb{R}$ and $\{A_j\}_{j=1}^n \subset SL_2(\mathbb{R})$, we define the one-dimensional Schrödinger operator $H = H(\theta_1, \theta_2, \dots, \theta_n, A_1, A_2, \dots, A_n)$ in $L^2(\mathbb{R})$ as follows.

$$(Hy)(x) = -y''(x), \quad x \in \mathbb{R} \setminus \Gamma, \quad (1.1)$$

$$\text{Dom}(H) = \left\{ y \in H^2(\mathbb{R} \setminus \Gamma) \left| \begin{array}{c} \begin{pmatrix} y(x+0) \\ y'(x+0) \end{pmatrix} = e^{i\theta_j} A_j \begin{pmatrix} y(x-0) \\ y'(x-0) \end{pmatrix} \\ \text{for } x \in \Gamma_j, \quad j = 1, 2, \dots, n \end{array} \right. \right\}. \quad (1.2)$$

This operator H is self-adjoint (see [13, Proposition 2.1]). Since the spectrum of H is independent of $\{\theta_j\}_{j=1}^n \subset \mathbb{R}$ (see [14, Proposition 1.1(e)]), we may put

$$\theta_1 = \theta_2 = \dots = \theta_n = 0,$$

which does not cause any loss of generality. Since H has 2π -periodic point interactions, the spectrum of H has the band structure. According to the Floquet-Bloch theory, we label each band of the spectrum of H . For $j \in \mathbb{N}$, we designate the j th band of $\sigma(H)$ as

$$B_j = [\lambda_{2j-2}, \lambda_{2j-1}]. \quad (1.3)$$

The sequence $\{\lambda_n\}_{n=0}^\infty \subset \mathbb{R}$ satisfies the inequalities

$$\lambda_0 < \lambda_1 \leq \lambda_2 < \lambda_3 \leq \lambda_4 < \cdots \leq \lambda_{2j-2} < \lambda_{2j-1} \leq \lambda_{2j} < \cdots \rightarrow \infty.$$

So, the consecutive bands B_j and B_{j+1} are separated by an open interval

$$G_j := (\lambda_{2j-1}, \lambda_{2j}),$$

which is called the j th gap of $\sigma(H)$.

In [13, 14, 15], we mainly dealt with two problems. One of the problems is to give a characterization of the band edges of $\sigma(H)$ by the *rotation number*. The other is to determine the indices of the absent spectral gaps in a class of H .

We quote the main theorem in [14]. For this purpose, we introduce the rotation number. First, we consider the Schrödinger equation

$$-y''(x, \lambda) = \lambda y(x, \lambda), \quad x \in \mathbb{R} \setminus \Gamma, \quad (1.4)$$

$$\begin{pmatrix} y(x+0, \lambda) \\ y'(x+0, \lambda) \end{pmatrix} = A_j \begin{pmatrix} y(x-0, \lambda) \\ y'(x-0, \lambda) \end{pmatrix}, \quad x \in \Gamma_j, \quad j = 1, 2, \dots, n, \quad (1.5)$$

where λ is a real parameter. We define the Prüfer transform of a nontrivial solution $y(x, \lambda)$ to (1.4) and (1.5) as follows. Let (r, ω) be the polar coordinates of (y, y') :

$$y = r \sin \omega, \quad y' = r \cos \omega.$$

Then we call the function $\omega = \omega(x, \lambda)$ the Prüfer transform of $y(x, \lambda)$. For each $j = 1, 2, \dots, n$, we write

$$A_j = \begin{pmatrix} a_j & b_j \\ c_j & d_j \end{pmatrix}. \quad (1.6)$$

Then, $\omega(x, \lambda)$ satisfies the equation

$$\omega'(x, \lambda) = \cos^2 \omega(x, \lambda) + \lambda \sin^2 \omega(x, \lambda), \quad x \in \mathbb{R} \setminus \Gamma \quad (1.7)$$

as well as the boundary conditions

$$\begin{aligned} & \sin \omega(x+0, \lambda)(c_j \sin \omega(x-0, \lambda) + d_j \cos \omega(x-0, \lambda)) \\ &= \cos \omega(x+0, \lambda)(a_j \sin \omega(x-0, \lambda) + b_j \cos \omega(x-0, \lambda)), \end{aligned} \quad (1.8)$$

$$\operatorname{sgn}(\sin \omega(x+0, \lambda)) = \operatorname{sgn}(a_j \sin \omega(x-0, \lambda) + b_j \cos \omega(x-0, \lambda)), \quad (1.9)$$

$$\operatorname{sgn}(\cos \omega(x+0, \lambda)) = \operatorname{sgn}(c_j \sin \omega(x-0, \lambda) + d_j \cos \omega(x-0, \lambda)) \quad (1.10)$$

for $x \in \Gamma_j$ and $j = 1, 2, \dots, n$, where

$$\operatorname{sgn}(x) = \begin{cases} 1 & \text{if } x > 0, \\ 0 & \text{if } x = 0, \\ -1 & \text{if } x < 0. \end{cases}$$

To determine the principal value of $\omega(x+0, \lambda)$ by the boundary conditions (1.8), (1.9), and (1.10), we must select a branch of $\omega(x+0, \lambda)$ for $x \in \Gamma$. We choose the branch of $\omega(x+0, \lambda)$ as

$$\omega(x+0, \lambda) - \omega(x-0, \lambda) \in [-\pi, \pi) \quad \text{for } x \in \Gamma. \quad (1.11)$$

Thanks to this selection, $\omega(x+0, \lambda)$ is uniquely determined. We pick $\omega_0 \in \mathbb{R}$. Let $\omega = \omega(x, \lambda, \omega_0)$ be the solution of (1.7) – (1.10) subject to the initial condition

$$\omega(+0, \lambda) = \omega_0. \quad (1.12)$$

We define the rotation number of H as

$$\rho(\lambda) = \lim_{n \rightarrow \infty} \frac{\omega(2n\pi + 0, \lambda, \omega_0) - \omega_0}{2n\pi}. \quad (1.13)$$

We recall (1.3). In [14], we proved the following theorem which relates $\rho(\lambda)$ to the spectrum of H .

Theorem 1.1. *The following statements (a), (b), and (c) hold true.*

(a) *The limit on the right-hand side of (1.13) exists and is independent of the initial value ω_0 .*

(b) *The function $\rho(\lambda)$ is non-decreasing on $\mathbb{R}$.*

(c) *We put*

$$l = \#\{j \in \{1, 2, \dots, n\} \mid (b_j < 0) \text{ or } (b_j = 0, d_j < 0)\}, \quad (1.14)$$

where $\#A$ stands for the number of the elements of A for a finite set A . Then, for $j \in \mathbb{N}$, we have

$$\lambda_{2j-2} = \max \left\{ \lambda \in \mathbb{R} \mid \rho(\lambda) = \frac{j-1}{2} - \frac{l}{2} \right\}, \quad (1.15)$$

$$\lambda_{2j-1} = \min \left\{ \lambda \in \mathbb{R} \mid \rho(\lambda) = \frac{j}{2} - \frac{l}{2} \right\}. \quad (1.16)$$

We note that (1.15) and (1.16) critically depend on the choice of the branch of $\omega(x+0, \lambda)$ for $x \in \Gamma$ (see [14, Section 4]).

The rotation number has a close relationship to the *density of states*. In order to see that, we introduce the density of states for H . For $k \in \mathbb{N}$, we put $I_k = \Gamma^c \cap (0, 2\pi k)$. Let us

introduce the generalized Kronig-Penney Hamiltonian in $L^2((0, 2\pi k))$ with the Dirichlet boundary conditions

$$y(+0) = y(2\pi k - 0) = 0.$$

We define the operator $H_{2\pi k, D}$ as

$$(H_{2\pi k, D}y)(x) = -y''(x), \quad x \in I_k,$$

$$\text{Dom}(H_{2\pi k, D}) = \left\{ y \in H^2(\mathbb{R} \setminus \Gamma) \left| \begin{array}{l} \begin{pmatrix} y(x+0) \\ y'(x+0) \end{pmatrix} = A_j \begin{pmatrix} y(x-0) \\ y'(x-0) \end{pmatrix} \\ \text{for } x \in \Gamma_j \cap (0, 2\pi k), \quad j = 1, 2, \dots, n, \\ y(+0) = y(2\pi k - 0) = 0 \end{array} \right. \right\}.$$

For $n \in \mathbb{N} \cup \{0\}$, let $\lambda_{k,n}$ be the $(n+1)$ st eigenvalue of $H_{2\pi k, D}$. Put

$$\nu(k, \lambda) = \#\{n \in \mathbb{N} \cup \{0\} \mid \lambda_{k,n} \leq \lambda\}.$$

Then we have the following theorem.

Theorem 1.2. *We have*

$$\lim_{k \rightarrow \infty} \frac{\nu(k, \lambda)}{2\pi k} = \frac{\rho(\lambda)}{\pi} + \frac{l}{2\pi}. \quad (1.17)$$

In the physics literatures, the left-hand side of (1.17) is referred to as the density of states. We will give the outline of the proof of Theorem 1.2 in Section 2; the complete proof is found in [14]. On the other hand, we did not describe Theorem 1.2 in [14]. So, we give the complete proof of it in Section 2.

Our study [14] is also motivated by the works [9, 12], which we recall below. Johnson and Moser found that the rotation number for the one-dimensional Schrödinger operators with almost periodic potentials has a close relation to its spectrum. They dealt with the Schrödinger operator $L = -d^2/dx^2 + q(x)$, where q is an almost periodic function with a frequency module $\mathcal{M}$. They proved that the rotation number $\alpha(\lambda)$ for L exists and defines a continuous function in $\{\lambda \in \mathbb{C} \mid \text{Im} \lambda \leq 0\}$. Furthermore, $\alpha(\lambda)$ is constant in an open interval I in a spectral gap and $2\alpha(\lambda) \in \mathcal{M}$ for $\lambda \in I$. In the special case where q is periodic of period 2π , they found that the j th band $\tilde{B}_j$ of $\sigma(L)$ is expressed as

$$\tilde{B}_j = \left\{ \lambda \mid \overline{\frac{j-1}{2} < \alpha(\lambda) < \frac{j}{2}} \right\} \quad (1.18)$$

for $j \in \mathbb{N}$. This means that

$$\tilde{\lambda}_{2j-2} = \max \left\{ \lambda \in \mathbb{R} \mid \alpha(\lambda) = \frac{j-1}{2} \right\},$$

$$\tilde{\lambda}_{2j-1} = \min \left\{ \lambda \in \mathbb{R} \mid \alpha(\lambda) = \frac{j}{2} \right\},$$

where $\tilde{B}_j = [\tilde{\lambda}_{2j-2}, \tilde{\lambda}_{2j-1}]$. Let $N(x, \lambda)$ be the number of the zeroes in $[0, x]$ of a nontrivial solution to $(L\varphi)(x) = \lambda\varphi(x)$. Then they described that

$$\lim_{x \rightarrow \infty} \frac{N(x, \lambda)}{x} = \lim_{x \rightarrow \infty} \frac{\nu(x, \lambda)}{x} = \frac{\alpha(\lambda)}{\pi},$$

where $\nu = \nu(x, \lambda)$ is the number of eigenvalues of $(Ly)(x, \lambda) = \lambda y(x, \lambda)$ in $[0, x]$ with the boundary conditions $y(0) = y(x) = 0$. In contrast to these results, our theorems involve the number of the interactions in the fundamental region.

Next, we introduce the results in [13, 15]. The aim of those papers is to determine the indices of the absent spectral gaps of $H(\theta_1, \theta_2, A_1, A_2)$. In [13], we dealt with the case where

$$A_1, A_2 \in SO(2) \setminus \{E, -E\}, \quad (1.19)$$

E being the 2×2 unit matrix. We put

$$A_j = \begin{pmatrix} \cos \gamma_j & -\sin \gamma_j \\ \sin \gamma_j & \cos \gamma_j \end{pmatrix} \quad \text{and} \quad \gamma_j \in (0, \pi) \cup (\pi, 2\pi)$$

for $j = 1, 2$. We define

$$\Lambda = \{m \in \mathbb{N} \mid G_m = \emptyset\}.$$

In [13], we have the following theorem.

Theorem 1.3. *Adopt the assumption (1.19). Let $\kappa_1 \neq \pi$.*

(a) *Suppose that $\gamma_1 - \gamma_2 \not\equiv 0$ and $\gamma_1 + \gamma_2 \not\equiv 0 \pmod{\pi}$. Then we have*

$$\Lambda = \emptyset.$$

(b) *Suppose that $\gamma_1 + \gamma_2 \equiv 0 \pmod{\pi}$. Then we have*

$$\Lambda = \begin{cases} \{3\} & \text{if } \frac{\kappa_1}{\pi} \notin \mathbb{Q}, \\ \{3\} \cup \{pk + 1 \mid k \in \mathbb{N}\} & \text{if } \frac{\kappa_1}{2\pi} = \frac{q}{p}, \quad (p, q) \in \mathbb{N}^2, \quad \text{and} \quad \gcd(p, q) = 1. \end{cases}$$

(c) *Assume that $\gamma_1 - \gamma_2 \equiv 0$ and $\gamma_1 + \gamma_2 \not\equiv 0 \pmod{\pi}$. We put $\eta_j = \pi^2 j^2 / 4(\pi - \kappa_1)^2$ for $j \in \mathbb{N}$. Then it holds that*

$$\bigcup_{k=1}^{\infty} B_k \cap B_{k+1} = \left\{ \eta_j \mid -2 \left(\sqrt{\eta_j} + \frac{1}{\sqrt{\eta_j}} \right)^{-1} \cot \kappa_1 \sqrt{\eta_j} = \tan \gamma_1 \text{ and } j \in \mathbb{N} \right\}.$$

In [15], we dealt with the case where

$$A_1 A_2 = \pm E \quad \text{and} \quad A_1, A_2 \in SL(2, \mathbb{R}) \setminus \{E, -E\}. \quad (1.20)$$

For convenience we rewrite the elements of A_1 as

$$A_1 = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Then we have the following theorem [15, Theorem 1.2].

Theorem 1.4. *Adopt the assumption (1.20). Let $\kappa_1 \neq \pi$.*

(a) *Assume that $\kappa_1/\pi \notin \mathbb{Q}$. Then we have*

$$\Lambda = \begin{cases} \{k+1\} & \text{if } d=a, \quad b \neq 0, \quad -c/b = k^2/4 \text{ for some } k \in \mathbb{N}, \\ \emptyset & \text{otherwise.} \end{cases}$$

(b) *Suppose that $\kappa_1/2\pi = q/p$, $(p, q) \in \mathbb{N}^2$, and $\gcd(p, q) = 1$. Then we have*

$$\Lambda = \begin{cases} \{pj \mid j \in \mathbb{N}\} & \text{if } b=0, \\ \{1+pj \mid j \in \mathbb{N}\} \cup \{1+k\} & \text{if } d=a, \quad b \neq 0, \quad -c/b = k^2/4, \\ & \text{for some } k \in \mathbb{N}, \quad k \not\equiv 0 \pmod{p}, \\ \{1+pj \mid j \in \mathbb{N}\} & \text{otherwise.} \end{cases}$$

Using Theorem 1.1, we can newly get a theorem. We discuss the spectral gaps of the Schrödinger operator formally expressed as

$$L_4 = -\frac{d^2}{dx^2} + \sum_{l=-\infty}^{\infty} (\beta_1 \delta(x - \kappa_1 - 2\pi l) + \beta_2 \delta'(x - 2\pi l)),$$

where $\kappa_1 \in (0, 2\pi)$ and $\beta_1, \beta_2 \in \mathbb{R} \setminus \{0\}$ are parameters. In our notations this operator is expressed as $L_4 = H(0, 0, M_1, M_2)$, where

$$M_1 = \begin{pmatrix} 1 & 0 \\ \beta_1 & 1 \end{pmatrix} \quad \text{and} \quad M_2 = \begin{pmatrix} 1 & \beta_2 \\ 0 & 1 \end{pmatrix}.$$

We have the following theorem for the operator L_4 .

Theorem 1.5. *We suppose that $\kappa_1 \neq \pi$ and*

$$(\beta_1, \beta_2) \notin \left\{ \left(\frac{n\pi}{|\pi - \kappa_1|} \tan \frac{\kappa_1 n\pi}{2|\pi - \kappa_1|}, -\frac{4|\pi - \kappa_1|}{n\pi} \tan \frac{\kappa_1 n\pi}{2|\pi - \kappa_1|} \right) \mid n \in \mathbb{N} \right\}.$$

Then we have the following statements (i) and (ii).

(i) *If either $\kappa_1 \notin \{\pi/2, 3\pi/2\}$ or $\beta_1 \neq \beta_2$ holds, then*

$$\Lambda = \emptyset.$$

(ii) *If $\kappa_1 \in \{\pi/2, 3\pi/2\}$ and $\beta_1 = \beta_2$, then*

$$\Lambda = \begin{cases} \{2\} & \text{if } \beta_1 > 0, \\ \{3\} & \text{if } \beta_1 < 0. \end{cases}$$

The study of L_4 is motivated by the work [17]. In [17], Yoshitomi investigated the spectral gaps of the operators

$$P_0 = -\frac{d^2}{dx^2} + \sum_{l=-\infty}^{\infty} (\beta_1 \delta(x - \kappa - 2\pi l) + \beta_2 \delta(x - 2\pi l)) \quad \text{in } L^2(\mathbb{R}),$$

and

$$P_1 = -\frac{d^2}{dx^2} + \sum_{l=-\infty}^{\infty} (\beta_1 \delta'(x - \kappa - 2\pi l) + \beta_2 \delta'(x - 2\pi l)) \quad \text{in } L^2(\mathbb{R}),$$

where $\kappa \in (0, 2\pi)$. For $j \in \mathbb{N}$ and $k \in \{0, 1\}$, he described that $\sigma(P_k)$ has an absent gap if and only if $\beta_1 + \beta_2 = 0$ and $\kappa/\pi \in \mathbb{Q}$ hold. Furthermore, his theorems say that if $\beta_1 + \beta_2 = 0$ and $\kappa/2\pi = m/n$, $(n, m) \in \mathbb{N}^2$, and $\gcd(m, n) = 1$, then the j th gap of $\sigma(P_k)$ is absent if and only if $j - k \in n\mathbb{N}$. We prove Theorem 1.5 in Section 3.

2. Proof of Theorem 1.2 and 1.3

In this section, we describe the proof of Theorem 1.2 and 1.3. We recall (1.6). Let

$$q_j = \#\{k \in \{1, 2, \dots, j\} \mid (b_k < 0) \text{ or } (b_k = 0, d_k < 0)\},$$

$$q_0 = 0,$$

and

$$\eta_j = \begin{cases} \operatorname{Arctan}(b_j/d_j) - q_{j-1}\pi & \text{if } b_j > 0, d_j > 0, \\ \operatorname{Arctan}(b_j/d_j) + \pi - q_{j-1}\pi, & \text{if } b_j > 0, d_j < 0, \\ \pi/2 - q_{j-1}\pi & \text{if } b_j > 0, d_j = 0, \\ \operatorname{Arctan}(b_j/d_j) - \pi - q_{j-1}\pi, & \text{if } b_j < 0, d_j < 0, \\ \operatorname{Arctan}(b_j/d_j) - q_{j-1}\pi & \text{if } b_j < 0, d_j > 0, \\ -\pi/2 - q_{j-1}\pi & \text{if } b_j < 0, d_j = 0, \\ -q_{j-1}\pi & \text{if } b_j = 0, d_j > 0, \\ -\pi - q_{j-1}\pi & \text{if } b_j = 0, d_j < 0 \end{cases}$$

for $j = 1, 2, \dots, n$, where $\operatorname{Arctan}(x) \in (-\pi/2, \pi/2)$ for $x \in \mathbb{R}$. Since

$$q_j = \begin{cases} q_{j-1} + 1 & \text{if } (b_j < 0) \text{ or } (b_j = 0, d_j < 0), \\ q_{j-1} & \text{otherwise,} \end{cases}$$

we have

$$\eta_j \in [-q_j\pi, -q_j\pi + \pi]. \quad (2.1)$$

We pick a $\gamma \in (0, \pi)$ such that

$$\eta_j < -q_j\pi + \gamma \quad \text{for } j = 1, 2, \dots, n.$$

Then we have the following lemma.

Lemma 2.1. *There exists $\lambda_0 \in \mathbb{R}$ such that*

$$-\pi(q_j + pq_n) \leq \omega(\kappa_j + 2\pi p + 0, \lambda, \omega_0) \leq -\pi(q_j + pq_n) + \gamma$$

for any $p \in \mathbb{N} \cup \{0\}$, $j = 1, 2, \dots, n$, $\lambda \leq \lambda_0$, and $\omega_0 \in [0, \gamma]$.

To prove this lemma, we recall a fundamental fact on the Prüfer transform from [3, Chapter 8, Theorem 2.1]. Let $c < d$. For $\beta \in [0, \pi)$, let $\theta = \theta(x, \lambda, c, \beta)$ be the solution to the initial value problem

$$\frac{d}{dx}\theta = \cos^2 \theta + \lambda \sin^2 \theta \quad \text{on } \mathbb{R}, \quad (2.2)$$

$$\theta|_{x=c} = \beta. \quad (2.3)$$

Then, it holds that

$$\lim_{\lambda \rightarrow -\infty} \theta(d, \lambda, c, \beta) = 0. \quad (2.4)$$

Moreover, the function $\theta(d, \cdot, c, \beta)$ is strictly monotone increasing on $\mathbb{R}$.

We describe the outline of the proof of Lemma 2.1.

Outline of the proof of Lemma 2.1. We fix $\omega_0 \in [0, \gamma]$. First, we shall show the following statements by induction on $j = 1, 2, \dots, n$.

$$\text{The limit } \beta_j := \lim_{\lambda \rightarrow -\infty} \omega(\kappa_j - 0, \lambda, \omega_0) \in \mathbb{R} \text{ exists, and we have } \beta_j = -q_{j-1}\pi. \quad (2.5)$$

$$\text{The function } \omega(\kappa_j - 0, \cdot, \omega_0) \text{ is strictly monotone increasing on } \mathbb{R}. \quad (2.6)$$

It follows by (2.4) that (2.5) and (2.6) are valid for $j = 1$. We pick $m \in \{1, 2, \dots, n\}$, arbitrarily. Suppose that (2.5) and (2.6) hold for $j = m$. Then we can show that the limit

$$\alpha_m := \lim_{\lambda \rightarrow -\infty} \omega(\kappa_m + 0, \lambda, \omega_0)$$

exists and

$$\alpha_m = \eta_m. \quad (2.7)$$

By (1.8), we have

$$\tan \omega(\kappa_m + 0, \lambda, \omega_0) = \frac{a_m \tan \omega(\kappa_m - 0, \lambda, \omega_0) + b_m}{c_m \tan \omega(\kappa_m - 0, \lambda, \omega_0) + d_m}. \quad (2.8)$$

Combining the monotonicity of $\omega(\kappa_m - 0, \cdot, \omega_0)$ and $a_m d_m - b_m c_m = 1$ with (2.8), we find that $\omega(\kappa_m + 0, \cdot, \omega_0)$ is strictly monotone increasing on $\mathbb{R}$.

Since $\omega(\kappa_{m+1} - 0, \lambda, \omega_0) = \theta(\kappa_{m+1}, \lambda, \kappa_m, \omega(\kappa_m + 0, \lambda, \omega_0))$, (2.6) is valid for $j = m + 1$. Using the monotonicity of $\omega(\kappa_m, \cdot, \omega_0)$, we infer that there exists $\lambda_m \in \mathbb{R}$ such that

$$-q_m \pi \leq \omega(\kappa_m + 0, \lambda, \omega_0) \leq -q_m \pi + \gamma \quad (2.9)$$

for $\lambda \leq \lambda_m$. By the comparison theorem [3, Chapter 8] and (2.9), we have

$$\theta(\kappa_{m+1}, \lambda, \kappa_m, -q_m \pi) \leq \omega(\kappa_{m+1} - 0, \lambda, \omega_0) < \theta(\kappa_{m+1}, \lambda, \kappa_m, -q_m \pi + \gamma)$$

for $\lambda \leq \lambda_m$. Since the equation (2.2) is π -periodic, we derive

$$\lim_{\lambda \rightarrow -\infty} \theta(\kappa_{m+1}, \lambda, \kappa_m, -q_m \pi) = \lim_{\lambda \rightarrow -\infty} \theta(\kappa_{m+1}, \lambda, \kappa_m, -q_m \pi + \gamma) = -q_m \pi,$$

so that

$$\beta_{m+1} = -q_m \pi.$$

So, we have proved (2.5) and (2.6) for $j = m + 1$. Therefore, (2.5) and (2.6) are valid for $j = 1, 2, \dots, n$.

Put $\lambda_0 = \min_{1 \leq j \leq n} \lambda_j$. We have

$$-\pi q_j \leq \omega(\kappa_j + 0, \lambda, \omega_0) < -\pi q_j + \gamma \quad (2.10)$$

for $j = 1, 2, \dots, n$, and $\lambda \leq \lambda_0$.

Using the comparison theorem and $\omega_0 \in [0, \gamma]$, we notice that

$$\omega(\kappa_j + 0, \lambda, 0) \leq \omega(\kappa_j + 0, \lambda, \omega_0) \leq \omega(\kappa_j + 0, \lambda, \gamma).$$

Therefore the estimate (2.10) is uniform with respect to $\omega_0 \in [0, \gamma]$.

Since the equations (1.6) – (1.9) is 2π -periodic with respect to x , we have the desired assertion from (2.10). $\square$

Proof of Theorem 1.1. By a similar way to the proof of [7, Theorem 2.1], it follows that (a) and (b) hold. So, we have only to show the statement (c). We recall (1.14). Then, we notice that $q_n = l$. By Lemma 2.1, we have

$$-\pi pl \leq \omega(2\pi p + 0, \lambda, \omega_0) \leq -\pi pl + \gamma$$

for $0 \leq \omega_0 \leq \gamma$, $\lambda \leq \lambda_0$, and $p \in \mathbb{N}$. This together with (1.13) implies that

$$\lim_{\lambda \rightarrow -\infty} \rho(\lambda) = -\frac{l}{2}. \quad (2.11)$$

Combining (2.11) with the discussion in the proof of [4, Proposition 2.1.], we get the assertion (c). $\square$

Proof of Theorem 1.3. By (2.5) and (2.6), we have

$$\lim_{\lambda \rightarrow -\infty} \omega(\kappa_j - 0, \lambda, \omega_0) = -q_{j-1}\pi,$$

and the function $\omega(\kappa_j - 0, \cdot, \omega_0)$ is strictly monotone increasing on $\mathbb{R}$. Since the equation (1.7) – (1.10) is 2π -periodic with respect to x , we have

$$\lim_{\lambda \rightarrow -\infty} \omega(2\pi p - 0, \lambda, \omega_0) = -q_{n-1}\pi - \pi(p-1)l$$

and the function $\omega(2\pi p - 0, \cdot, \omega_0)$ is strictly monotone increasing on $\mathbb{R}$ for $p \in \mathbb{N}$. Because of the monotonicity of $\omega(2\pi p - 0, \cdot, \omega_0)$, there exists $\lambda_{p,m} \in \mathbb{R}$ satisfying

$$\omega(2\pi p - 0, \lambda_{p,m}, \omega_0) = -\pi\{q_{n-1} + (p-1)l\} + m\pi$$

for each $m \in \mathbb{N}$. In a similar way to [3, Chapter 8, Theorem 2.1], we see that $\lambda_{p,m}$ is the $(m+1)$ st eigenvalue of $H_{2\pi p, D}$.

We fix $\lambda \in \mathbb{R}$, arbitrarily. Define

$$m_p^* = \#\{m \in \mathbb{N} \mid \lambda_{p,m} \leq \lambda\} + 1.$$

Then we have

$$\lambda_{p,m_p^*} \leq \lambda < \lambda_{p,m_p^*+1}.$$

By the monotonicity of $\omega(2\pi p - 0, \cdot, \omega_0)$, we have

$$-\pi\{q_{n-1} + (p-1)l\} + m_p^*\pi < \omega(2\pi p + 0, \lambda, \omega_0) < -\pi\{q_{n-1} + (p-1)l\} + (m_p^* + 1)\pi.$$

This inequality reduces

$$m_p^* < \frac{\omega(2\pi p + 0, \lambda, \omega_0)}{\pi} + q_{n-1} + (p-1)l < m_p^* + 1.$$

So we derive

$$m_p^* = \left\lceil \frac{\omega(2\pi p + 0, \lambda, \omega_0)}{\pi} \right\rceil + q_{n-1} + (p-1)l.$$

By the definition of $\gamma(p, \lambda)$ and m_p^* , we have

$$\gamma(p, \lambda) = m_p^* = \left\lceil \frac{\omega(2\pi p + 0, \lambda, \omega_0)}{\pi} \right\rceil + q_{n-1} + (p-1)l. \quad (2.12)$$

On the other hand, we notice that

$$\begin{aligned} & \frac{\omega(2\pi p + 0, \lambda, \omega_0)/\pi}{2p\pi} + \frac{q_{n-1} + (p-1)l - 1}{2p\pi} \\ & \leq \frac{[\omega(2\pi p + 0, \lambda, \omega_0)/\pi]}{2p\pi} + \frac{q_{n-1} + (p-1)l}{2p\pi} \\ & \leq \frac{\omega(2\pi p + 0, \lambda, \omega_0)/\pi}{2p\pi} + \frac{q_{n-1} + (p-1)l}{2p\pi} \end{aligned} \quad (2.13)$$

Using (2.12), (2.13), and (1.11), we get (1.17). $\square$

3. Proof of Theorem 1.5

In this section, we prove Theorem 1.5. In the first place, we define the monodromy matrix. For this purpose, we consider the equations

$$-y''(x, \lambda) = \lambda y(x, \lambda), \quad x \in \mathbb{R} \setminus \Gamma, \quad (3.1)$$

$$\begin{pmatrix} y(x+0, \lambda) \\ y'(x+0, \lambda) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \beta_1 & 1 \end{pmatrix} \begin{pmatrix} y(x-0, \lambda) \\ y'(x-0, \lambda) \end{pmatrix}, \quad x \in \Gamma_1, \quad (3.2)$$

$$\begin{pmatrix} y(x+0, \lambda) \\ y'(x+0, \lambda) \end{pmatrix} = \begin{pmatrix} 1 & \beta_2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} y(x-0, \lambda) \\ y'(x-0, \lambda) \end{pmatrix}, \quad x \in \Gamma_2, \quad (3.3)$$

where λ is real parameter. These equations have two solutions $y_1(x, \lambda)$ and $y_2(x, \lambda)$ which are uniquely determined by the initial conditions

$$y_1(+0, \lambda) = 1, \quad y_1'(+0, \lambda) = 0,$$

and

$$y_2(+0, \lambda) = 0, \quad y_2'(+0, \lambda) = 1,$$

respectively. Then, the monodromy matrix of (3.1) - (3.3) is defined as

$$M(\lambda) = \begin{pmatrix} y_1(2\pi + 0, \lambda) & y_2(2\pi + 0, \lambda) \\ y_1'(2\pi + 0, \lambda) & y_2'(2\pi + 0, \lambda) \end{pmatrix} \quad (3.4)$$

As described in [17, Lemma 4] (see also [13, 15]), we have

$$\mathcal{B} := \bigcup_{k=1}^{\infty} B_k \cap B_{k+1} = \{\lambda \in \mathbb{R} \mid M(\lambda) = E \text{ or } M(\lambda) = -E\}.$$

Put

$$\tau = 2\pi - \kappa_1.$$

By a direct calculation, we get

$$y_1(2\pi + 0, \lambda) = (1 + \beta_1\beta_2) \cos \tau\sqrt{\lambda} \cos \kappa_1\sqrt{\lambda} + \left(\frac{\beta_1}{\sqrt{\lambda}} - \beta_2\sqrt{\lambda} \right) \sin \tau\sqrt{\lambda} \cos \kappa_1\sqrt{\lambda} - \beta_2\sqrt{\lambda} \cos \tau\sqrt{\lambda} \sin \kappa_1\sqrt{\lambda} - \sin \tau\sqrt{\lambda} \sin \kappa_1\sqrt{\lambda}, \quad (3.5)$$

$$y'_1(2\pi + 0, \lambda) = \beta_1 \cos \tau\sqrt{\lambda} \cos \kappa_1\sqrt{\lambda} - \sqrt{\lambda} \sin \tau\sqrt{\lambda} \cos \kappa_1\sqrt{\lambda} - \sqrt{\lambda} \cos \tau\sqrt{\lambda} \sin \kappa_1\sqrt{\lambda}, \quad (3.6)$$

$$y_2(2\pi + 0, \lambda) = \beta_2 \cos \tau\sqrt{\lambda} \cos \kappa_1\sqrt{\lambda} + \frac{1}{\sqrt{\lambda}} \sin \tau\sqrt{\lambda} \cos \kappa_1\sqrt{\lambda} + \frac{1 + \beta_1\beta_2}{\sqrt{\lambda}} \cos \tau\sqrt{\lambda} \sin \kappa_1\sqrt{\lambda} + \left(\frac{\beta_1}{\lambda} - \beta_2 \right) \sin \tau\sqrt{\lambda} \sin \kappa_1\sqrt{\lambda}, \quad (3.7)$$

$$y'_2(2\pi + 0, \lambda) = \cos \tau\sqrt{\lambda} \cos \kappa_1\sqrt{\lambda} + \frac{\beta_1}{\sqrt{\lambda}} \cos \tau\sqrt{\lambda} \sin \kappa_1\sqrt{\lambda} - \sin \tau\sqrt{\lambda} \sin \kappa_1\sqrt{\lambda}. \quad (3.8)$$

In order to establish Theorem 1.5, we show the following theorem.

Theorem 3.1. *We suppose that $\kappa_1 \neq \pi$ and*

$$(\beta_1, \beta_2) \notin \left\{ \left(\frac{n\pi}{|\pi - \kappa_1|} \tan \frac{\kappa_1 n\pi}{2|\pi - \kappa_1|}, -\frac{4|\pi - \kappa_1|}{n\pi} \tan \frac{\kappa_1 n\pi}{2|\pi - \kappa_1|} \right) \mid n \in \mathbb{N} \right\}. \quad (3.9)$$

Then we have the following statements (i) and (ii).

(i) *If either $\kappa_1 \notin \{\pi/2, 3\pi/2\}$ or $\beta_1 \neq \beta_2$ holds, then we have*

$$\mathcal{B} = \emptyset.$$

(ii) *If $\kappa_1 \in \{\pi/2, 3\pi/2\}$ and $\beta_1 = \beta_2$, then we have*

$$\mathcal{B} = \{1\}.$$

We prove this theorem by using the following lemma.

Lemma 3.2. *Assume that $\kappa_1 \neq \pi$ and $M(\lambda) = \pm E$. Then we have the following statements.*

(i) *If $\lambda \neq -\beta_1/\beta_2$, then $\lambda = \beta_2/\beta_1$ and $\cos \kappa_1\sqrt{\lambda} = \cos \tau\sqrt{\lambda} = 0$.*

(ii) *If $\lambda = -\beta_1/\beta_2$, then there exists $n \in \mathbb{N}$ such that*

$$\beta_1 = \frac{n\pi}{|\pi - \kappa_1|} \tan \frac{\kappa_1 n\pi}{2|\pi - \kappa_1|},$$

and

$$\beta_2 = -\frac{4|\pi - \kappa_1|}{n\pi} \tan \frac{\kappa_1 n\pi}{2|\pi - \kappa_1|}.$$

Proof. We suppose that $M(\lambda) = \pm E$. We first show that $\lambda \neq 0$. We have

$$M(0) = \begin{pmatrix} 1 + \beta_1\beta_2 + \beta_1\tau & \beta_2 + \tau + (1 + \beta_1\beta_2)\kappa_1 + \beta_1\kappa_1\tau \\ 0 & 1 + \beta_1\kappa_1 \end{pmatrix}.$$

This means $M(0) \neq \pm E$ because of $1 + \beta_1\kappa_1 \neq 1$. This is why $\lambda \neq 0$.

Since $M(\lambda) = \pm E$, we have

$$y_1(2\pi + 0, \lambda) - y'_2(2\pi + 0, \lambda) = y'_1(2\pi + 0, \lambda) = y_2(2\pi + 0, \lambda) = 0.$$

By $\{y'_1(2\pi + 0, \lambda)/\lambda + y_2(2\pi + 0, \lambda)\}\sqrt{\lambda} \cos \kappa_1 \sqrt{\lambda} = 0$, it turns out that

$$\begin{aligned} & \left(\frac{\beta_1}{\sqrt{\lambda}} + \beta_2 \sqrt{\lambda} \right) \cos \tau \sqrt{\lambda} \cos^2 \kappa_1 \sqrt{\lambda} + \beta_1 \beta_2 \cos \tau \sqrt{\lambda} \cos \kappa_1 \sqrt{\lambda} \sin \kappa_1 \sqrt{\lambda} \\ & + \left(\frac{\beta_1}{\sqrt{\lambda}} - \beta_2 \sqrt{\lambda} \right) \sin \tau \sqrt{\lambda} \cos \kappa_1 \sqrt{\lambda} \sin \kappa_1 \sqrt{\lambda} = 0. \end{aligned} \quad (3.10)$$

On the other hand, it follows by $(y_1(2\pi + 0, \lambda) - y'_2(2\pi + 0, \lambda)) \sin \kappa_1 \sqrt{\lambda} = 0$ that

$$\begin{aligned} & \beta_1 \beta_2 \cos \tau \sqrt{\lambda} \cos \kappa_1 \sqrt{\lambda} \sin \kappa_1 \sqrt{\lambda} + \left(\frac{\beta_1}{\sqrt{\lambda}} - \beta_2 \sqrt{\lambda} \right) \sin \tau \sqrt{\lambda} \cos \kappa_1 \sqrt{\lambda} \sin \kappa_1 \sqrt{\lambda} \\ & - \left(\beta_2 \sqrt{\lambda} + \frac{\beta_1}{\sqrt{\lambda}} \right) \cos \tau \sqrt{\lambda} \sin^2 \kappa_1 \sqrt{\lambda} = 0. \end{aligned} \quad (3.11)$$

Substituting (3.11) from (3.10), we have

$$\left(\frac{\beta_1}{\sqrt{\lambda}} + \beta_2 \sqrt{\lambda} \right) \cos \tau \sqrt{\lambda} = 0,$$

namely

$$\frac{\beta_1}{\sqrt{\lambda}} + \beta_2 \sqrt{\lambda} = 0 \quad \text{or} \quad \cos \tau \sqrt{\lambda} = 0. \quad (3.12)$$

We show the statement (i). We suppose that $\lambda \neq -\beta_1/\beta_2$. Then it follows by (3.12) that $\cos \tau \sqrt{\lambda} = 0$. This combined with $\lambda \neq 0$ and $y'_1(2\pi + 0, \lambda) = 0$ means $\cos \kappa_1 \sqrt{\lambda} = 0$. Substituting $\cos \kappa_1 \sqrt{\lambda} = \cos \tau \sqrt{\lambda} = 0$ for $y_2(2\pi + 0, \lambda) = 0$, we have $\lambda = \beta_2/\beta_1$. Therefore we get (i).

Next, we show the statement (ii). We suppose that $\lambda = -\beta_1/\beta_2$. Then we have $\beta_1/\sqrt{\lambda} + \beta_2 \sqrt{\lambda} = 0$. Substituting $\beta_1/\sqrt{\lambda} = -\beta_2 \sqrt{\lambda}$ for $(y_1(2\pi + 0, \lambda) - y_2(2\pi + 0, \lambda))/\beta_2 = 0$, we have

$$\sin \tau \sqrt{\lambda} \cos \kappa_1 \sqrt{\lambda} - \frac{\beta_1}{2\sqrt{\lambda}} \cos \tau \sqrt{\lambda} \cos \kappa_1 \sqrt{\lambda} = 0. \quad (3.13)$$

We prove $\cos \kappa_1 \sqrt{\lambda} \neq 0$ by contradiction. Seeking a contradiction, we assume $\cos \kappa_1 \sqrt{\lambda} = 0$. Then it follows by $y'_1(2\pi + 0, \lambda) = 0$ and $\lambda \neq 0$ that $\cos \tau \sqrt{\lambda} = 0$. Substituting $\cos \kappa_1 \sqrt{\lambda} = \cos \tau \sqrt{\lambda} = 0$ for $y_2(2\pi + 0, \lambda) = 0$, we have $\lambda = \beta_1/\beta_2$. This contradicts $\lambda = -\beta_1/\beta_2$. Therefore we have $\cos \kappa_1 \sqrt{\lambda} \neq 0$.

By (3.13) and $\cos \kappa_1 \sqrt{\lambda} \neq 0$, it follows that

$$\sin \tau \sqrt{\lambda} = \frac{\beta_1}{2\sqrt{\lambda}} \cos \tau \sqrt{\lambda}. \quad (3.14)$$

Inserting $\beta_1/\lambda = -\beta_2$ and (3.14) into (3.6), we have

$$\sin \kappa_1 \sqrt{\lambda} = \frac{\beta_1}{2\sqrt{\lambda}} \cos \kappa_1 \sqrt{\lambda}. \quad (3.15)$$

By (3.14) and (3.15), it turns out that $\sin(\tau - \kappa_1)\sqrt{\lambda} = 0$. This implies that $\beta_1/\beta_2 < 0$ because of $\lambda = -\beta_1/\beta_2$ and $\tau - \kappa_1 \neq 0$. Substituting $\lambda = -\beta_1/\beta_2$ and $\tau = 2\pi - \kappa_1$ for $\sin(\tau - \kappa_1)\sqrt{\lambda} = 0$, we obtain

$$\sin 2(\pi - \kappa_1) \sqrt{-\frac{\beta_1}{\beta_2}} = 0.$$

Namely, there exists $n \in \mathbb{N}$ such that

$$-\frac{\beta_1}{\beta_2} = \frac{n^2}{4(\pi - \kappa_1)^2}. \quad (3.16)$$

On the other hand, Equation (3.15) means

$$\beta_1 = \frac{n\pi}{|\pi - \kappa_1|} \tan \frac{\kappa_1 n\pi}{2|\pi - \kappa_1|}.$$

This combined with (3.16) implies

$$\beta_2 = -\frac{4|\pi - \kappa_1|}{n\pi} \tan \frac{\kappa_1 n\pi}{2|\pi - \kappa_1|}.$$

□

Next, we show Theorem 3.1.

Proof of Theorem 3.1. We suppose $\kappa_1 \neq \pi$ and (3.9). We define

$$S = \begin{cases} \{\beta_2/\beta_1\} & \text{if } \cos \kappa_1 \sqrt{\beta_2/\beta_1} = \cos \tau \sqrt{\beta_2/\beta_1} = 0, \\ \emptyset & \text{otherwise.} \end{cases}$$

Then, Lemma 3.2 says

$$S \subset \mathcal{B}.$$

Since $S \supset \mathcal{B}$, we have $\mathcal{B} = \emptyset$ if $S = \emptyset$. Next we consider the case where $S \neq \emptyset$. We have $S = \{\xi\}$, where $\xi = \beta_2/\beta_1$. Since

$$M(\xi) = -\sin \kappa_1 \sqrt{\xi} \sin \tau \sqrt{\xi} \begin{pmatrix} 1 & \beta_2 - \frac{\beta_1}{\lambda} \\ 0 & 1 \end{pmatrix},$$

and

$$\beta_2 - \frac{\beta_1}{\lambda} = \beta_2 - \frac{\beta_1}{\frac{\beta_2}{\beta_1}} = \frac{(\beta_2 - \beta_1)(\beta_2 + \beta_1)}{\beta_2},$$

$M(\xi) = \pm E$ is equivalent to

$$\beta_2 - \beta_1 = 0 \quad \text{or} \quad \beta_2 + \beta_1 = 0, \quad (3.17)$$

whence $\xi \in \mathcal{B}$ if and only if (3.17) holds. This together with $\{\xi\} = S \supset \mathcal{B}$ implies that

$$\mathcal{B} = \begin{cases} \{\xi\} & \text{if } \beta_2 - \beta_1 = 0 \quad \text{or} \quad \beta_2 + \beta_1 = 0, \\ \emptyset & \text{otherwise.} \end{cases}$$

If $\beta_1 + \beta_2 = 0$, then we have $S = \emptyset$, so that $\mathcal{B} = \emptyset$. If $\beta_2 - \beta_1 = 0$, then we obtain

$$\mathcal{B} = S = \begin{cases} \{1\} & \text{if } \kappa_1 = \frac{\pi}{2}, \frac{3}{2}\pi, \\ \emptyset & \text{otherwise.} \end{cases}$$

□

Finally, we prove Theorem 1.5.

Proof of Theorem 1.5. Theorem 3.1 (i) directly follows Theorem 1.5 (i). So, our last work is to prove (ii). We suppose $\kappa_1 - \pi/2$ and $\beta_1 = \beta_2$. Then, Theorem 3.1 (ii) reads $\mathcal{B} = \{1\}$.

We calculate the rotation number $\rho(1)$. Substituting $\lambda = 1$ for (1.7), we have

$$\frac{d}{dx}\omega(x, \lambda) = 1, \quad x \in \mathbb{R} \setminus \Gamma. \quad (3.18)$$

Since the rotation number is independent of the initial value ω_0 , we may put $\omega_0 = 0$. Equation (3.18) means $\omega(\kappa_1 - 0, 1, 0) = \pi/2$. It follows from (1.8)–(1.11) that

$$\omega(\kappa_1 + 0, 1, 0) = \begin{cases} \text{Arctan}\left(\frac{1}{\beta_1}\right) & \text{if } \beta_1 > 0, \\ \pi + \text{Arctan}\left(\frac{1}{\beta_1}\right) & \text{if } \beta_1 < 0. \end{cases}$$

Using Equation (3.18) again, we have

$$\omega(2\pi - 0, 1, 0) = \begin{cases} \text{Arctan}\left(\frac{1}{\beta_1}\right) + (2\pi - \kappa_1) & \text{if } \beta_1 > 0, \\ \pi + \text{Arctan}\left(\frac{1}{\beta_1}\right) + (2\pi - \kappa_1) & \text{if } \beta_1 < 0. \end{cases}$$

Using (1.8)–(1.11) in the case where $x = 2\pi - 0$, we have $\omega(2\pi + 0, 1, 0) = 2\pi$. Since the equation (1.7) is π -periodic in ω , we have $\omega(2\pi t + 0, 1, 0) = 2\pi t$ for $t \in \mathbb{N}$. Therefore we have $\rho(1) = 1$.

We recall (1.14). Since

$$l = \begin{cases} 1 & \text{if } \beta_1 > 0, \\ 0 & \text{if } \beta_1 < 0, \end{cases}$$

then we arrive at the goal owing to Theorem 1.1.

In a similar way, we obtain (ii) in the case where $\kappa_1 = 3\pi/2$ and $\beta_1 = \beta_2$. □

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Hiroaki Niikuni
Department of Mathematics and Information Sciences
Tokyo Metropolitan University
Minami-Ohsawa 1-1
Hachioji Tokyo 192-0397
Japan
e-mail: dreamsphere@infoseek.jp