

On a compressible fluid model of Korteweg type in a maximal regularity class

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1 Introduction

This article shows local and global existence theorems in a maximal regularity class for a compressible fluid model of Korteweg type as follows:

$$\left\{ \begin{array}{ll} \partial_t \rho + \operatorname{div}(\rho \mathbf{u}) = 0 & \text{in } \Omega \times (0, T_0), \\ \rho(\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u}) = \operatorname{Div}(\mathbf{S}(\mathbf{u}) + \mathbf{K}(\rho) - P(\rho)\mathbf{I}) & \text{in } \Omega \times (0, T_0), \\ \mathbf{n} \cdot \nabla \rho = 0, \quad \mathbf{u} = 0 & \text{on } \Gamma \times (0, T_0), \\ (\rho, \mathbf{u})|_{t=0} = (\rho_0 + \rho_\infty, \mathbf{u}_0) & \text{in } \Omega, \end{array} \right. \quad (1.1)$$

where Ω is a domain in $\mathbf{R}^N$, $N \geq 2$, with boundary Γ and T_0 is a positive number.

Here $\rho = \rho(x, t)$ and $\mathbf{u} = (u_1(x, t), \dots, u_N(x, t))^{\top 1}$ are respectively the fluid density and the fluid velocity at $x = (x_1, \dots, x_N) \in \Omega$ and $t \in (0, T_0)$; P is a given function describing the pressure and $\mathbf{I}$ is the $N \times N$ identity matrix; $\mathbf{S}(\mathbf{u})$ is the viscous stress tensor given by $\mathbf{S}(\mathbf{u}) = \mu \mathbf{D}(\mathbf{u}) + (\nu - \mu) \operatorname{div} \mathbf{u} \mathbf{I}$, where $\mu, \nu > 0$ are viscosity coefficients and $\mathbf{D}(\mathbf{u})$ is the doubled deformation tensor, i.e. $\mathbf{D}(\mathbf{u})$ is an $N \times N$ matrix whose (i, j) component is given by $\partial_i u_j + \partial_j u_i$ for $\partial_i = \partial/\partial x_i$; $\mathbf{K}(\rho)$ is the so-called Korteweg tensor, i.e.

$$\begin{aligned} \mathbf{K}(\rho) &= \frac{\kappa}{2} (\Delta \rho^2 - |\nabla \rho|^2) \mathbf{I} - \kappa \nabla \rho \otimes \nabla \rho \\ &= \kappa \left(\rho \Delta \rho + \frac{|\nabla \rho|^2}{2} \right) \mathbf{I} - \kappa \nabla \rho \otimes \nabla \rho, \end{aligned} \quad (1.2)$$

where $\kappa > 0$ is a capillarity coefficient and $\nabla \rho \otimes \nabla \rho$ is an $N \times N$ matrix whose (i, j) component is given by $(\partial_i \rho)(\partial_j \rho)$; $\mathbf{n}$ is the unit outward normal vector to Γ and $\mathbf{a} \cdot \mathbf{b} = \sum_{i=1}^N a_i b_i$ for N -vectors $\mathbf{a} = (a_1, \dots, a_N)^{\top}$ and $\mathbf{b} = (b_1, \dots, b_N)^{\top}$; $\rho_0 = \rho_0(x)$ and $\mathbf{u}_0 = (u_{01}(x), \dots, u_{0N}(x))^{\top}$ are given initial data, while ρ_∞ is a positive constant.

Throughout this article, we assume

¹ $\mathbf{M}^{\top}$ denotes the transpose of $\mathbf{M}$.

Assumption 1.1. (1) *The coefficients μ , ν , and κ are positive constants.*

(2) *The pressure $P : (\rho_\infty/8, 8\rho_\infty) \rightarrow \mathbf{R}$ is smooth enough.*

Korteweg formulated in 1901 some tensor that included gradients of density in order to model fluid capillarity effects, and Dunn and Serrin [2] derived (1.2) in view of rational mechanics by introducing the thermodynamics of interstitial working. Concerning the mathematical analysis of Korteweg-type model, we refer e.g. to [5, 1, 4, 15] for the whole space problem and to [6, 7, 8, 9] for boundary value problems. On the other hand, [10] employs the Korteweg-type model in order to analyze the structure of liquid-vapor phase transition in numerical analysis.

2 Preliminaries

Let $p \in (1, \infty)$, and let $q \in (1, \infty)$ or $q = \infty$. We here introduce function spaces used throughout this article as follows:

- $L_q(G)$ and $H_q^m(G)$, $m \in \mathbf{N}$, are respectively the usual Lebesgue spaces and the Sobolev spaces, where G is a domain in $\mathbf{R}^N$. The norm of $L_q(G)$ is denoted by $\|\cdot\|_{L_q(G)}$, while the norm of $H_q^m(G)$ is denoted by $\|\cdot\|_{H_q^m(G)}$.
- Let $(\cdot, \cdot)_{\theta, p}$ be the real interpolation functor for $\theta \in (0, 1)$. Then the Besov spaces $B_{q, p}^{3-2/p}(G)$ and $B_{q, p}^{2-2/p}(G)$ are defined as

$$B_{q, p}^{3-2/p}(G) = (H_q^1(G), H_q^3(G))_{1-1/p, p}, \quad B_{q, p}^{2-2/p}(G) = (L_q(G), H_q^2(G))_{1-1/p, p}.$$

- Let X be a Banach space and I be an interval in $\mathbf{R}$. Then $L_p(I, X)$ and $H_p^1(I, X)$ are respectively the X -valued Lebesgue spaces and the X -valued Sobolev spaces. The norm of $L_p(I, X)$ is denoted by $\|\cdot\|_{L_p(I, X)}$, while the norm of $H_p^1(I, X)$ is denoted by $\|\cdot\|_{H_p^1(I, X)}$.
- Let $T \in (0, \infty)$ or $T = \infty$. Then ${}_0H_p^1((0, T), X)$ is given by

$${}_0H_p^1((0, T), X) = \{f \in H_p^1((0, T), X) : f|_{t=0} = 0 \text{ in } X\}.$$

Next, we introduce the definition of uniform C^3 domains.

Definition 2.1 ([3, 14]). *Let D be a domain in $\mathbf{R}^N$ with boundary ∂D . Then D is called a uniform C^3 domain, if there exist positive constants α , β , and K such that the following assertion holds: for any $x_0 = (x_{01}, \dots, x_{0N}) \in \partial D$, there are coordinate*

number j and a C^3 function $h(x')$ ($x' = (x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_N)$) on $B'_\alpha(x'_0)$, with $x'_0 = (x_{01}, \dots, x_{0j-1}, x_{0j+1}, \dots, x_{0N})$,

$$B'_\alpha(x'_0) = \{x \in \mathbf{R}^N \mid |x' - x'_0| < \alpha\}, \quad \|h\|_{H_\infty^3(B'_\alpha(x'_0))} \leq K,$$

such that

$$\begin{aligned} D \cap B_\beta(x_0) &= \{x \in \mathbf{R}^N : x_j > h(x'), x' \in B'_\alpha(x'_0)\} \cap B_\beta(x_0), \\ \partial D \cap B_\beta(x_0) &= \{x \in \mathbf{R}^N : x_j = h(x'), x' \in B'_\alpha(x'_0)\} \cap B_\beta(x_0). \end{aligned}$$

Here $B_\beta(x_0) = \{x \in \mathbf{R}^N : |x - x_0| < \beta\}$.

Remark 2.2. Typical examples of uniform C^3 domains are as follows: bounded domains; exterior domains; half-spaces, layers, tubes, and their perturbed domains.

3 Local solvability

3.1 Linearization

Let us start with the linearization of (1.1). Replace ρ by $\rho + \rho_\infty$ in (1.1). Then the first equation becomes

$$\partial_t \rho + \rho_\infty \operatorname{div} \mathbf{u} = -\rho \operatorname{div} \mathbf{u} - \mathbf{u} \cdot \nabla \rho; \quad (3.1)$$

the second equation becomes

$$\begin{aligned} \rho_\infty \partial_t \mathbf{u} - \operatorname{Div}(\mathbf{S}(\mathbf{u}) + \kappa \rho_\infty \Delta \rho \mathbf{I}) &= -\rho \partial_t \mathbf{u} - (\rho + \rho_\infty) \mathbf{u} \cdot \nabla \mathbf{u} \\ &\quad + \operatorname{Div}(\mathbf{K}(\rho + \rho_\infty) - \kappa \rho_\infty \Delta \rho \mathbf{I}) \\ &\quad - P'(\rho + \rho_\infty) \nabla \rho, \end{aligned} \quad (3.2)$$

where $P'(s) = (dP/ds)(s)$. To prove the local solvability, we further rewrite (3.1) and (3.2) as follows:

$$\begin{aligned} \partial_t \rho + (\rho_0 + \rho_\infty) \operatorname{div} \mathbf{u} &= -(\rho - \rho_0) \operatorname{div} \mathbf{u} - \mathbf{u} \cdot \nabla \rho \\ &=: F(\rho, \mathbf{u}); \end{aligned}$$

for $\tilde{\mu} = \mu/\kappa$ and $\tilde{\nu} = \nu/\kappa$,

$$\begin{aligned} (\rho_0 + \rho_\infty) \partial_t \mathbf{u} - \kappa \operatorname{Div}(\tilde{\mu} \mathbf{D}(\mathbf{u}) + (\tilde{\nu} - \tilde{\mu}) \operatorname{div} \mathbf{u} \mathbf{I} + (\rho_0 + \rho_\infty) \Delta \rho \mathbf{I}) \\ = -(\rho - \rho_0) \partial_t \mathbf{u} - (\rho + \rho_\infty) \mathbf{u} \cdot \nabla \mathbf{u} \\ + \operatorname{Div}(\mathbf{K}(\rho + \rho_\infty) - \kappa(\rho_0 + \rho_\infty) \Delta \rho \mathbf{I}) - P'(\rho + \rho_\infty) \nabla \rho \\ =: G(\rho, \mathbf{u}). \end{aligned}$$

Setting $\gamma = \rho_0 + \rho_\infty$, we have achieved the following equivalent system of (1.1):

$$\left\{ \begin{array}{l} \partial_t \rho + \gamma \operatorname{div} \mathbf{u} = F(\rho, \mathbf{u}) \quad \text{in } \Omega \times (0, T_0), \\ \partial_t \mathbf{u} - \gamma^{-1} \kappa \operatorname{Div}(\tilde{\mu} \mathbf{D}(\mathbf{u})) + (\tilde{\nu} - \tilde{\mu}) \operatorname{div} \mathbf{u} \mathbf{I} + \gamma \Delta \rho \mathbf{I} = \gamma^{-1} G(\rho, \mathbf{u}) \quad \text{in } \Omega \times (0, T_0), \\ \mathbf{n} \cdot \nabla \rho = 0, \quad \mathbf{u} = 0 \quad \text{on } \Gamma \times (0, T_0), \\ (\rho, \mathbf{u})|_{t=0} = (\rho_0, \mathbf{u}_0) \quad \text{in } \Omega. \end{array} \right. \quad (3.3)$$

3.2 Linearized problem

For a positive number S , we consider a linearized problem associated with (3.3) as follows:

$$\left\{ \begin{array}{l} \partial_t \rho + \gamma_1 \operatorname{div} \mathbf{u} = f \quad \text{in } \Omega \times (0, S), \\ \partial_t \mathbf{u} - \gamma_4^{-1} \operatorname{Div}(\gamma_2 \mathbf{D}(\mathbf{u})) + (\gamma_3 - \gamma_2) \operatorname{div} \mathbf{u} \mathbf{I} + \gamma_1 \Delta \rho \mathbf{I} = \mathbf{g} \quad \text{in } \Omega \times (0, S), \\ \mathbf{n} \cdot \nabla \rho = 0, \quad \mathbf{u} = 0 \quad \text{on } \Gamma \times (0, S), \\ (\rho, \mathbf{u})|_{t=0} = (\rho_0, \mathbf{u}_0) \quad \text{in } \Omega, \end{array} \right. \quad (3.4)$$

where Ω and $\gamma_i = \gamma_i(x)$ ($i = 1, 2, 3, 4$) satisfy the following assumption.

Assumption 3.1. (1) *The domain Ω is a uniform C^3 domain in $\mathbf{R}^N$, $N \geq 2$, and its boundary is denoted by Γ .*

(2) *The coefficients $\gamma_i = \gamma_i(x)$ ($i = 1, 2, 3, 4$) are uniformly Lipschitz continuous functions on $\overline{\Omega}$, i.e. there exists a positive constant γ_L such that $|\gamma_i(x) - \gamma_i(y)| \leq \gamma_L |x - y|$ for any $x, y \in \overline{\Omega}$ and for $i = 1, 2, 3, 4$. In addition, there exist positive constants γ_* , γ^* such that $\gamma_* \leq \gamma_i(x) \leq \gamma^*$ for any $x \in \overline{\Omega}$ and for $i = 1, 2, 3, 4$.*

Let $q \in (1, \infty)$ and $X_q = H_q^1(\Omega) \times L_q(\Omega)^N$. We define an operator A_q by

$$A_q(\rho, \mathbf{u}) = (-\gamma_1 \operatorname{div} \mathbf{u}, \gamma_4^{-1} \operatorname{Div}(\gamma_2 \mathbf{D}(\mathbf{u})) + (\gamma_3 - \gamma_2) \operatorname{div} \mathbf{u} \mathbf{I} + \gamma_1 \Delta \rho \mathbf{I}),$$

with the domain $D(A_q)$:

$$D(A_q) = \{(\rho, \mathbf{u}) \in \times H_q^3(\Omega) \times H_q^2(\Omega)^N \mid \mathbf{n} \cdot \nabla \rho = 0, \mathbf{u} = 0 \text{ on } \Gamma\}.$$

Note that $D(A_q) \subset X_q$ and $A_q : D(A_q) \rightarrow X_q$. One then has

Lemma 3.2 ([12]). *Let $q \in (1, \infty)$ and suppose that Assumption 3.1 holds. Then A_q generates an analytic C_0 -semigroup $\{e^{A_q t}\}_{t \geq 0}$ on X_q . In addition, there exist constants $\delta_1 \geq 1$ and $C_{N,q,\delta_1} > 0$ such that for any $t > 0$*

$$\|e^{A_q t}(\rho_0, \mathbf{u}_0)\|_{X_q} \leq C_{N,q,\delta_1} e^{(\delta_1/2)t} \|(\rho_0, \mathbf{u}_0)\|_{X_q} \quad ((\rho_0, \mathbf{u}_0) \in X_q),$$

$$\begin{aligned}\|\partial_t e^{A_q t}(\rho_0, \mathbf{u}_0)\|_{X_q} &\leq C_{N,q,\delta_1} e^{(\delta_1/2)t} t^{-1} \|(\rho_0, \mathbf{u}_0)\|_{X_q} \quad ((\rho_0, \mathbf{u}_0) \in X_q), \\ \|\partial_t e^{A_q t}(\rho_0, \mathbf{u}_0)\|_{X_q} &\leq C_{N,q,\delta_1} e^{(\delta_1/2)t} \|(\rho_0, \mathbf{u}_0)\|_{D(A_q)} \quad ((\rho_0, \mathbf{u}_0) \in D(A_q)),\end{aligned}$$

where $\|\cdot\|_{D(A_q)}$ denotes the graph norm of A_q .

Setting $D_{q,p}(\Omega) = (X_q, D(A_q))_{1-1/p,p}$ for $p, q \in (1, \infty)$, we have

Lemma 3.3 ([12]). *Let $(p, q) \in (1, \infty)$ and suppose that Assumption 3.1 holds. Then, for any $(\rho_0, \mathbf{u}_0) \in D_{q,p}(\Omega)$, $(\rho, \mathbf{u}) = e^{A_q t}(\rho_0, \mathbf{u}_0)$ is a unique solution to the system (3.4) under the condition of $(f, \mathbf{g}) = (0, 0)$. In addition,*

$$\begin{aligned}&\|\partial_t \rho\|_{L_p((0,S), H_q^1(\Omega))} + \|\rho\|_{L_p((0,S), H_q^3(\Omega))} \\ &+ \|\partial_t \mathbf{u}\|_{L_p((0,S), L_q(\Omega)^N)} + \|\mathbf{u}\|_{L_p((0,S), H_q^2(\Omega)^N)} \\ &\leq C_{N,p,q,\delta_1} e^{\delta_1 S} \|(\rho_0, \mathbf{u}_0)\|_{D_{q,p}(\Omega)}\end{aligned}$$

for some positive constant C_{N,p,q,δ_1} independent of S , where δ_1 is the same constant as in Lemma 3.2.

The following lemma is also proved in [12].

Lemma 3.4 (Maximal regularity). *Let $p, q \in (1, \infty)$ and suppose that Assumption 3.1 holds. Then, for $(\rho_0, \mathbf{u}_0) = (0, 0)$ and for any f and $\mathbf{g}$ with*

$$f \in L_p((0, S), H_q^1(\Omega)), \quad \mathbf{g} \in L_p((0, S), L_q(\Omega)^N),$$

the system (3.4) admits a unique solution $(\rho, \mathbf{u})$ with

$$\begin{aligned}\rho &\in {}_0H_p^1((0, S), H_q^1(\Omega)) \cap L_p((0, S), H_q^3(\Omega)), \\ \mathbf{u} &\in {}_0H_p^1((0, S), L_q(\Omega)^N) \cap L_p((0, S), H_q^2(\Omega)^N).\end{aligned}$$

In addition, the solution $(\rho, \mathbf{u})$ satisfies the estimate:

$$\begin{aligned}&\|\partial_t \rho\|_{L_p((0,S), H_q^1(\Omega))} + \|\rho\|_{L_p((0,S), H_q^3(\Omega))} \\ &+ \|\partial_t \mathbf{u}\|_{L_p((0,S), L_q(\Omega)^N)} + \|\mathbf{u}\|_{L_p((0,S), H_q^2(\Omega)^N)} \\ &\leq C_{N,p,q,\delta_2} e^{\delta_2 S} \left(\|f\|_{L_p((0,S), H_q^1(\Omega))} + \|\mathbf{g}\|_{L_p((0,S), L_q(\Omega)^N)} \right)\end{aligned}$$

for positive constants δ_2 and C_{N,p,q,δ_2} independent of S .

3.3 Local existence theorem

Let $(p, q) \in (2, \infty) \times (N, \infty)$ and $(\rho_0, \mathbf{u}_0) \in B_{q,p}(\Omega)^{3-2/p}(\Omega) \times B_{q,p}^{2-2/p}(\Omega)^N$ with the following conditions:

$$\mathbf{n} \cdot \nabla \rho_0 = 0, \quad \mathbf{u}_0 = 0 \quad \text{on } \Gamma, \quad (3.5)$$

$$\frac{\rho_\infty}{2} \leq \rho_0(x) + \rho_\infty \leq 2\rho_\infty \quad (x \in \overline{\Omega}), \quad (3.6)$$

and let $(\rho_*, \mathbf{u}_*) = e^{A_q t}(\rho_0, \mathbf{u}_0)$ for $\gamma_1 = \gamma$, $\gamma_2 = \tilde{\mu}$, $\gamma_3 = \tilde{\nu}$, and $\gamma_4 = \gamma\kappa^{-1}$. Then, setting $\rho = \sigma + \rho_*$ and $\mathbf{u} = \mathbf{v} + \mathbf{u}_*$ in (3.3), we observe that

$$\left\{ \begin{array}{l} \partial_t \sigma + \gamma \operatorname{div} \mathbf{v} = F(\sigma + \rho_*, \mathbf{v} + \mathbf{u}_*) \quad \text{in } \Omega \times (0, T_0), \\ \partial_t \mathbf{v} - \gamma^{-1} \kappa \operatorname{Div}(\tilde{\mu} \mathbf{D}(\mathbf{v})) + (\tilde{\nu} - \tilde{\mu}) \operatorname{div} \mathbf{v} \mathbf{I} + \gamma \Delta \sigma \mathbf{I} \\ \quad = \gamma^{-1} G(\sigma + \rho_*, \mathbf{v} + \mathbf{u}_*) \quad \text{in } \Omega \times (0, T_0), \\ \mathbf{n} \cdot \nabla \rho = 0, \quad \mathbf{u} = 0 \quad \text{on } \Gamma \times (0, T_0), \\ (\sigma, \mathbf{v})|_{t=0} = (0, 0) \quad \text{in } \Omega. \end{array} \right. \quad (3.7)$$

To use the contraction mapping principle, we introduce the following notation:

- For $T > 0$, ${}_0Z_T := {}_0Z_T^1 \times {}_0Z_T^2$ with

$$\begin{aligned} {}_0Z_T^1 &= {}_0H_p^1((0, T), H_q^1(\Omega)) \cap L_p((0, T), H_q^3(\Omega)), \\ {}_0Z_T^2 &= {}_0H_p^1((0, T), L_q(\Omega)^N) \cap L_p((0, T), H_q^2(\Omega)^N). \end{aligned}$$

Here the norm $\|\cdot\|_{{}_0Z_T}$ of ${}_0Z_T$ is given by

$$\begin{aligned} \|(\rho, \mathbf{u})\|_{{}_0Z_T} &= \|\rho\|_{{}_0H_p^1((0, T), H_q^1(\Omega))} + \|\rho\|_{L_p((0, T), H_q^3(\Omega))} \\ &\quad + \|\mathbf{u}\|_{{}_0H_p^1((0, T), L_q(\Omega)^N)} + \|\mathbf{u}\|_{L_p((0, T), H_q^2(\Omega)^N)}. \end{aligned}$$

- For $T > 0$ and $r > 0$,

$$\begin{aligned} {}_0Z_T(r) &:= \left\{ (\tau, \mathbf{w}) \in {}_0Z_T : \|(\tau, \mathbf{w})\|_{{}_0Z_T} \leq r, \right. \\ &\quad \left. \frac{\rho_\infty}{4} \leq \tau(x, t) + \rho_*(x, t) + \rho_\infty \leq 4\rho_\infty \text{ for any } (x, t) \in \overline{\Omega} \times [0, T] \right\}. \end{aligned}$$

Let $(\tau, \mathbf{w}) \in {}_0Z_T(L)$ for a suitable positive number L and for $T > 0$, and replace $(\sigma, \mathbf{v})$ by $(\tau, \mathbf{w})$ in the right-hand sides of (3.7). Then, by the maximal regularity stated in Lemma 3.4, we can define a contraction mapping $\Phi : {}_0Z_T(L) \ni (\tau, \mathbf{w}) \mapsto (\sigma, \mathbf{v}) \in {}_0Z_T(L)$ for a sufficiently small $T \in (0, T_0)$. We thus obtain by the contraction mapping principle a local existence theorem in the maximal regularity class as follows:

Theorem 3.5. *Assume that Ω is a uniform C^3 domain in $\mathbf{R}^N$, $N \geq 2$, with boundary Γ . Let $(p, q) \in (2, \infty) \times (N, \infty)$, and let R be an arbitrary positive number. Then there exist positive constants L and $T \in (0, T_0)$ such that, for any $(\rho_0, \mathbf{u}_0) \in B_{q,p}^{3-2/p}(\Omega) \times B_{q,p}^{2-2/p}(\Omega)^N$ satisfying $\|(\rho_0, \mathbf{u}_0)\|_{B_{q,p}(\Omega)^{3-2/p} \times B_{q,p}^{2-2/p}(\Omega)^N} \leq R$ with (3.5) and (3.6), the system (3.7) admits a unique solution $(\sigma, \mathbf{v})$ on $(0, T)$ in ${}_0Z_T(L)$.*

4 Global solvability

Throughout this section, we assume

Assumption 4.1. (1) The domain Ω is a bounded domain in $\mathbf{R}^N$, $N \geq 2$, with C^3 boundary Γ .

(2) $P'(\rho_\infty) > 0$.

4.1 Linearization

Let $\mu_\infty = \mu/\rho_\infty$, $\nu_\infty = \nu/\rho_\infty$, and $\gamma_\infty = \rho_\infty^{-1}P'(\rho_\infty)$. Then we rewrite (3.1) and (3.2) as follows:

$$\partial_t \rho + \rho_\infty \operatorname{div} \mathbf{u} = -\operatorname{div}(\rho \mathbf{u}) =: \mathbf{F}(\rho, \mathbf{u})$$

and

$$\begin{aligned} & \partial_t \mathbf{u} - \operatorname{Div}(\mu_\infty \mathbf{D}(\mathbf{u}) + (\nu_\infty - \mu_\infty) \operatorname{div} \mathbf{u} \mathbf{I} + \kappa \Delta \rho \mathbf{I}) + \gamma_\infty \nabla \rho \\ &= \rho_\infty^{-1} \left\{ -\rho \partial_t \mathbf{u} - (\rho + \rho_\infty) \mathbf{u} \cdot \nabla \mathbf{u} \right. \\ &+ \operatorname{Div}(\mathbf{K}(\rho + \rho_\infty) - \kappa \rho_\infty \Delta \rho \mathbf{I}) - (P'(\rho + \rho_\infty) - P'(\rho_\infty)) \nabla \rho \left. \right\} \\ &=: \mathbf{G}(\rho, \mathbf{u}). \end{aligned}$$

Thus we have achieved the following equivalent system of (1.1) with $T_0 = \infty$:

$$\left\{ \begin{array}{ll} \partial_t \rho + \rho_\infty \operatorname{div} \mathbf{u} = \mathbf{F}(\rho, \mathbf{u}) & \text{in } \Omega \times (0, \infty), \\ \partial_t \mathbf{u} - \operatorname{Div}(\mu_\infty \mathbf{D}(\mathbf{u}) + (\nu_\infty - \mu_\infty) \operatorname{div} \mathbf{u} \mathbf{I} + \kappa \Delta \rho \mathbf{I}) + \gamma_\infty \nabla \rho = \mathbf{G}(\rho, \mathbf{u}) & \text{in } \Omega \times (0, \infty), \\ \mathbf{n} \cdot \nabla \rho = 0, \quad \mathbf{u} = 0 & \text{on } \Gamma \times (0, \infty), \\ (\rho, \mathbf{u})|_{t=0} = (\rho_0, \mathbf{u}_0) & \text{in } \Omega. \end{array} \right. \quad (4.1)$$

4.2 Linearized problem

We consider a linearized problem associated with (4.1) as follows:

$$\left\{ \begin{array}{ll} \partial_t \rho + \rho_\infty \operatorname{div} \mathbf{u} = f & \text{in } \Omega \times (0, \infty), \\ \partial_t \mathbf{u} - \operatorname{Div}(\mu_\infty \mathbf{D}(\mathbf{u}) + (\nu_\infty - \mu_\infty) \operatorname{div} \mathbf{u} \mathbf{I} + \kappa \Delta \rho \mathbf{I}) + \gamma_\infty \nabla \rho = \mathbf{g} & \text{in } \Omega \times (0, \infty), \\ \mathbf{n} \cdot \nabla \rho = 0, \quad \mathbf{u} = 0 & \text{on } \Gamma \times (0, \infty), \\ (\rho, \mathbf{u})|_{t=0} = (\rho_0, \mathbf{u}_0) & \text{in } \Omega. \end{array} \right. \quad (4.2)$$

To construct an analytic C_0 -semigroups associated with (4.2), we set for $q \in (1, \infty)$

$$\mathbf{H}_q^1(\Omega) = \left\{ \rho \in H_q^1(\Omega) : \int_{\Omega} \rho \, dx = 0 \right\}, \quad \mathbf{X}_q = \mathbf{H}_q^1(\Omega) \times L_q(\Omega)^N,$$

and also

$$\mathbf{H}_q^3(\Omega) = H_q^3(\Omega) \cap \mathbf{H}_q^1(\Omega).$$

Their norms are given by

$$\begin{aligned} \|\rho\|_{\mathbf{H}_q^1(\Omega)} &= \|\rho\|_{H_q^1(\Omega)}, \quad \|(\rho, \mathbf{u})\|_{\mathbf{H}_q^1(\Omega) \times L_q(\Omega)^N} = \|\rho\|_{H_q^1(\Omega)} + \|\mathbf{u}\|_{L_q(\Omega)^N}, \\ \|\rho\|_{\mathbf{H}_q^3(\Omega)} &= \|\rho\|_{H_q^3(\Omega)}. \end{aligned}$$

In addition, an operator $\mathbf{A}_q$ is defined by

$$\mathbf{A}_q(\rho, \mathbf{u}) = (-\rho_{\infty} \operatorname{div} \mathbf{u}, \operatorname{Div}(\mu_{\infty} \mathbf{D}(\mathbf{u}) + (\nu_{\infty} - \mu_{\infty}) \operatorname{div} \mathbf{u} \mathbf{I} + \kappa \Delta \rho \mathbf{I}) + \gamma_{\infty} \nabla \rho),$$

with the domain $\mathbf{D}(\mathbf{A}_q)$:

$$\mathbf{D}(\mathbf{A}_q) = \{(\rho, \mathbf{u}) \in \mathbf{H}_q^3(\Omega) \times H_q^2(\Omega)^N : \mathbf{n} \cdot \nabla \rho = 0, \mathbf{u} = 0 \text{ on } \Gamma\}.$$

Note that $\mathbf{D}(\mathbf{A}_q) \subset \mathbf{X}_q$ and $\mathbf{A}_q : \mathbf{D}(\mathbf{A}_q) \rightarrow \mathbf{X}_q$.

Now we introduce the generation of an analytic C_0 -semigroup $\{e^{\mathbf{A}_q t}\}_{t \geq 0}$ on $\mathbf{X}_q$ and its exponential stability. To this end, we consider the following resolvent problem:

$$\begin{cases} \lambda \rho + \rho_{\infty} \operatorname{div} \mathbf{u} = f & \text{in } \Omega, \\ \lambda \mathbf{u} - \operatorname{Div}(\mu_{\infty} \mathbf{D}(\mathbf{u}) + (\nu_{\infty} - \mu_{\infty}) \operatorname{div} \mathbf{u} \mathbf{I} + \kappa \Delta \rho \mathbf{I}) + \gamma_{\infty} \nabla \rho = \mathbf{g} & \text{in } \Omega, \\ \mathbf{n} \cdot \nabla \rho = 0, \quad \mathbf{u} = 0 & \text{on } \Gamma, \end{cases} \quad (4.3)$$

where λ is the resolvent parameter varying in $\mathbf{C}_{+, \delta} = \{z \in \mathbf{C} : \Re z > \delta\}$ for $\delta \in \mathbf{R}$. By [12] and a small perturbation method, we have

Lemma 4.2. *Let $q \in (1, \infty)$ and suppose that Assumption 4.1 holds. Then there exists a positive number δ_3 such that, for any $\lambda \in \mathbf{C}_{+, \delta_3}$ and $(f, \mathbf{g}) \in \mathbf{X}_q$, the system (4.3) admits a unique solution $(\rho, \mathbf{u}) \in \mathbf{H}_q^3(\Omega) \times H_q^2(\Omega)^N$. In addition, the solution $(\rho, \mathbf{u})$ satisfies the estimate:*

$$|\lambda| \|(\rho, \mathbf{u})\|_{\mathbf{X}_q} + \|(\rho, \mathbf{u})\|_{\mathbf{H}_q^3(\Omega) \times H_q^2(\Omega)^N} \leq C_{N, q, \delta_3} \|(f, \mathbf{g})\|_{\mathbf{X}_q}$$

for some positive constant C_{N, q, δ_3} independent of $\lambda \in \mathbf{C}_{+, \delta_3}$.

Let $\overline{\mathbf{C}_+} = \overline{\mathbf{C}_{+, 0}} = \{z \in \mathbf{C} : \Re z \geq 0\}$. Combining Lemma 4.2 with a homotopic argument² and the closed graph theorem then yields

²We refer e.g. to [3, Section 7].

Lemma 4.3. *Let $q \in (1, \infty)$ and suppose that Assumption 4.1 holds. Then, for any $\lambda \in \overline{\mathbf{C}_+}$ and $(f, \mathbf{g}) \in \mathbf{X}_q$, the system (4.3) admits a unique solution $(\rho, \mathbf{u}) \in H_q^3(\Omega) \times H_q^2(\Omega)^N$. In addition, the solution satisfies the estimate:*

$$|\lambda| \|(\rho, \mathbf{u})\|_{\mathbf{X}_q} + \|(\rho, \mathbf{u})\|_{H_q^3(\Omega) \times H_q^2(\Omega)^N} \leq C_{N,q} \|(f, \mathbf{g})\|_{\mathbf{X}_q}$$

for some positive constant $C_{N,q}$ independent of $\lambda \in \overline{\mathbf{C}_+}$.

By Lemma 4.3 and the standard theory of analytic C_0 -semigroups, we have

Lemma 4.4. *Let $q \in (1, \infty)$ and suppose that Assumption 4.1 holds. Then $\mathbf{A}_q$ generates an analytic C_0 -semigroup $\{e^{\mathbf{A}_q t}\}_{t \geq 0}$ on $\mathbf{X}_q$. In addition, there exist constants $\delta_4 \in (0, 1)$ and $C_{N,q,\delta_4} > 0$ such that for any $t > 0$*

$$\begin{aligned} \|e^{\mathbf{A}_q t}(\rho_0, \mathbf{u}_0)\|_{\mathbf{X}_q} &\leq C_{N,q,\delta_4} e^{-2\delta_4 t} \|(\rho_0, \mathbf{u}_0)\|_{\mathbf{X}_q} & ((\rho_0, \mathbf{u}_0) \in \mathbf{X}_q), \\ \|\partial_t e^{\mathbf{A}_q t}(\rho_0, \mathbf{u}_0)\|_{\mathbf{X}_q} &\leq C_{N,q,\delta_4} e^{-2\delta_4 t} t^{-1} \|(\rho_0, \mathbf{u}_0)\|_{\mathbf{X}_q} & ((\rho_0, \mathbf{u}_0) \in \mathbf{X}_q), \\ \|\partial_t e^{\mathbf{A}_q t}(\rho_0, \mathbf{u}_0)\|_{\mathbf{X}_q} &\leq C_{N,q,\delta_4} e^{-2\delta_4 t} \|(\rho_0, \mathbf{u}_0)\|_{\mathbf{D}(\mathbf{A}_q)} & ((\rho_0, \mathbf{u}_0) \in \mathbf{D}(\mathbf{A}_q)), \end{aligned}$$

where $\|\cdot\|_{\mathbf{D}(\mathbf{A}_q)}$ denotes the graph norm of $\mathbf{A}_q$.

Similarly to [13], we have by setting $\mathbf{D}_{q,p}(\Omega) = (\mathbf{X}_q, \mathbf{D}(\mathbf{A}_q))_{1-1/p,p}$

Lemma 4.5. *Let $p, q \in (1, \infty)$ and suppose that Assumption 4.1 holds. Then, for any $(\rho, \mathbf{u}) \in \mathbf{D}_{q,p}(\Omega)$, $(\rho, \mathbf{u}) = e^{\mathbf{A}_q t}(\rho_0, \mathbf{u}_0)$ is a unique solution to the system (4.2) under the condition of $(f, \mathbf{g}) = (0, 0)$. In addition,*

$$\begin{aligned} &\|e^{\delta_4 t} \partial_t \rho\|_{L_p((0,\infty), H_q^1(\Omega))} + \|e^{\delta_4 t} \rho\|_{L_p((0,\infty), H_q^3(\Omega))} \\ &+ \|e^{\delta_4 t} \partial_t \mathbf{u}\|_{L_p((0,\infty), L_q(\Omega)^N)} + \|e^{\delta_4 t} \mathbf{u}\|_{L_p((0,\infty), H_q^2(\Omega)^N)} \\ &\leq C_{N,p,q,\delta_4} \|(\rho_0, \mathbf{u}_0)\|_{\mathbf{D}_{q,p}(\Omega)} \end{aligned}$$

for some positive constant C_{N,p,q,δ_4} , where δ_4 is the same constant as in Lemma 4.4.

Next, we introduce a maximal regularity with exponential stability for (4.2). To this end, we start with the standard maximal regularity following from [12] with a small perturbation method as follows:

Lemma 4.6. *Let $p, q \in (1, \infty)$ and suppose that Assumption 4.1 holds. Then there exists a positive number δ_5 such that, for $(\rho_0, \mathbf{u}_0) = (0, 0)$ and for any f and $\mathbf{g}$ with*

$$e^{-\delta_5 t} f \in L_p((0, \infty), H_q^1(\Omega)), \quad e^{-\delta_5 t} \mathbf{g} \in L_p((0, \infty), L_q(\Omega)^N),$$

the system (4.2) admits a unique solution $(\rho, \mathbf{u})$ with

$$\rho \in H_{p,\text{loc}}^1((0, \infty), H_q^1(\Omega)) \cap L_{p,\text{loc}}((0, \infty), H_q^3(\Omega)),$$

$$\mathbf{u} \in H_{p,\text{loc}}^1((0, \infty), L_q(\Omega)^N) \cap L_{p,\text{loc}}((0, \infty), H_q^2(\Omega)^N).$$

In addition, the solution $(\rho, \mathbf{u})$ satisfies the estimate:

$$\begin{aligned} & \|e^{-\delta_5 t} \partial_t \rho\|_{L_p((0, \infty), H_q^1(\Omega))} + \|e^{-\delta_5 t} \rho\|_{L_p((0, \infty), H_q^3(\Omega))} \\ & + \|e^{-\delta_5 t} \partial_t \mathbf{u}\|_{L_p((0, \infty), L_q(\Omega)^N)} + \|e^{-\delta_5 t} \mathbf{u}\|_{L_p((0, \infty), H_q^2(\Omega)^N)} \\ & \leq C_{N,p,q,\delta_5} \left(\|e^{-\delta_5 t} f\|_{L_p((0, \infty), H_q^1(\Omega))} + \|e^{-\delta_5 t} \mathbf{g}\|_{L_p((0, \infty), L_q(\Omega)^N)} \right) \end{aligned}$$

for some positive constant C_{N,p,q,δ_5} .

Similarly to [11], we can prove by Lemmas 4.4 and 4.6

Lemma 4.7 (Maximal regularity with exponential stability). *Let $p, q \in (1, \infty)$ and suppose that Assumption 4.1 holds. Then there exists a positive number $\delta_6 \in (0, 1)$ such that, for $(\rho_0, \mathbf{u}_0) = (0, 0)$ and for any f and $\mathbf{g}$ with*

$$e^{\delta_6 t} f \in L_p((0, \infty), H_q^1(\Omega)), \quad e^{\delta_6 t} \mathbf{g} \in L_p((0, \infty), L_q(\Omega)^N),$$

the system (4.2) admits a unique solution $(\rho, \mathbf{u})$ with

$$\begin{aligned} \rho & \in {}_0H_p^1((0, \infty), H_q^1(\Omega)) \cap L_p((0, \infty), H_q^3(\Omega)), \\ \mathbf{u} & \in {}_0H_p^1((0, \infty), L_q(\Omega)^N) \cap L_p((0, \infty), H_q^2(\Omega)^N). \end{aligned}$$

In addition, the solution $(\rho, \mathbf{u})$ satisfies the estimate:

$$\begin{aligned} & \|e^{\delta_6 t} \partial_t \rho\|_{L_p((0, \infty), H_q^1(\Omega))} + \|e^{\delta_6 t} \rho\|_{L_p((0, \infty), H_q^3(\Omega))} \\ & + \|e^{\delta_6 t} \partial_t \mathbf{u}\|_{L_p((0, \infty), L_q(\Omega)^N)} + \|e^{\delta_6 t} \mathbf{u}\|_{L_p((0, \infty), H_q^2(\Omega)^N)} \\ & \leq C_{N,p,q,\delta_6} \left(\|e^{\delta_6 t} f\|_{L_p((0, \infty), H_q^1(\Omega))} + \|e^{\delta_6 t} \mathbf{g}\|_{L_p((0, \infty), L_q(\Omega)^N)} \right) \end{aligned}$$

for some positive constant C_{N,p,q,δ_6} .

4.3 Global existence theorem

Let $p, q \in (2, \infty) \times (N, \infty)$ and $(\rho_0, \mathbf{u}_0) \in D_{q,p}(\Omega)$ with $\|(\rho_0, \mathbf{u}_0)\|_{D_{q,p}(\Omega)} \leq \varepsilon_1$ for some positive number $\varepsilon_1 \in (0, 1)$ determined below, and let $(\rho_*, \mathbf{u}_*) = e^{\mathbf{A}t}(\rho_0, \mathbf{u}_0)$. Then, setting $\rho = \sigma + \rho_*$ and $\mathbf{u} = \mathbf{v} + \mathbf{u}_*$ in (4.1), we observe that

$$\left\{ \begin{aligned} & \partial_t \sigma + \rho_\infty \operatorname{div} \mathbf{v} = F(\sigma + \rho_*, \mathbf{v} + \mathbf{u}_*) \quad \text{in } \Omega \times (0, \infty), \\ & \partial_t \mathbf{v} - \operatorname{Div}(\mu_\infty \mathbf{D}(\mathbf{v})) + (\nu_\infty - \mu_\infty) \operatorname{div} \mathbf{v} \mathbf{I} + \kappa \Delta \sigma \mathbf{I} + \gamma_\infty \nabla \sigma \\ & \quad = \mathbf{G}(\sigma + \rho_*, \mathbf{v} + \mathbf{u}_*) \quad \text{in } \Omega \times (0, \infty), \\ & \mathbf{n} \cdot \nabla \sigma = 0, \quad \mathbf{v} = 0 \quad \text{on } \Gamma \times (0, \infty), \\ & (\sigma, \mathbf{v})|_{t=0} = (0, 0) \quad \text{in } \Omega. \end{aligned} \right. \quad (4.4)$$

To use the contraction mapping principle, we introduce the following notation:

- ${}_0Z_\infty := {}_0Z_\infty^1 \times {}_0Z_\infty^2$ with

$$\begin{aligned} {}_0Z_\infty^1 &= {}_0H_p^1((0, \infty), H_p^1(\Omega)) \cap L_p((0, \infty), H_q^3(\Omega)), \\ {}_0Z_\infty^2 &= {}_0H_p^1((0, \infty), L_q(\Omega)^N) \cap L_p((0, \infty), H_q^2(\Omega)^N). \end{aligned}$$

In addition, for $\delta > 0$,

$$\begin{aligned} \|(\rho, \mathbf{u})\|_{{}_0Z_\infty^\delta} &:= \|e^{\delta t} \partial_t \rho\|_{L_p((0, \infty), H_q^1(\Omega))} + \|e^{\delta t} \rho\|_{L_p((0, \infty), H_q^3(\Omega))} \\ &\quad + \|e^{\delta t} \partial_t \mathbf{u}\|_{L_p((0, \infty), L_q(\Omega)^N)} + \|e^{\delta t} \rho\|_{L_p((0, \infty), H_q^2(\Omega)^N)}. \end{aligned}$$

- For $\delta > 0$ and $r > 0$,

$$\begin{aligned} {}_0Z_\infty^\delta(r) &:= \left\{ (\tau, \mathbf{w}) \in {}_0Z_\infty : \|(\tau, \mathbf{w})\|_{{}_0Z_\infty^\delta} \leq r, \mathbf{w} = 0 \text{ on } \Gamma, \right. \\ &\quad \left. \frac{\rho_\infty}{4} \leq \tau(x, t) + \rho_*(x, t) + \rho_\infty \leq 4\rho_\infty \text{ for any } (x, t) \in \overline{\Omega} \times [0, \infty) \right\}. \end{aligned}$$

Let $(\tau, \mathbf{w}) \in {}_0Z_\infty^\delta(\varepsilon_2)$ for a suitable positive number $\delta \in (0, 1)$ and for $\varepsilon_2 \in (0, 1)$, and replace $(\sigma, \mathbf{v})$ by $(\tau, \mathbf{w})$ in the right-hand sides of (4.4). Then, by the maximal regularity with exponential stability stated in Lemma 4.7, we can define a contraction mapping $\Phi : {}_0Z_\infty^\delta(\varepsilon_2) \ni (\tau, \mathbf{w}) \mapsto (\sigma, \mathbf{v}) \in {}_0Z_\infty^\delta(\varepsilon_2)$ for sufficiently small positive numbers ε_1 and ε_2 . We thus obtain by the contraction mapping principle a global existence theorem in the maximal regularity class as follows:

Theorem 4.8. *Let $(p, q) \in (2, \infty) \times (N, \infty)$ and suppose that Assumption 4.1 holds. Then there exist positive numbers δ , ε_1 , and ε_2 such that, for any $(\rho_0, \mathbf{u}_0) \in \mathbf{D}_{q,p}(\Omega)$ with $\|(\rho_0, \mathbf{u}_0)\|_{\mathbf{D}_{q,p}(\Omega)} \leq \varepsilon_1$, the system (4.4) admits a unique global solution $(\sigma, \mathbf{v})$ in ${}_0Z_\infty^\delta(\varepsilon_2)$.*

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