

On Clarkson-Boas-type inequalities

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Introduction

In the context of uniform convexity, Clarkson's inequalities for L_p were proved in [2]. Boas [1] considered their generalization in parameters, which was 'completed' by Koskela [10]; we call these inequalities given by Boas and Koskela 'of Clarkson-Boas-type' (see also Kato [6] for their high-dimensional versions).

In this expository note, some recent results on Clarkson's and Clarkson-Boas-type inequalities are given especially in connection with type, cotype properties:

(i) By applying vector-valued interpolation to the Littlewood matrices as operators between $L_p(L_q)$ -valued $l_r^{2^n}$ -spaces, 'Clarkson's inequality' for $L_p(L_q)$ (and for some other Banach spaces as corollaries) is obtained in the high-dimensional setting (Kato & Miyazaki [7]): This might provide, in particular, one of the most concise proofs of classical Clarkson's inequalities (cf. Miyazaki & Kato [14]). The same argument, applied to the '*Rademacher matrices*', yields type inequalities with the best 'type constant' 1 for $L_p(L_q)$ (Kato, Miyazaki & Takahashi [8]). Our idea comes from Pietsch's work [15] (cf. [14], [11], [12]).

(ii) Further application of interpolation with decomposition argument of operators yields 'generalized Clarkson's inequalities' (high-dimensional Clarkson-Boas-type inequalities) for $L_p(L_q)$. This 'completes' and generalizes Boas' another inequality ([1]). Such high-dimensional versions of Clarkson-Boas-type inequalities are closely related with the Grothendieck inequality (see Tonge [16]). As a straightforward application the von Neumann-Jordan constant ([3]; cf. also [5]) for the spaces considered here is determined ([7]).

(iii) In general, Banach spaces with 'type or cotype constant' 1 are characterized as those satisfying Clarkson-Boas-type inequalities (Kato & Takahashi [9]).

1. Preliminaries. $p', q', \dots$ denote the conjugate exponents of $p, q, \dots$

1.1. Clarkson's inequalities (Clarkson [2]). (i) Let $1 < p \leq 2$. Then, for all f and g in L_p ,

$$(CI-1) \quad (\|f+g\|_p^{p'} + \|f-g\|_p^{p'})^{1/p'} \leq 2^{1/p'} (\|f\|_p^p + \|g\|_p^p)^{1/p},$$

$$(CI-2) \quad (\|f+g\|_p^p + \|f-g\|_p^p)^{1/p} \leq 2^{1/p} (\|f\|_p^{p'} + \|g\|_p^{p'})^{1/p'}.$$

(ii) Let $2 \leq p < \infty$. Then, for all f and g in L_p ,

$$(CI-3) \quad (\|f+g\|_p^p + \|f-g\|_p^p)^{1/p} \leq 2^{1/p} (\|f\|_p^{p'} + \|g\|_p^{p'})^{1/p'},$$

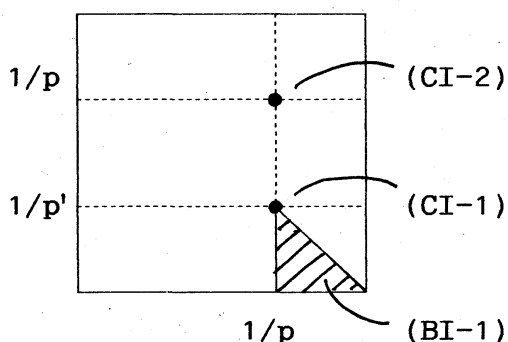
$$(CI-4) \quad (\|f+g\|_p^p + \|f-g\|_p^p)^{1/p} \leq 2^{1/p'} (\|f\|_p^p + \|g\|_p^p)^{1/p}.$$

1.2. Boas' inequality (Boas [1], Theorem 1). (i) Let $1 < r \leq p \leq s < \infty$ and $s' \leq r$. Then, for all f and g in L_p ,

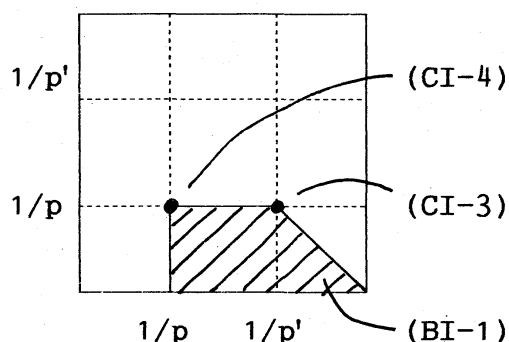
$$(BI-1) \quad (\|f+g\|_p^s + \|f-g\|_p^s)^{1/s} \leq 2^{1/r'} (\|f\|_p^r + \|g\|_p^r)^{1/r}.$$

(BI-1) includes (CI-1), (CI-3) and (CI-4). The situation is well expressed in the following unit squares with the coordinates $1/r$ (horizontal) and $1/s$ (vertical):

(i) the case $1 < p \leq 2$:



(ii) the case $2 < p < \infty$:



Let $A_n = (\varepsilon_{ij})$ be the Littlewood matrices, that is,

$$A_1 = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \quad A_{n+1} = \begin{pmatrix} A_n & A_n \\ A_n & -A_n \end{pmatrix} \quad (n = 1, 2, \dots).$$

1.3. Generalized Clarkson's inequalities (Kato [6]; cf. [10], [16], [11], [12], [14]). Let $1 < p < \infty$ and $1 \leq r, s \leq \infty$. Then, for an arbitrary positive integer n and all $f_1, f_2, \dots, f_{2^n} \in L_p$,

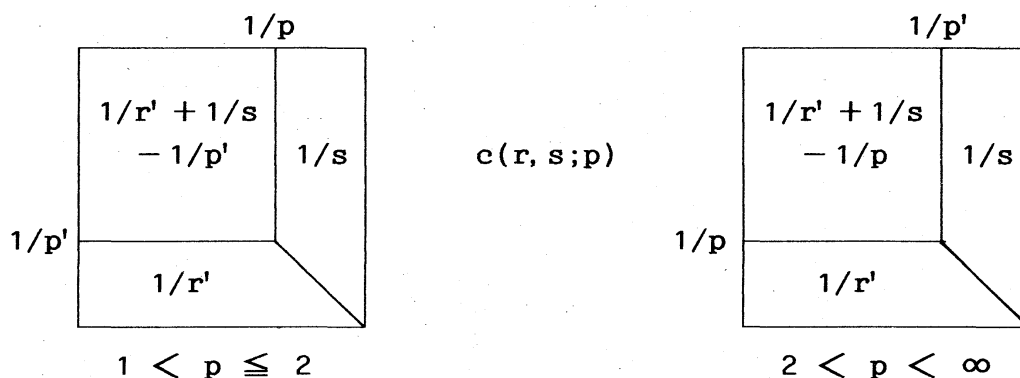
$$(GCI) \quad \left\{ \sum_{i=1}^{2^n} \left\| \sum_{j=1}^{2^n} \varepsilon_{ij} f_j \right\|_p^s \right\}^{1/s} \leq 2^{nc(r,s;p)} \left\{ \sum_{j=1}^{2^n} \|f_j\|_p^r \right\}^{1/r},$$

where

$$c(r, s; p) = \begin{cases} \frac{1}{r'} + \frac{1}{s} - \min\left(\frac{1}{p}, \frac{1}{p'}\right) & \text{if (i) } \min(p, p') \leq r \leq \infty, \\ & 1 \leq s \leq \max(p, p'), \\ \frac{1}{s} & \text{if (ii) } 1 \leq r \leq \min(p, p'), \\ & 1 \leq s \leq r', \\ \frac{1}{r'} & \text{if (iii) } s' \leq r \leq \infty, \\ & \max(p, p') \leq s \leq \infty. \end{cases}$$

Equality is attained in (GCI) for all $1 \leq r, s \leq \infty$. In other words,

$$\|A_n: l_r^{2^n}(L_p) \rightarrow l_s^{2^n}(L_p)\| = 2^{nc(r,s;p)}.$$



1. 4. **Remark.** A more generalized inequality including (GCI) is considered in Maligranda and Persson [11] (see also [12]).

1. 5. **Definition.** A Banach space X is said to be of (Rademacher)

type p ($1 \leq p \leq 2$) resp. of cotype q ($2 \leq q \leq \infty$) if X satisfies with some $M > 0$ and $1 \leq s < \infty$

$$(TpI) \quad \left\{ \int_0^1 \left\| \sum_{j=1}^n r_j(t) x_j \right\|^s dt \right\}^{1/s} \leq M \left\{ \sum_{j=1}^n \|x_j\|^p \right\}^{1/p}$$

resp.

$$(CqI) \quad \left\{ \sum_{j=1}^n \|x_j\|^q \right\}^{1/q} \leq M \left\{ \int_0^1 \left\| \sum_{j=1}^n r_j(t) x_j \right\|^s dt \right\}^{1/s}$$

for all finite system $\{x_j\}$ in X , where $r_j(t)$ are the Rademacher functions, i. e., $r_j(t) = \text{sgn}(\sin 2^j \pi t)$. (Usually, s is taken to be 1, 2; or p resp. q : Recall Khinchin-Kahane's inequality.) The smallest constant M in (TpI) resp. (CqI) are denoted by $T_{p(s)}(X)$ resp. $C_{q(s)}(X)$.

Now, we define *Rademacher matrices* $R_n = (r_{ij}^{(n)})$ inductively by

$$R_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \quad R_{n+1} = \begin{pmatrix} 1 & \vdots & 1 & R_n \\ -1 & \vdots & -1 & R_n \end{pmatrix} \quad (n = 1, 2, \dots).$$

Then, since

$$\begin{aligned} \left\{ \int_0^1 \left\| \sum_{j=1}^n r_j(t) f_j \right\|^s dt \right\}^{1/s} &= \left\{ \frac{1}{2^n} \sum_{\theta_j = \pm 1} \left\| \sum_{j=1}^n \theta_j f_j \right\|^s \right\}^{1/s} \\ &= \left\{ \frac{1}{2^n} \sum_{i=1}^{2^n} \left\| \sum_{j=1}^n r_{ij}^{(n)} f_j \right\|^s \right\}^{1/s}, \end{aligned}$$

type and cotype inequalities are described by the norms of the Rademacher matrices (see [8], Proposition 2.3).

2. Clarkson's inequality of 2^n -dimension, type, cotype constants for $L_p(L_q)$ and interpolation.

2.1. Theorem (Clarkson's inequality (CI-1) of 2^n -dimension for L_p and interpolation; Kato [6], Miyazaki & Kato [14]). Let

$1 \leq p \leq 2$. Then, for all $f_1, f_2, \dots, f_{2^n} \in L_p$,

$$(CI-1') \quad \left\{ \sum_{i=1}^{2^n} \left\| \sum_{j=1}^{2^n} \varepsilon_{ij} f_j \right\|_p^{p'} \right\}^{1/p'} \leq 2^{n/p'} \left\{ \sum_{j=1}^{2^n} \|f_j\|_p^p \right\}^{1/p}$$

or equivalently,

$$\|A_n : l_p^{2^n}(L_p) \rightarrow l_{p'}^{2^n}(L_p)\| \leq 2^{n/p'}.$$

Indeed, it is immediate to see that

$$M_1 = \|A_n : l_1^{2^n}(L_1) \rightarrow l_\infty^{2^n}(L_1)\| = 1 \quad (\text{the case } p = 1),$$

$$M_2 = \|A_n : l_2^{2^n}(L_2) \rightarrow l_2^{2^n}(L_2)\| = 2^{n/2} \quad (\text{the case } p = 2).$$

Put $\theta = 2/p'$ ($0 < \theta < 1$), where $1 < p < 2$. Then, by interpolation,

$$\|A_n : l_p^{2^n}(L_p) \rightarrow l_{p'}^{2^n}(L_p)\| \leq M_1^{1-\theta} M_2^\theta \leq 2^{n/p'},$$

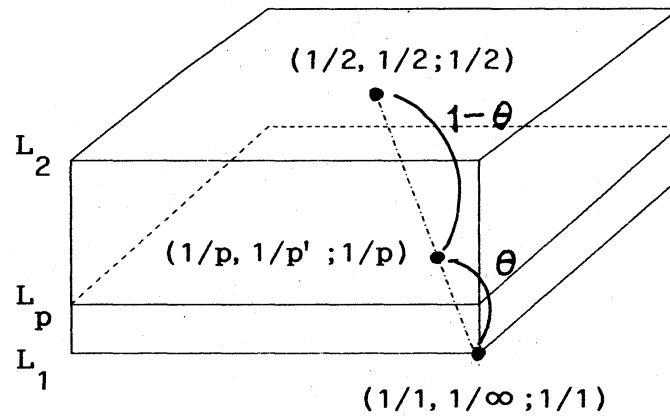
as is desired ([14]). (Note that

$$(l_1^{2^n}(L_1), l_2^{2^n}(L_2))_{[\theta]} = l_p^{2^n}(L_p) \quad \text{with equal norms,}$$

$$(l_\infty^{2^n}(L_1), l_2^{2^n}(L_2))_{[\theta]} = l_{p'}^{2^n}(L_p) \quad \text{with equal norms}$$

since $(1 - \theta)/1 + \theta/2 = 1/p$ and $(1 - \theta)/\infty + \theta/2 = 1/p'$.)

Note. The following figure may visually explain what we have done just above:



Note $(1/p, 1/p'; 1/p) = (1 - \theta)(1/1, 1/\infty; 1/1) + \theta(1/2, 1/2; 1/2)$,
or figuratively, $(1/p, 1/p'; L_p) = (1 - \theta)(1/1, 1/\infty; L_1) + \theta(1/2, 1/2; L_2)$, where $\theta = 2/p'$.

2.2. Theorem (Clarkson's inequality (CI-1) of 2^n -dimension for $L_p(L_q)$ and interpolation; Kato & Miyazaki [7]). Let $1 < p, q < \infty$ and $t = \min\{p, q, p', q'\}$. Then, for an arbitrary positive integer n and all $f_1, f_2, \dots, f_{2^n}$ in $L_p(L_q) = L_p(\mu; L_q(\nu))$,

$$(CI-1'') \quad \left\{ \sum_{i=1}^{2^n} \left\| \sum_{j=1}^{2^n} \varepsilon_{ij} f_j \right\|_{p(q)}^{t'} \right\}^{1/t'} \leq 2^{n/t'} \left\{ \sum_{j=1}^{2^n} \|f_j\|_{p(q)}^t \right\}^{1/t},$$

or equivalently

$$\|A_n : l_t^{2^n}(L_p(L_q)) \rightarrow l_{t'}^{2^n}(L_p(L_q))\| \leq 2^{n/t'}.$$

In this case, more skillful use of interpolation is required (see [7]). By applying the same argument to 'Rademacher matrices', 'the type inequality' for $L_p(L_q)$ is obtained:

2.3. Theorem (Type inequality for $L_p(L_q)$; Kato, Miyazaki & Takahashi [8]). Let $1 < p, q < \infty$ and $t = \min\{p, q, p', q'\}$. Then,

$$\|R_n : l_t^n(L_p(L_q)) \rightarrow l_{t'}^n(L_p(L_q))\| \leq 2^{n/t'} :$$

In other words, for all $f_1, f_2, \dots, f_n$ in $L_p(L_q)$,

$$\left\{ \frac{1}{2^n} \sum_{i=1}^{2^n} \left\| \sum_{j=1}^n r_{ij}^{(n)} f_j \right\|_{p(q)}^{t'} \right\}^{1/t'} \leq \left\{ \sum_{j=1}^n \|f_j\|_{p(q)}^t \right\}^{1/t};$$

that is, $L_p(L_q)$ is of type t and its 'type t constant',

$T_{t(t')}(L_p(L_q))$, is 1. Here, t' in the left side may be replaced by any s with $1 \leq s \leq t'$, i.e., $T_{t(s)}(X) = 1.$

Here, the constants t and t' are optimal as far as 'the type constant' is 1.

2.4. Remark. Theorem 2.3 with duality argument yields analogous results on cotype inequalities for $L_p(L_q)$ ([8]). Type and cotype inequalities for Sobolev spaces given in Milman [13] and Cobos [4] are immediately derived as its corollaries.

3. Generalized Clarkson's inequalities for $L_p(L_q)$

3.1. Theorem (Generalized Clarkson's inequalities for $L_p(L_q)$;

Kato & Miyazaki [7]). Let $1 < p, q < \infty$ and $1 \leq r, s \leq \infty$.

Then, for an arbitrary positive integer n and all $f_1, f_2, \dots, f_{2^n}$ in $L_p(L_q)$,

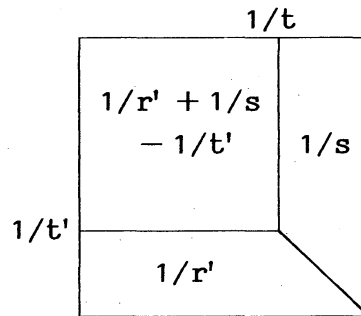
$$(GCI') \quad \left\{ \sum_{i=1}^{2^n} \left\| \sum_{j=1}^{2^n} \varepsilon_{ij} f_j \right\|_{p(q)}^s \right\}^{1/s} \leq 2^{nc(r,s;p,q)} \left\{ \sum_{j=1}^{2^n} \|f_j\|_{p(q)}^r \right\}^{1/r},$$

where, letting $t = \min\{p, q, p', q'\}$ and $1/t + 1/t' = 1$,

$$c(r, s; p, q) = \begin{cases} \frac{1}{r'} + \frac{1}{s} - \frac{1}{t'} & \text{if (i) } t \leq r \leq \infty, \\ & 1 \leq s \leq t', \\ \frac{1}{s} & \text{if (ii) } 1 \leq r \leq t, \\ & 1 \leq s \leq r', \\ \frac{1}{r'} & \text{if (iii) } s' \leq r \leq \infty, \\ & t' \leq s \leq \infty. \end{cases}$$

Equality is attained in (GCI') for all $1 \leq r, s \leq \infty$; and hence

$$\|A_n: l_r^{2^n}(L_p(L_q)) \rightarrow l_s^{2^n}(L_p(L_q))\| = 2^{nc(r,s;p,q)}.$$



$c(r, s; p, q)$

3.2. Remarks. (i) (GCI') covers all the 'rest cases' of Boas' inequality for $L_p(L_q)$ ([1], Theorem 2): Let $1 < p, q < \infty$. Let $1 < r \leq p \leq s < \infty$ and let $s' \leq r \leq s\{\min(q, q') - 1\}$. Then, for all f and g in $L_p(L_q)$,

$$(BI-2) \quad (\|f+g\|_{p(q)}^s + \|f-g\|_{p(q)}^s)^{1/s} \leq 2^{1/r'} (\|f\|_{p(q)}^r + \|g\|_{p(q)}^r)^{1/r}$$

(ii) (GCI) for L_p , $l_p(L_p)$, and $W_p^k(\Omega)$ are corollaries of (GCI').

The last one includes Milman's ([13]) and Cobos' result ([4]).

The von Neumann-Jordan constant for a Banach space X ([3]), $C_{NJ}(X)$, is the smallest constant C satisfying

$$\frac{1}{C} \leq \frac{\|x+y\|^2 + \|x-y\|^2}{2(\|x\|^2 + \|y\|^2)} \leq C$$

for all x and y in X with $\|x\|^2 + \|y\|^2 \neq 0$. For any Banach space X , $1 \leq C_{NJ}(X) \leq 2$; and it is a Hilbert space if and only if $C_{NJ}(X) = 1$ ([5]). For L_p , $C_{NJ}(L_p) = 2^{\max(1/p, 1/p') - 1}$ ([3]).

3.3. Corollary ([7]). Let $1 \leq p, q \leq \infty$ and let $t = \min\{p, q, p', q'\}$. Then,

$$C_{NJ}(L_p(L_q)) = C_{NJ}(l_p(L_q)) = 2^{\max(1/t, 1/t') - 1}$$

and

$$C_{NJ}(l_p(L_p)) = C_{NJ}(W_p^k(\Omega)) = 2^{\max(1/p, 1/p') - 1}.$$

4. Banach spaces satisfying Clarkson's inequalities

4.1. Theorem (Kato & Takahashi [9]). Let X be a Banach space.

(i) Let $1 < p \leq 2$ and $p \leq s \leq p'$. Then, X satisfies the Clarkson-Boas-type inequality

$$(CBI-1) \quad (\|x+y\|^s + \|x-y\|^s)^{1/s} \leq 2^{1/s} (\|x\|^p + \|y\|^p)^{1/p}$$

if and only if X is of type p and $T_{p(s)}(X) = 1$. In particular, X satisfies Clarkson's inequalities

$$(CI-1^*) \quad (\|x+y\|^{p'} + \|x-y\|^{p'})^{1/p'} \leq 2^{1/p'} (\|x\|^p + \|y\|^p)^{1/p}$$

resp.

$$(CI-2^*) \quad (\|x+y\|^p + \|x-y\|^p)^{1/p} \leq 2^{1/p} (\|x\|^{p'} + \|y\|^{p'})^{1/p'}$$

if and only if X is of type p , and $T_{p(p')}(X) = 1$ resp.

$$T_{p(p)}(X) = 1.$$

(ii) Let $2 \leq q < \infty$ and $q' \leq s \leq q$. Then, X satisfies the Clarkson-Boas-type inequality

$$(CBI-2) \quad (\|x+y\|^q + \|x-y\|^q)^{1/q} \leq 2^{1/s'} (\|x\|^s + \|y\|^s)^{1/s}$$

if and only if X is of cotype q and $C_{q(s)}(X) = 1$. In particular, X satisfies Clarkson's inequalities

$$(CI-3^*) \quad (\|x+y\|^q + \|x-y\|^q)^{1/q} \leq 2^{1/q} (\|x\|^{q'} + \|y\|^{q'})^{1/q'}$$

resp.

$$(CI-4^*) \quad (\|x+y\|^q + \|x-y\|^q)^{1/q} \leq 2^{1/q'} (\|x\|^q + \|y\|^q)^{1/q}$$

if and only if X is of cotype q , and $\underline{C}_{q(q')}(X) = 1$ resp.

$$\underline{C}_{q(q)}(X) = 1.$$

Note. The above theorem implies in particular that the notions of G_0 - and G_n -Fourier type for a Banach space in Milman [13] are equivalent. (See [9] for some other related results.)

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