

Numerical Radius Norm for Module Maps モジュール写像の数域半径ノルム

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In the present note we will explain a couple of results related to numerical radius norm for module maps on C^* -algebras.

(1) One is to extend the Ando-Okubo's theorem concerning Schur multipliers in the infinite dimensional setting.

(2) The other is to characterize a completely bounded module map. This is joint with Masaru Nagisa (Chiba University).

1. Schur products and Schur multipliers

Let $M_n(\mathbb{C})$ be the $n \times n$ matrix algebra over $\mathbb{C}$. For $a = [a_{ij}], b = [b_{ij}] \in M_n(\mathbb{C})$, the *Schur product* $\circ$ is defined by

$$a \circ b = [a_{ij}b_{ij}].$$

The *Schur multiplier* $S_a : M_n(\mathbb{C}) \rightarrow M_n(\mathbb{C})$ for $a \in M_n(\mathbb{C})$ is defined by $S_a(x) = a \circ x$. The *Schur norm* for $a \in M_n(\mathbb{C})$ is defined by

$$\|a\|_s = \|S_a\| = \sup\{\|a \circ x\| \mid \|x\| = 1\}.$$

The following is due to Haagerup in which the Schur multiplier appears naturally in operator algebras.

Example [Haagerup, 5] Let G be a locally compact group, $C(G)$ the continuous functions and $R(G)$ the group von-Neumann algebra. For $\varphi \in C(G)$, if

$$M_\varphi : R(G) \ni \lambda_g \mapsto \varphi(g)\lambda_g \in R(G)$$

is normal (σ -weak - σ -weak continuous), then

$$\|M_\varphi\|_{cb} = \sup\{\|[\varphi(g_j^{-1}g_i)]\|_s \mid g_i \in G, i \leq n, n \in \mathbb{N}\}.$$

The next result is unpublished.

Theorem A[Haagerup, 5] Let $a = [a_{ij}] \in M_n(\mathbb{C})$. Then the following are equivalent:

- 1) $\|S_a\| \leq 1$.
- 2) There are $0 \leq r_1, r_2 \in M_n(\mathbb{C})$ such that

$$\begin{bmatrix} r_1 & a \\ a^* & r_2 \end{bmatrix} \geq 0, \quad r_1 \circ I \leq I \text{ and } r_2 \circ I \leq I.$$

- 3) a has a factorization $a = b^*c$ such that $b^*b \circ I \leq I$, $c^*c \circ I \leq I$.
- 4) There are vectors $\{\xi_i\}, \{\eta_i\} \subset \ell_n^2$, ($i = 1, \dots, n$) such that $\|\xi_i\|, \|\eta_i\| \leq 1$ and $a_{ij} = (\xi_j | \eta_i)$.

We consider *the numerical radius norm* $w(\cdot)$ on $\mathbb{B}(\mathcal{H})$:

$$w(a) = \sup_{\xi \neq 0} \frac{|(a\xi | \xi)|}{\|\xi\|^2}.$$

It is easy to see that $w(a) \leq \|a\| \leq 2w(a)$.

We also consider the induced norm for S_a with respect to the numerical radius norm that will be denoted by $\|S_a\|_w$:

$$\|S_a\|_w \equiv \sup_{x \neq 0} \frac{w(a \circ x)}{w(x)}.$$

The following is due to Ando and Okubo which looks similar to Theorem A but each condition is finer than the above.

Theorem B[Ando-Okubo, 2] Let $a = [a_{ij}] \in M_n(\mathbb{C})$. Then the following are equivalent.

- 1) $_w$ $\|S_a\|_w \leq 1$.

2)_w There is a $0 \leq r \in M_n(\mathbb{C})$ such that

$$\begin{bmatrix} r & a \\ a^* & r \end{bmatrix} \geq 0, \quad \text{and } r \circ I \leq I.$$

3)_w a has a factorization $a = b^*db$ such that $b^*b \circ I \leq I$, $d^*d \leq I$.

4)_w There are vectors $\{\xi_i\} \subset \ell_n^2$, ($i = 1, \dots, n$) and a contraction $d \in M_n(\mathbb{C})$ such that $\|\xi_i\| \leq 1$ and $a_{ij} = (d\xi_j | \xi_i)$.

Remark The Haagerup's theorem is derived from the Ando-Okubo's theorem, because

$$\|S_a\| = \|S \begin{bmatrix} 0 & a \\ 0 & 0 \end{bmatrix}\|_w.$$

To show 1) $\Rightarrow$ 2), $\|S_a\| \leq 1$ implies

$$\|S \begin{bmatrix} 0 & a \\ 0 & 0 \end{bmatrix}\|_w \leq 1.$$

By the implication 1)_w $\Rightarrow$ 2)_w, there exists

$0 \leq r = \begin{bmatrix} r_{11} & r_{21} \\ r_{12} & r_{22} \end{bmatrix} \in M_n(\mathbb{C})$ such that

$$\begin{bmatrix} r_{11} & r_{21} & 0 & a \\ r_{12} & r_{22} & 0 & 0 \\ 0 & 0 & r_{11} & r_{21} \\ a^* & 0 & r_{12} & r_{22} \end{bmatrix} \geq 0.$$

This implies that $\begin{bmatrix} r_{11} & a \\ a^* & r_{22} \end{bmatrix} \geq 0$.

2. Module maps on operator systems

Let $\varphi : M_n(\mathbb{C}) \longrightarrow M_n(\mathbb{C})$. Then the followings are equivalent:

1) There exists $a \in M_n(\mathbb{C})$ such that $\varphi = S_a$.

$$2) \quad \varphi(\lambda x \mu) = \lambda \varphi(x) \mu \quad \text{for } x \in M_n(\mathbb{C}),$$

$$\lambda = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix} \quad \text{and} \quad \mu = \begin{bmatrix} \mu_1 & & 0 \\ & \ddots & \\ 0 & & \mu_n \end{bmatrix} \in M_n(\mathbb{C})$$

(i.e. ℓ_n^∞ -module map)

Let A be a C^* -algebra and V an operator system in $\mathbb{B}(\mathcal{H})$, i.e., V is a self-adjoint subspace in $\mathbb{B}(\mathcal{H})$ with the identity. Let T be a bounded linear map from $(V, \|\cdot\|)$ to $(\mathbb{B}(\mathcal{H}), \|\cdot\|)$. We denote by $T \otimes \text{id}_n$ the linear map

$$\mathbf{M}_n(V) \ni [x_{ij}] \mapsto [T(x_{ij})] \in \mathbf{M}_n(\mathbb{B}(\mathcal{H})).$$

If $\sup_n \|T \otimes \text{id}_n\|$ is bounded, then we say T is completely bounded and denote the supremum by $\|T\|_{cb}$. If $T \otimes \text{id}_n$ is positive for all n , then we say T is completely positive.

We denote by $\|T\|_w$ the operator norm of T viewed as a bounded linear map from $(V, w(\cdot))$ to $(\mathbb{B}(\mathcal{H}), w(\cdot))$, i.e.,

$$\|T\|_w = \sup\{w(T(x)) \mid w(x) \leq 1, x \in V\}.$$

Every completely bounded map from an operator space to $\mathbb{B}(\mathcal{H})$ is also completely bounded with respect to numerical radius norm. We use the following notation:

$$\|T\|_{wcb} = \sup_{n \in \mathbb{N}} \|T \otimes \text{id}_n\|_w.$$

We call that an action of A on $\mathcal{H}$ is locally cyclic if, for any n and $\xi_1, \xi_2, \dots, \xi_n \in \mathcal{H}$, there exists a vector $\eta \in \mathcal{H}$ such that

$$\xi_i \in \text{the norm closure of } \{a\eta \mid a \in A\}.$$

We remark that, for $x = (x_{ij}) \in \mathbf{M}_n(V) \subset \mathbb{B}(\mathcal{H}^n)$ and $a = (a_{kl}) \in M_{nm}(A)$, we can see

$$a^* \cdot x \cdot a = \left(\sum_{k,l} a_{ki}^* \cdot x_{kl} \cdot a_{lj} \right) \in \mathbf{M}_m(V) \subset \mathbb{B}(\mathcal{H}^m)$$

and we get

$$w(a^* \cdot x \cdot a) \leq \|a\|^2 w(x).$$

The condition (locally cyclic) implies that $\|\cdot\|_w = \|\cdot\|_{wcb}$.

Proposition 1 *Let A be a unital C^* -algebra, V an A -bimodule operator system and T a bounded A -bimodule map from V to $\mathbb{B}(\mathcal{H})$. If the action of A on $\mathcal{H}$ is locally cyclic, then we have*

$$\|T\|_w = \|T\|_{wcb}.$$

Let A be a unital C^* -algebra and V an A -bimodule operator system in $\mathbb{B}(\mathcal{H})$. Set

$$N = \left\{ \begin{pmatrix} x & y \\ z & w \end{pmatrix} \mid x, w \in \mathbb{B}(\mathcal{H}), y, z \in V \right\}.$$

Then N is an A -bimodule operator system by the action

$$a \cdot \begin{pmatrix} x & y \\ z & w \end{pmatrix} \cdot b = \begin{pmatrix} axb & ayb \\ azb & awb \end{pmatrix} \quad \text{for } a, b \in A.$$

Theorem 2 *Let A be a unital C^* -algebra, V an A -bimodule operator system in $\mathbb{B}(\mathcal{H})$ and T a completely bounded A -bimodule map from V to $\mathbb{B}(\mathcal{H})$. Then we have*

$$\begin{aligned} & \|T\|_{wcb} \\ &= \inf \left\{ \|S\| \mid \begin{pmatrix} S & T \\ T^* & S \end{pmatrix} : N \longrightarrow \mathbf{M}_2(\mathbb{B}(\mathcal{H})) \right. \\ & \qquad \qquad \qquad \left. A\text{-bimodule completely positive} \right\} \\ &= \inf \left\{ \|S\| \mid \begin{pmatrix} S & T \\ T^* & S \end{pmatrix} : N \longrightarrow \mathbf{M}_2(\mathbb{B}(\mathcal{H})) \right. \\ & \qquad \qquad \qquad \left. \text{completely positive} \right\} \end{aligned}$$

where

$$\begin{pmatrix} S & T \\ T^* & S \end{pmatrix} \begin{pmatrix} x & y \\ z & w \end{pmatrix} = \begin{pmatrix} S(x) & T(y) \\ T(z^*)^* & S(w) \end{pmatrix},$$

$$\text{for } \begin{pmatrix} x & y \\ z & w \end{pmatrix} \in N = \left\{ \begin{pmatrix} x & y \\ z & w \end{pmatrix} \mid x, w \in \mathbb{B}(\mathcal{H}), y, z \in V \right\}.$$

To show this, we need the Wittstock's Hahn Banach type theorem for completely bounded maps. The key operator space N_0 is that

$$N_0 = \left\{ \begin{bmatrix} a+x & y \\ z & a-x \end{bmatrix} \mid a \in A, x \in \mathbb{B}(\mathcal{H}), y, z \in V \right\}$$

and we consider

$$\varphi_0 \left(\begin{bmatrix} a+x & y \\ z & a-x \end{bmatrix} \right) = a + \frac{1}{2}(T(y) + T(z^*)^*).$$

We can see that φ_0 is a completely positive A -bimodule map. Using the Wittstock's Hahn-Banach theorem [13], we get the unital completely positive A -bimodule map φ from N to $\mathbb{B}(\mathcal{H})$ which is an extension of φ_0 .

Set

$$S(x) = 2\varphi \left(\begin{bmatrix} x & 0 \\ 0 & 0 \end{bmatrix} \right)$$

Then we have a desired map.

Remark V. I. Paulsen and C. Y. Suen [9] introduced the norm $|||T|||$ for a completely bounded map T from a C^* -algebra A to $\mathbb{B}(\mathcal{H})$ as follows:

$$|||T||| = \inf \left\{ \|S\| \mid \begin{pmatrix} S & T \\ T^* & S \end{pmatrix} : \mathbf{M}_2(A) \longrightarrow \mathbf{M}_2(\mathbb{B}(\mathcal{H})) \right. \\ \left. \text{completely positive} \right\}.$$

By the injectivity of $\mathbb{B}(\mathcal{H})$, we can get

$$|||T||| = \|T\|_{wcb}.$$

We set $\mathcal{A}^{(*)} = \{x^* \in \mathbb{B}(\mathcal{H}) \mid x \in \mathcal{A}\}$.

Theorem 3 *Let $\mathcal{A}$ be a norm closed unital algebra on $\mathcal{H}$ and T a completely bounded left $\mathcal{A}^{(*)}$ -, right $\mathcal{A}$ -module map from $\mathbb{K}(\mathcal{H})$ to $\mathbb{B}(\mathcal{H})$. Then there exist $t = (t_{ij}) \in \mathbb{B}(\ell^2(I))$ and $\{v_i \mid i \in I\} \subset \mathcal{A}'$ such that*

$$\|t\| = \|T\|_{wcb}, \quad \sum_{i \in I} v_i^* v_i \leq 1$$

$$T(x) = \sum_{i,j \in I} v_i^* t_{ij} x v_j \quad (x \in \mathbb{K}(\mathcal{H})).$$

To see this, we may regard T as a normal completely bounded $\mathcal{A}^{(*)} - \mathcal{A}$ -module map on $\mathbb{B}(\mathcal{H})$. Then there exist a $*$ -representation π of $\mathbb{K}(\mathcal{H})$ on a Hilbert space $\mathcal{K}$, an isometry $w : \mathcal{H} \rightarrow \mathcal{K}$ and an operator $s \in \pi(\mathbb{K}(\mathcal{H}))'$ such that

$$\|T\|_{wcb} = \|s\|, \quad T(\cdot) = w^* s \pi(\cdot) w.$$

Since all irreducible representations of $\mathbb{K}(\mathcal{H})$ are unitarily equivalent to the identity representation, we may assume that

$$\begin{aligned} \mathcal{K} &= \mathcal{H} \otimes \ell^2(I), \quad \pi(x) = x \otimes 1, \quad s = (s_{ij} 1_{\mathbb{B}(\mathcal{H})})_{i,j \in I} \\ (s_{ij} &\in \mathbb{C}), \quad w = (w_i)_{i \in I} \in \mathbb{B}(\mathcal{H}, \mathcal{H} \otimes \ell^2(I)), \\ T(x) &= w^* s (x \otimes 1) w = \sum_{i,j \in I} w_i^* s_{ij} x w_j \text{ for } x \in \mathbb{K}(\mathcal{H}). \end{aligned}$$

We can replace $\{w_i\}$ by $\{v_i\} \subset \mathcal{A}'$.

Corollary 4 [Smith, 11] *Let $\mathcal{A}$ and $\mathcal{B}$ be norm closed unital algebras on $\mathcal{H}$ and T a completely bounded left $\mathcal{A}$ - right $\mathcal{B}$ -module map from $\mathbb{K}(\mathcal{H})$ to $\mathbb{B}(\mathcal{H})$. Then there exist $\{a_i \mid i \in I\} \subset \mathcal{A}'$ and $\{b_i \mid i \in I\} \subset \mathcal{B}'$ such that*

$$T(x) = \sum_{i \in I} a_i x b_i, \quad \left\| \sum_{i \in I} a_i a_i^* \right\| \left\| \sum_{i \in I} b_i^* b_i \right\| = \|T\|_{cb}^2$$

for $x \in \mathbb{K}(\mathcal{H})$.

We define the left action of $\mathcal{A} \oplus \mathcal{B}^{(*)}$ and the right action of $\mathcal{A}^{(*)} \oplus \mathcal{B}$ on $\mathbf{M}_\infty(\mathbb{K}(\mathcal{H}))$ and the left $\mathcal{A} \oplus \mathcal{B}^{(*)}$ -, right $\mathcal{A}^{(*)} \oplus \mathcal{B}$ -module completely bounded

map $\tilde{T}$ from $\mathbf{M}_2(\mathbf{K}(\mathcal{H}))$ to $\mathbf{M}_2(\mathbf{B}(\mathcal{H}))$ as follows:

$$(a_1 \oplus b_1^*) \begin{pmatrix} x & y \\ z & w \end{pmatrix} (a_2^* \oplus b_2) = \begin{pmatrix} a_1 x a_2^* & a_1 y b_2 \\ b_1^* z a_2^* & b_1^* w b_2 \end{pmatrix}$$

$$\tilde{T} \left(\begin{pmatrix} x & y \\ z & w \end{pmatrix} \right) = \begin{pmatrix} 0 & T(y) \\ 0 & 0 \end{pmatrix}$$

where $x, y, z, w \in \mathbf{K}(\mathcal{H})$ and $a_1, a_2 \in \mathcal{A}$, $b_1, b_2 \in \mathcal{B}$. Then we show that $\|\tilde{T}\|_{wcb} = \|T\|_{cb}$. Apply the previous theorem for $\tilde{T}$. Then we have the desired form for T .

For a Hilbert space $\mathcal{H}$, we choose a completely orthonormal system $\{e_i \mid i \in I\}$. We denote by ℓ^∞ the maximal abelian subalgebra of $\mathbf{B}(\mathcal{H})$ generated by $\{e_i \otimes e_i \mid i \in I\}$, where $(e_i \otimes e_j)(\xi) = (\xi|e_j)e_i$ for $\xi \in \mathcal{H}$. Let T be a bounded ℓ^∞ -bimodule map from $\mathbf{K}(\mathcal{H})$ to $\mathbf{B}(\mathcal{H})$. By the module property of T , we have the $I \times I$ -matrix $a = (a_{ij})$ over $\mathbb{C}$ such that

$$T(e_i \otimes e_j) = a_{ij}(e_i \otimes e_j).$$

Since the set $\{a_{ij}\}_{i,j \in I}$ is bounded, we can define the bounded linear operator a_T from ℓ^1 to ℓ^∞ given by

$$a_T((\lambda_j)_{j \in I}) = \left(\sum_{j \in I} a_{ij} \lambda_j \right)_{i \in I} \quad \text{for } (\lambda_j)_{j \in I} \in \ell^1.$$

We will extend the Ando-Okubo's theorem.

Theorem 4 *Let T be an ℓ^∞ -bimodule bounded linear map from $\mathbf{K}(\mathcal{H})$ to $\mathbf{B}(\mathcal{H})$. Then the following are equivalent:*

- (1) $\|T\|_w \leq 1$.
- (2) $\|T\|_{wcb} \leq 1$.
- (3) *There exists a completely positive contraction S from $\mathbf{K}(\mathcal{H})$ to $\mathbf{B}(\mathcal{H})$*

such that $\begin{pmatrix} S & T \\ T^ & S \end{pmatrix} : \mathbf{M}_2(\mathbf{K}(\mathcal{H})) \rightarrow \mathbf{M}_2(\mathbf{B}(\mathcal{H}))$ is ℓ^∞ -bimodule completely positive.*

- (4) *There exist a bounded linear operator v from ℓ^1 to ℓ^2 and $b \in \mathbf{B}(\ell^2)$ such that $a_T = v^* b v$ and $\|v\|^2 \|b\| \leq 1$.*
- (5) *There exist $\{\xi_i \mid i \in I\} \subset \mathcal{H}$ and $b \in \mathbf{B}(\mathcal{H})$ such that*

$$\|\xi_i\| \leq 1, \|b\| \leq 1, a_{ij} = (b\xi_j | \xi_i).$$

Note added in proof.

C.-Y. Suen has already shown Theorem 2 in a similar setting in the paper: Induced completely bounded norms and inflated Schur product, Acta Sci. Math.(Szeged) 66, (2000), 273-286.

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