

Satellite Links with Brunnian Properties

Taizo Kanenobu (金信 泰造)

We shall work throughout in the PL category. An n -link with m components is a locally flat oriented submanifold of the oriented $(n+2)$ -sphere S^{n+2} homeomorphic to m disjoint copies of S^n . An n -knot is an n -link with one component. A trivial n -link is one whose components bound disjoint locally flat $(n+1)$ -disks B^{n+1} in S^{n+2} . An n -link L is splittable if there exists an $(n+2)$ -disk B^{n+2} in S^{n+2} satisfying $L \cap B^{n+2} \neq \emptyset$, $L \cap \partial B^{n+2} = \emptyset$, and $L \cap (S^{n+2} - B^{n+2}) \neq \emptyset$, where ∂B^{n+2} is the boundary of B^{n+2} .

Let $\mathcal{A}$ be the family of those subsets S of $I = \{1, 2, \dots, m\}$ for which the sublink $L_S = \bigcup_{i \in S} L_i$ of an n -link $L = L_1 \cup L_2 \cup \dots \cup L_m$ does not split. Then we call L has the Brunnian property of type $\mathcal{A}$. For the convenience we assume that $\emptyset, \{i\} \notin \mathcal{A}$ for all $i \in I$. In this family of subsets $\mathcal{A}$, the following condition must be satisfied:

(*) If $S, T \in \mathcal{A}$ and $S \cap T \neq \emptyset$, then $S \cup T \in \mathcal{A}$.

Conversely we prove:

Theorem. Suppose $n \geq 1$ and $m \geq 2$. Let $\mathcal{A}$ be a family of subsets of I satisfying the condition (*). Then there exists an n -link with m components with the Brunnian property of type $\mathcal{A}$.

This theorem is previously obtained by H. Debrunner [3] for $n \geq 2$, using a ribbon n -link. Our example is a satellite link, which is defined in a similar way that a satellite knot is defined in [11, pp.110-113] and [7]. As partial results, the following are known: Let $\mathcal{B}_k$, $2 \leq k \leq m$, be the family of all the subsets of I consisting of k or more elements. An n -link with the Brunnian property of type $\mathcal{B}_k$ is one such that no sublink with k or more components is splittable but every sublink with less than k components is completely splittable. For $n = 1$ and $k = m$, such links were given by H. Brunn [1], see also [11, pp.67-69]; for $n = 1$ and $k \leq m$, by H. Debrunner [2]. R. H. Fox [3, problem 38] asked whether examples existed for $n = 2$ and $k \geq m$, and T. Yanagawa [13] answered by constructing such examples using ribbon 2-link. For $n \geq 1$ and $k = m$, see also [11, pp.197-199].

A group G is indecomposable (relative to free product) if $G = A * B$ implies $A = 1$ or $B = 1$. To prove that a link is unsplittable, we use the following fact, cf. [10, Theorem 27.1]:

Proposition. An n -link L is nonsplittable if its group $\pi_1(S^{n+2} - L)$ is indecomposable. If $n = 1$, then the converse is valid.

Hence any 1-knot group is indecomposable. Moreover an n -knot group with a nontrivial center ([5]) is indecomposable ([9, p.195]).

Proof of Theorem. Let $O = O_1 \cup O_2 \cup \dots \cup O_m$ be a trivial n -link with m components. Let $x_i \in \pi_1(S^{n+2} - O)$ be a meridian of O_i . Let $S = \{i_1, i_2, \dots, i_k\}$ I , $1 \leq i_1 < i_2 < \dots < i_k \leq m$. We write F_S for the free group with basis $\{x_{i_1}, x_{i_2}, \dots, x_{i_k}\}$; thus $\pi_1(S^{n+2} - O_S) = F_S$. Let $\alpha_S = [x_{i_1}, x_{i_2}, \dots, x_{i_k}]$, where $[x_{i_1}, x_{i_2}] = x_{i_1}^{-1} x_{i_2}^{-1} x_{i_1} x_{i_2}$ and $[x_{i_1}, \dots, x_{i_{j-1}}, x_{i_j}] = [[x_{i_1}, \dots, x_{i_{j-1}}], x_{i_j}]$.

Let $\mathcal{O} = \{S_1, S_2, \dots, S_r\}$ and let $\alpha_i = \alpha_{S_i}$. If $n = 1$, then α_i can be represented by mutually disjoint simple closed curves ℓ_i in $S^3 - N(O)$, where $N(O)$ is a tubular neighborhood of O in S^3 , such that the $(\#S_i + 1)$ -component link $O_{S_i} \cup \ell_i$ has the Brunnian property of type $\mathcal{B}_{\#S_i+1}$ and that the r -component link $\ell_1 \cup \ell_2 \cup \dots \cup \ell_r$ is trivial. We can always find such ℓ_i as illustrated in the figure, which consists of four circles $O_1 \cup O_2 \cup O_3 \cup O_4$ and three curves $\ell_1 \cup \ell_2 \cup \ell_3$, where ℓ_1, ℓ_2 and ℓ_3 represent $[x_1, x_2, x_3]$, $[x_3, x_4]$ and $[x_2, x_4]$, respectively, see [7, p.67]. If $n \geq 2$, we can also find mutually disjoint simple closed curves ℓ_i in $S^{n+2} - N(O)$; each α_i is represented by a unique isotopy class of ℓ_i ([6, Corollary 8.1.2 and Theorem 10.1]).

Hence in any case, if let V_i be disjoint tubular neighborhoods of the ℓ_i in $S^{n+2} - N(O)$, then $S^{n+2} - \text{int } V_i$ is homeomorphic to $S^n \times D^2$, where $\text{int } V_i$ is the interior of V_i . Let K be an n -knot such that $\pi_1(S^{n+2} - K)$ is not infinite cyclic and indecomposable. Let $h_1 : S^{n+2} - \text{int } V_i \rightarrow N(K)$ be a homeomorphism. Then $S^{n+2} - \text{int } h_1(V_i)$ is homeomorphic to $S^n \times D^2$, $2 \leq i \leq r$. In the same way, we inductively define homeomorphisms $h_j : S^{n+2} - \text{int } V_j^{j-1} \rightarrow N(K)$, $1 \leq j \leq r$, where $V_i^0 = V_i$ and $V_i^j = h(V_i^{j-1})$, $j+1 \leq i \leq r$. Let $\ell_i^0 = \ell_i$ and $\ell_i^j = h_j(\ell_i^{j-1})$. Let $L_0 = O$ and $L^j = h_j(L^{j-1})$, where $L_i^0 = O_i$ and $L_i^j = h_j(L_i^{j-1})$. We show that the iterated satellite link $L = L^r$ ($L_i = L_i^r$) has the Brunnian property of type $\mathcal{A}$. If $S_i \not\subset T \subset I$, then ℓ_i and O_T split and if $S_i \subset S_j = \emptyset$, then ℓ_i and O_{S_j} split. Thus, if $T \notin \mathcal{A}$, then L_T is splittable. Moreover, to show the contrary, we have only to prove that L is nonsplittable assuming $I \in \mathcal{A}$.

Let $S_1 = I$ and $m > \#S_2 \geq \dots \geq \#S_r$. Applying the van Kampen theorem, we have the diagrams of inclusion homomorphisms:

$$\begin{array}{ccc}
 & \pi_1(\partial N(K)) & \\
 \eta \swarrow & & \searrow \theta^j \\
 \pi_1(S^{n+2} - \text{int } N(K)) & & \pi_1(N(K) - L^j) \\
 \xi^j \searrow & & \swarrow \omega^j \\
 & \pi_1(S^{n+2} - L^j) &
 \end{array}$$

for $1 \leq j \leq r$. Note that $\pi_1(S^{n+2} - \text{int } N(K)) \cong \pi_1(S^{n+2} - K)$. Since O_{S_j} and ℓ_i split for $1 \leq i \leq j$, $L_{S_{j+1}}^j = O_{S_{j+1}}$, and so $\pi_1(S^{n+2} - L_{S_{j+1}}^j) = F_{S_{j+1}}$. By deleting the components which are not contained in S_{j+1} , we have an epimorphism $\psi^j : \pi_1(S^{n+2} - L^j) \rightarrow F_{S_{j+1}}$.

If both η and θ^j are injective, then both ξ^j and ω^j are also injective ([9, Sec. 4.2]), that is $\pi_1(S^{n+2} - L^j)$ is the free product of $\pi_1(S^{n+2} - K)$ and $\pi_1(N(K) - L^j)$ with an amalgamated subgroup $\pi_1(\partial N(K))$ [9, p.207]. Further suppose that both $\pi_1(S^{n+2} - K)$ and $\pi_1(N(K) - L^j)$ are indecomposable, then $\pi_1(S^{n+2} - L^j)$ is indecomposable [9, p.246].

Case 1. $n = 1$. Let $\pi_1(\partial N(K)) = \langle \mu, \lambda \mid [\mu, \lambda] = 1 \rangle$, where μ is a meridian and λ is a longitude. Since K is knotted, η is injective ([11, Theorem 4B2]). Let $f^j : \pi_1(N(K) - L^j) \rightarrow \pi_1(S^3 - L^{j-1} \cup \ell_j^{j-1})$ be an isomorphism and $\zeta^{j-1} : \pi_1(S^3 - L^{j-1} \cup \ell_j^{j-1}) \rightarrow \pi_1(S^3 - L^{j-1})$ be an inclusion homomorphism. Then $\psi^{j-1} \zeta^{j-1} f^j \theta^j(\mu) = \alpha_j$, which has infinite order in F_{S_j} ([9, Sec. 1.4]). Furthermore $f^j \theta^j(\lambda)$ is a meridian of ℓ_j^{j+1} , and so θ^j is injective. Thus ξ^j and ω^j are injective.

Since any proper sublink of $O \cup \ell_1$ is trivial and ℓ_1 represents a nontrivial element α_1 in F_I , $O \cup \ell_1$ is nonsplittable, and so $\pi_1(N(K) - L^j) \cong \pi_1(S^3 - O \cup \ell_1)$ is indecomposable. In the same way, $L_{S_j}^{j-1} \cup \ell_j^{j-1} = O_{S_j} \cup \ell_j$ is

nonsplittable. Suppose that L^{j-1} is nonsplittable. Then $L^{j-1} \cup \ell_j^{j-1} = (L_{S_j}^{j-1} \cup \ell_j^{j-1}) \cup L^{j-1}$ is also nonsplittable, and so $\pi_1(N(K) - L^j) \cong \pi_1(S^3 - L^{j-1} \cup \ell_j^{j-1})$ is indecomposable. Hence by induction on j , $\pi_1(S^3 - L)$ is indecomposable.

Case 2. $n \geq 2$. Let $\pi_1(\partial N(K)) = \langle \mu \mid \rangle$. Then $\eta(\mu)$ is a meridian of $N(K)$, and so η is injective. Since the inclusion homomorphism $\pi_1((S^{n+2} - \text{int } V_j^{j-1}) - L^j) \rightarrow \pi_1(S^{n+2} - L^j)$ is isomorphic, we have an isomorphism $g^j : \pi_1(N(K) - L^j) \rightarrow \pi_1(S^{n+2} - L^j)$ and $\psi^j g^j \theta^j(\mu) = \alpha_{j+1}$, which has infinite order in $F_{S_{j+1}}$, and so θ^j is injective. Thus ξ^j and ω^j are injective.

If $\pi_1(N(K) - L^1) \cong \pi_1(S^{n+2} - L^1)$ is indecomposable, then by induction on j , $\pi_1(S^{n+2} - L)$ is indecomposable. Hence the proof is reduced to the lemma below.

Sublemma. Let $H_m = \langle x_1, x_2, \dots, x_m \mid [x_1, x_2, \dots, x_m] = 1 \rangle$. If $m \geq 2$, then H_m is indecomposable.

Proof. We prove by induction on m . H_2 is free abelian of rank 2, and is indecomposable. Assume that H_{m-1} is indecomposable. Let $H_m = A * B$. Since $\beta = [x_1, x_2, \dots, x_{m-1}]$ and x_m commute, either both β and x_m are in a conjugate of A or B , or β and x_m are both powers of the same element [9, Corollary 4.1.6]. Considering the exponent sums on generators, the latter case cannot occur. Thus by an inner

automorphism of H_m , we may suppose that $\beta \in A$ and $x_m \in A$. Let N be the normal subgroup generated by β and x_m in A . Then we have $H_{m-1} \cong A/N * B$, cf. [8, Problem 4.1.5]. By inductive hypothesis, we obtain $A/N = 1$ or $B = 1$. If $A/N = 1$, then $H_m \cong A * H_{m-1}$, and so the rank (i.e., minimum number of generators) of A is one [8, p.192], a contradiction. This completes the proof.

Lemma. If $n \geq 2$, then $G = \pi_1(S^{n+2} - L^1)$ is indecomposable.

Proof. Let $G = C * D$. Then by the condition, $\pi_1(S^{n+2} - K)$ is contained in a conjugate of C or D [8, p.245]. We may suppose that $\pi_1(S^{n+2} - K)$ is contained in C . Then the HNN extension of G with an associated subgroup $\langle \mu \mid \rangle \cong \mathbb{Z}$ [8, p.179]

$$G^* = \langle G, x_{m+1} \mid x_{m+1}^{-1} \mu x_{m+1} = \mu \rangle$$

is a nontrivial free product $C^* * D$, where C^* is an HNN extension of C

$$C^* = \langle C, x_{m+1} \mid x_{m+1}^{-1} \mu x_{m+1} = \mu \rangle.$$

On the other hand, since $\mu = [x_1, x_2, \dots, x_m]$, G^* is the free product of H_{m+1} and $\pi_1(S^{n+2} - K)$ with an amalgamated subgroup $\langle \mu \mid \rangle$. Now both H_{m+1} and $\pi_1(S^{n+2} - K)$ are indecomposable, so is G^* , and this contradiction completes the proof.

Remark 1) A satellite n -link built from the trivial link O and the simple closed curve ℓ representing $\prod_{S \in \mathcal{L}_k} \alpha_S$, where $\mathcal{L}_k$ is the family of all the subsets of I consisting of k elements, has the Brunnian property of type $\mathcal{B}_k$.

2) If $n = 1$, then the Alexander polynomial of our link L is zero by [12, Theorem 5].

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Department of Mathematics
Kyushu University 33
Fukuoka, 812
Japan

