

Scattering theory from a geometric view point

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This article is based on the author's recent joint works with Erik Skibsted [IS1, IS2].

1 Assumptions

Let (M, g) be a connected and complete Riemannian manifold, and we consider the Schrödinger operator

$$H = H_0 + V; \quad H_0 = -\frac{1}{2}\Delta$$

on the Hilbert space $\mathcal{H} = L^2(M) = L^2(M, (\det g)^{1/2} dx)$. The Laplace-Beltrami operator $-\Delta$ is defined in local coordinates by

$$-\Delta = p_i^* g^{ij} p_j = (\det g)^{-1/2} p_i (\det g)^{1/2} g^{ij} p_j,$$

where

$$p_i = -i\partial_i, \quad g = g_{ij} dx^i \otimes dx^j, \quad \det g = \det(g_{ij}), \quad (g^{ij}) = (g_{ij})^{-1}.$$

Under the following Conditions 1.1–1.4 H is essentially self-adjoint on $C_c^\infty(M)$. We will denote the self-adjoint extension also by H .

Condition 1.1 (End structure). There exists a relatively compact open set $O \Subset M$ with smooth boundary ∂O such that the exponential map restricted to outward normal vectors on ∂O :

$$\exp_O := \exp|_{N^+\partial O}: N^+\partial O \rightarrow M$$

is diffeomorphic onto $E := M \setminus \overline{O}$.

A component of E is called an *end*, and such M a *manifold with ends*, cf. [K1]. Then there exists a function $r \in C^\infty(M)$ such that

$$r(x) = \text{dist}(x, O), \quad x \in E.$$

Note that r is not uniquely determined on O .

Recall that the geometric Hessian by $\nabla^2 f \in \Gamma(T^*M \otimes T^*M)$ for $f \in C^\infty(M)$ is defined in local coordinates by

$$(\nabla^2 f)_{ij} = \partial_i \partial_j f - \Gamma_{ij}^k \partial_k f; \quad \Gamma_{ij}^k = \frac{1}{2} g^{kl} (\partial_i g_{lj} + \partial_j g_{li} - \partial_l g_{ij}). \quad (1.1)$$

Condition 1.2 (Mourre type condition). There exist $\delta \in (0, 1]$ and $r_0 \geq 0$ such that for $x \in E$ with $r(x) \geq r_0$

$$\nabla^2 r^2 \geq (1 + \delta)g, \quad (1.2)$$

where the inequality is understood as that for quadratic forms on fibers of TM ,

Condition 1.3 (Quantum mechanics bound). There exists $\kappa \in (0, 1)$ such that

$$|d\Delta r^2|^2 = g^{ij}(\partial_i \Delta r^2)(\partial_j \Delta r^2) \leq C\langle r \rangle^{-1-\kappa}, \quad \langle r \rangle = (1 + r^2)^{1/2}. \quad (1.3)$$

The quantities in Conditions 1.2 and 1.3 appear in the Morre-type commutator computations: If we define

$$A = i[H_0, r^2] = \frac{1}{2}\{(\partial_i r^2)g^{ij}p_j + p_i^*g^{ij}(\partial_j r^2)\}, \quad (1.4)$$

then

$$i[H_0, A] = p_i^*(\nabla^2 r^2)^{ij}p_j + \frac{i}{4}(\partial_i \Delta r^2)g^{ij}p_j - \frac{i}{4}p_i^*g^{ij}(\partial_j \Delta r^2).$$

Condition 1.4 (Short-range potential). The potential $V \in L^\infty(M; \mathbb{R})$ satisfies for some $\eta \in (0, 1]$

$$|V(x)| \leq C\langle r \rangle^{-1-\eta}. \quad (1.5)$$

2 Free propagator

Set $K(t, x) = r(x)^2/2t$ and let A be as defined by (1.4). We define the free propagator $U(t): \mathcal{H} \rightarrow \mathcal{H}$, $t > 0$, by

$$U(t) = e^{iK(t, \cdot)} e^{-i\frac{\ln t}{2}A}.$$

Note that the function K is a solution to the Hamilton-Jacobi equation

$$\partial_t K = -\frac{1}{2}g^{ij}(\partial_i K)(\partial_j K) \quad \text{on } E. \quad (2.1)$$

In fact, r satisfies the eikonal equation

$$|\nabla r|^2 = g^{ij}(\partial_i r)(\partial_j r) = 1 \quad \text{on } E.$$

On the other hand, $e^{-i\frac{\ln t}{2}A}$ is written explicitly by

$$e^{-i\frac{\ln t}{2}A}u(x) = \exp\left(\int_1^t \frac{1}{4s}(-\Delta r^2)(\omega(s, x)) ds\right)u(\omega(t, x)), \quad (2.2)$$

where the flow $\omega = \omega(t, x)$, $(t, x) \in (0, \infty) \times M$, is given by

$$\partial_t \omega^i = -\frac{1}{2t}g^{ij}(\omega)(\partial_j r^2)(\omega), \quad \omega(1, x) = x. \quad (2.3)$$

In fact, if we differentiate $e^{-i\frac{\ln t}{2}A}u$ in t , then we obtain a transport equation and thus (2.2) by solving the equation. By (2.2) we can see that $e^{-i\frac{\ln t}{2}A}$ is the geodesic dilation on $\mathcal{H}$ with respect to r . In fact we note that, using the relation $-\Delta f = g^{ij}(\nabla^2 f)_{ij} = \text{tr}(\nabla^2 f)$,

$$\exp\left(\int_1^t \frac{1}{4s}(-\Delta r^2)(\omega(s, x)) ds\right) = J(\omega(t, x))^{1/2} \left(\frac{\det g(\omega(t, x))}{\det g(x)}\right)^{1/4}, \quad (2.4)$$

and that (2.3) is solved for $(t, x) \in (0, \infty) \times E$ by

$$\omega(t, x) = \exp_O\left[\frac{1}{t}(\exp_O)^{-1}(x)\right],$$

and for $(t, x) \in (0, \infty) \times O$ by something different and complicated. The first factor in the right-hand side of (2.4) is the Jacobian for $\omega(t, \cdot)$, and the second is the change of density for $\omega(t, \cdot)$.

In particular, we learn that $U(t)$ is unitary on both

$$\mathcal{H}_{\text{aux}} := L^2(E) \subset \mathcal{H} \quad \text{and} \quad (\mathcal{H}_{\text{aux}})^\perp = L^2(O) \subset \mathcal{H}.$$

3 Main results

Theorem 3.1 (Positive eigenvalues, [Do, K2, IS2]). *Suppose Conditions 1.1–1.4. Then the positive eigenvalues of H are absent: $\sigma_{\text{pp}}(H) \cap (0, \infty) = \emptyset$.*

Theorem 3.2 (Wave operator, [IS1]). *Under Conditions 1.1–1.4 there exist the strong limits*

$$\Omega_+ := \text{s-lim}_{t \rightarrow +\infty} e^{itH} U(t) P_{\text{aux}}, \quad \tilde{\Omega}_+ := \text{s-lim}_{t \rightarrow +\infty} U(t)^* e^{-itH} P_c,$$

where P_{aux} is the orthogonal projection onto $\mathcal{H}_{\text{aux}}$, and $P_c = \chi_{(0, \infty)}(H)$. Moreover the wave operator Ω_+ is complete, i.e.

$$\tilde{\Omega}_+ = \Omega_+^*, \quad \Omega_+^* \Omega_+ = P_{\text{aux}}, \quad \Omega_+ \Omega_+^* = P_c.$$

We denoted the characteristic function of $\mathcal{O} \subset \mathbb{R}$ by $\chi_{\mathcal{O}}$. It follows by a standard local compactness argument that the negative spectrum of H (if not empty) consists of eigenvalues of finite multiplicities accumulating at most at zero.

Corollary 3.3 (Intertwining property and spectrum). *One has the intertwining property:*

$$\Omega_+^* H \Omega_+ = \frac{1}{2} r^2 P_{\text{aux}}.$$

In particular, the singular continuous spectrum of H is absent, i.e., $\sigma_{\text{sc}}(H) = \emptyset$, and the continuous spectrum $\sigma_c(H) = [0, \infty)$.

The following corollary implies the existence of “the asymptotic speed”. For self-adjoint operators B and B_i , $i = 1, 2, \dots$, we denote

$$B = \text{s-}C_c(\mathbb{R})\text{-}\lim_{i \rightarrow +\infty} B_i,$$

if for any $f \in C_c(\mathbb{R})$ the following equality holds:

$$f(B) = \text{s-}\lim_{i \rightarrow +\infty} f(B_i).$$

Corollary 3.4 (Asymptotic observables). *In the continuous subspace $\mathcal{H}_c(H)$ there exists the $*$ -representation*

$$\omega_\infty^+ := \text{s-}C_c(M)\text{-}\lim_{t \rightarrow +\infty} e^{itH} \omega(t, \cdot) e^{-itH}. \quad (3.1)$$

In particular, the asymptotic speed

$$r(\omega_\infty^+) = \text{s-}C_c(\mathbb{R})\text{-}\lim_{t \rightarrow +\infty} e^{itH} \frac{r(\cdot)}{t} e^{-itH}$$

exists as a self-adjoint operator on $\mathcal{H}_c(H)$. This operator is positive with zero kernel. Moreover, for all $\varphi \in C_c(M)$

$$\varphi(\omega_\infty^+) = \Omega_+ M_\varphi \Omega_+^*, \quad H_c = 2^{-1} r(\omega_\infty^+)^2.$$

Here M_φ denotes the multiplication operator by φ . In local coordinates $\omega(t, \cdot)$ has d (dimension of M) components which we can substitute for any $f \in C_c(M)$, so the limit in (3.1) makes sense.

- Remarks 3.5.**
1. Theorem 3.1 is generalized under weaker conditions including asymptotically hyperbolic manifolds, [IS2].
 2. This type of the free propagator in Theorem 3.2 appeared first in [Y]. For later developments refer to [DeG, CHS, HS].
 3. The above results are independent of choice of r on O .
 4. As for Theorem 3.1, Conditions 1.2–1.4 are optimal in the sense that we can construct counterexamples to the existence of Ω_+ under the slight relaxation of the conditions allowing either $\delta = 0$ in (1.2), $\kappa = 0$ in (1.3) or $\eta = 0$ in (1.5).

4 Generator of the free propagator

We briefly see why the free propagator $U(t)$ works as a comparable system, and see also the relationship with the previous result on the wave operators on manifolds with ends, [IN], where the radial Laplacian was chosen as the free operator.

Let $G(t)$ be the time-dependent generator of $U(t)$:

$$\frac{d}{dt}U(t) = -iG(t)U(t).$$

By a formal computation we can see

$$G(t) = -\partial_t K + \frac{1}{2}\{(\partial_i K)g^{ij}(p_j - \partial_j K) + (p_i - \partial_i K)g^{ij}(\partial_j K)\},$$

so that

$$\begin{aligned} H - G(t) &= V + W(t) + \alpha(t); \\ W(t) &= \frac{1}{2}(p_i - \partial_i K)^* g^{ij} (p_j - \partial_j K), \\ \alpha(t) &= \alpha(t, x) = (\partial_t K) + \frac{1}{2}g^{ij}(\partial_i K)(\partial_j K). \end{aligned} \tag{4.1}$$

The right-hand side of (4.1) is interpreted to be *short-range*. In fact the first is so by Condition 1.4; The second term is so from a classical point of view in the sense that for any nontrapped classical trajectory $(x(t), p(t))$

$$0 \leq \frac{1}{2}g^{ij}(x(t))\{p_i(t) - \partial_i K(t, x(t))\}\{p_j(t) - \partial_j K(t, x(t))\} \leq C\langle t \rangle^{-1-\delta}, \tag{4.2}$$

cf. the fact that K is a solution to the Hamilton-Jacobi equation; As for the third term this is due to (2.1): For any $N > 0$

$$|\alpha(t, x)| \leq C_N t^{-2} \langle r \rangle^{-N}.$$

In the proof of Theorem 3.2 the translation of the classical estimate (4.2) into the quantum mechanics plays an essential role.

We remark that, since

$$G(t) = \frac{1}{2}p_r^* p_r - \frac{1}{2}\left(p_r - \frac{r}{t}\right)^* \left(p_r - \frac{r}{t}\right) \quad \text{on } E; \quad p_r := (\partial_k r)g^{kl}p_l,$$

which we can see with ease in the *geodesic spherical coordinates*, $G(t)$ differs from the one-dimensional radial Laplacian by a short-range term, cf. [IN]. Note that $r(t)/t$ classically approaches the radial momentum $p_r(t)$, cf. (4.2).

5 Example: Ends of warped-product type

Here we give an example of a manifold that satisfies Conditions 1.1–1.4.

Let $V = 0$, and suppose that there exists a relatively compact open subset $O \Subset M$ such that isometrically the closure $\bar{E} := M \setminus O \cong [0, \infty) \times S$ for some $(d-1)$ -dimensional manifold S , and that

$$g = dr \otimes dr + f(r)h_{\alpha\beta}(\sigma) d\sigma^\alpha \otimes d\sigma^\beta; \quad g_{rr} = 1, \quad g_{r\alpha} = g_{\alpha r} = 0, \tag{5.1}$$

where $(r, \sigma) \in [0, \infty) \times S$ denotes local coordinates and the Greek indices run over $2, \dots, d$.

Then Condition 1.1 is automatically satisfied. By (1.1), it follows

$$(\nabla^2 r^2)_{rr} = 2, \quad (\nabla^2 r^2)_{r\alpha} = (\nabla^2 r^2)_{\alpha r} = 0, \quad (\nabla^2 r^2)_{\alpha\beta} = r f' h_{\alpha\beta}.$$

Thus, if we set $f = e^{2\varphi}$, (1.2) is equivalent to

$$2r\varphi' \geq 1 + \delta, \tag{5.2}$$

and, by $\Delta r^2 = g^{ij}(\nabla^2 r^2)_{ij} = 2 + 2(d-1)r\varphi'$, (1.3) to

$$|(r\varphi')'| \leq C\langle r \rangle^{-(1+\kappa)/2}. \tag{5.3}$$

We see that the inequalities (5.2) and (5.3) allow, for example,

$$\begin{aligned} f(r) &= f_{1,\mu}(r) = r^2 \langle r \rangle^{2\mu}, \quad \mu \geq -(1-\delta)/2, \\ f(r) &= f_{2,\nu}(r) = r^2 e^{-2} \exp(2\langle r \rangle^\nu), \quad 0 \leq \nu \leq (1-\kappa)/2. \end{aligned}$$

Note that the Euclidean space corresponds to $f(r) = f_{1,0}(r) = f_{2,0}(r) = r^2$. We also note that in [IS2] the absence of embedded eigenvalues is discussed for a wider class of manifolds with ends including $f_{1,\mu}$ with $\mu > -1$ and $f_{2,\nu}$ with $0 \leq \nu \leq 1$.

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