

THE UNIQUE EXISTENCE OF A CLOSED ORBIT OF FITZHUGH-NAGUMO SYSTEM

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1. Introduction

Our purpose is to give a parameter range for which the ordinary differential equations governing the FitzHugh-Nagumo system have a unique non-trivial closed orbit. It is wider than those already known.

To explicate the ion mechanism for the excitation and the conduction of nerve Hodgkin and Huxley([H-H]) developed the system of four nonlinear ordinary differential equations as a model of nerve conduction in the squid giant axon(Loligo). R. FitzHugh([Fi]) and J. Nagumo et al.([Na]) simplified the system by introducing the following two dimensional autonomous system of ordinary differential equations:

$$\begin{cases} \dot{w} = v - \frac{1}{3}w^3 + w + I \\ \dot{v} = \rho(a - w - bv), \end{cases} \quad (\text{FHN})$$

where the dot ($\dot{}$) denotes differentiation and a, ρ, b are real constants such that

$$[\text{C1}] \quad a \in \mathbb{R}, \quad \rho > 0, \quad 0 < b < 1.$$

The variable w corresponds to the potential difference through the axon membrane and v represents the potassium activation. The quantity I is the current through the membrane. The system (FHN) for special values of I has been investigated by using numerical methods and phase space analysis in [Fi] or [Na].

The system (FHN) has a unique equilibrium point (x_I, y_I) for each $I \in \mathbb{R}$, where

$$\begin{aligned} x_I = & \sqrt[3]{\{3(I + a/b) + \sqrt{9(I + a/b)^2 + 4(1/b - 1)^3}\}/2} \\ & + \sqrt[3]{\{3(I + a/b) - \sqrt{9(I + a/b)^2 + 4(1/b - 1)^3}\}/2} \end{aligned}$$

and

$$y_I = (a - x_I)/b.$$

Instead of the parameter I we introduce a new parameter η . By the transformation $\eta = x_I$, $x = w - \eta$ and $y = v - a/b + \eta/b + \rho b(w - \eta)$, the system (FHN) is transformed to the following system:

$$\begin{cases} \dot{x} = y - \left\{ \frac{1}{3}x^3 + \eta x^2 + (\eta^2 + \rho b - 1)x \right\} \\ \dot{y} = -\frac{\rho b}{3} \left\{ x^3 + 3\eta x^2 + 3(\eta^2 + \frac{1}{b} - 1)x \right\} \end{cases} \quad (\text{FNS})$$

The system (FNS) is called the FitzHugh nerve system and has a unique equilibrium point $E(0,0)$. We shall give some results for the system (FNS) is equivalent to the system (FHN).

Note that, if $\rho b \geq 1$, then the system (FNS) has no non-trivial closed orbits. Thus instead of [C1], we can assume the condition

$$[\text{C2}] \quad a \in \mathbb{R}, \quad 0 < b < 1, \quad 0 < \rho < 1/b.$$

It was studied in such papers as [H1], [K-S] and [Su] for the system (FNS) with the condition [C2]. Let $\eta_0 = \sqrt{1 - \rho b}$. The following is our main result.

Theorem. *The system (FNS) has a unique non-trivial closed orbit if $\eta^2 < \eta_0^2$.*

This result improves those given in [K-S] and [H1]. In fact, the result that ‘If either $\eta^2 \leq 4^{-1}\eta_0^2$ or $\{\rho b^2 - 7b + 6 < 0 \text{ and } \eta^2 < \eta_0^2\}$, then the system (FNS) has a unique non-trivial closed orbit’ was given in [K-S]. In [H1] the result that ‘There is a positive constant $\eta_1 \leq \eta_0$ such that the system (FNS) has a unique non-trivial closed orbit for $|\eta| \leq \eta_1$ ’ was given. Therefore the result of the above theorem is clearly stronger than those in [K-S] and [H1].

2. Lemmas

In this section we prepare some lemmas to be used in the next section to prove the Theorem. We consider the Liénard system of the following form

$$\begin{cases} \dot{x} = y - F(x) \\ \dot{y} = -g(x), \end{cases} \quad (1)$$

where F is continuously differentiable and g is continuous. We assume the following conditions for the system (1):

$$[C3] \quad xg(x) > 0 \text{ if } x \neq 0,$$

$$[C4] \quad \text{There exist } a_1 < 0 \text{ and } a_2 > 0 \text{ such that } F(a_1) = F(a_2) = 0, \\ xF(x) \leq 0 \text{ for } a_1 < x < a_2, F(x) \text{ is nondecreasing for } x < a_1 \text{ and } x > a_2,$$

$$[C5] \quad \lim_{x \rightarrow \pm\infty} \int_0^x \left\{ F'(\xi) + |g(\xi)| \right\} d\xi = \pm\infty.$$

To prove the Theorem we shall use the following three lemmas.

Lemma 1. Assume that the system (1) satisfies the conditions [C3], [C4], [C5] and besides

$$[C6] \quad G(a_1) > G(a_2) \text{ and there exists a constant } \alpha \geq 0 \text{ such that } \frac{F(x)}{G^\alpha(x)} \text{ is} \\ \text{nondecreasing for } x \in (a_1, x_1) \cup (a_2, +\infty); \text{ moreover, there exists a constant} \\ \delta > 0 \text{ such that } \frac{F(x)}{G^\alpha(x)} \text{ is strictly increasing in } x \text{ with } 0 < |x| < \delta,$$

where $G(x) = \int_0^x g(\xi)d\xi$ and $x_1 < 0$ is a number satisfying the equation $G(a_2) = G(x_1)$.

Then the system (1) has a unique non-trivial closed orbit.

Proof of Lemma 1. Under the conditions [C3], [C4] and [C5] the system (1) has at least one non-trivial closed orbit. See [H1]. Moreover, by [Ze], under the conditions [C3], [C4] and [C6] the system (1) has at most one non-trivial closed orbit. $\square$

We can assume that the condition $\eta^2 < \eta_0^2$ in the Theorem holds with $\eta \geq 0$. The proof for the case $\eta < 0$ is essentially the same.

Lemma 2. Let

$$\Gamma(x) = \{2\eta_0^2 + 3(\frac{1}{b} - 1)\}x^2 + \eta\{2(\eta_0^2 - \eta^2) + 3(\eta_0^2 + \frac{1}{b} - 1)\}x \\ - 3(\eta^2 - \eta_0^2)(\eta^2 + \frac{1}{b} - 1).$$

Then $\Gamma(\epsilon(\eta - \eta_0)) > 0$ if $\eta^2 < \eta_0^2$, where $\epsilon = \frac{3(\eta + \eta_0)(\eta^2 + \frac{1}{b} - 1)}{2\eta_0\{2(\eta_0^2 - \eta^2) + 3(\eta_0^2 + \frac{1}{b} - 1)\}}.$

Proof of Lemma 2. If $\eta^2 < \eta_0^2$, we have

$$\begin{aligned}\Gamma(\epsilon(\eta - \eta_0)) &= \epsilon^2(\eta - \eta_0)^2 \left\{ 2\eta_0^2 + 3\left(\frac{1}{b} - 1\right) \right\} \\ &\quad + \epsilon\eta \left\{ 2(\eta_0^2 - \eta^2) + 3\left(\eta_0^2 + \frac{1}{b} - 1\right) \right\} (\eta - \eta_0) - 3(\eta^2 - \eta_0^2) \left(\eta^2 + \frac{1}{b} - 1\right) \\ &= \epsilon^2(\eta - \eta_0)^2 \left\{ 2\eta_0^2 + 3\left(\frac{1}{b} - 1\right) \right\} + 3\left(\frac{\eta}{2\eta_0} - 1\right)(\eta^2 - \eta_0^2) \left(\eta^2 + \frac{1}{b} - 1\right) > 0. \quad \square\end{aligned}$$

Lemma 3. *Let*

$$g(x) = \frac{\rho b}{3} \left\{ x^3 + 3\eta x^2 + 3\left(\eta^2 + \frac{1}{b} - 1\right)x \right\} \quad \text{and} \quad G(x) = \int_0^x g(\xi) d\xi.$$

Then $G(a) - G(-a) \geq 0$ for every $a > 0$.

Proof of Lemma 3. Since $G(a) - G(-a) = \frac{2}{3}\rho b \eta a^3 \geq 0$, the proof is completed. $\square$

3. Proof of Theorem

We shall prove the Theorem by using the above three lemmas. We set $F(x) = (1/3)x^3 + \eta x^2 + (\eta^2 - \eta_0^2)x$. Then, if $\eta^2 < \eta_0^2$, we see easily that the system (FNS) satisfies the conditions [C3], [C4] and [C5]. We shall check the condition [C6] in Lemma 1. We have for $\eta^2 < \eta_0^2$

$$a_1 = \frac{-3\eta - \sqrt{12\eta_0^2 - 3\eta^2}}{2} < 0 \quad \text{and} \quad a_2 = \frac{-3\eta + \sqrt{12\eta_0^2 - 3\eta^2}}{2} > 0.$$

Then we get

$$G(a_1) - G(a_2) = \frac{\rho b}{4} \left\{ \eta^3 + 2\eta_0^2 + 6\left(\frac{1}{b} - 1\right) \right\} \sqrt{12\eta_0^2 - 3\eta^2} > 0.$$

If $\eta^2 < \eta_0^2$, since $0 < \epsilon < 1$, we have $a_2 > \frac{3}{2}(\eta_0 - \eta) > \epsilon(\eta_0 - \eta)$. Let $x_1 < 0$ be a number satisfying the equation $G(a_2) = G(x_1)$. From the above fact, the monotonicity of G and Lemma 3, it follows that

$$G(x_1) = G(a_2) > G(\epsilon(\eta_0 - \eta)) \geq G(\epsilon(\eta - \eta_0)).$$

Using the fact that $a_1 < x_1 < \epsilon(\eta - \eta_0) < 0$, we shall show that $F(x)/G^\alpha(x)$ is nondecreasing for $x \in (a_1, \epsilon(\eta - \eta_0)) \cup (a_2, +\infty)$. This means that $F'(x)G(x) - \alpha F(x)g(x) \geq 0$ for $x \in (a_1, \epsilon(\eta - \eta_0)) \cup (a_2, +\infty)$.

From the calculation in [H1] we see that the above claim means that

$$\begin{aligned} \Phi(x, \alpha) &= (3 - 4\alpha)x^4 + 6\eta(3 - 4\alpha)x^3 + 3\left\{5(3 - 4\alpha)\eta^2 - (1 - 4\alpha)\eta_0^2 + 2(3 - 2\alpha)\left(\frac{1}{b} - 1\right)\right\}x^2 \\ &\quad + 12\eta\left\{2(2 - 3\alpha)\eta^2 - (1 - 3\alpha)\eta_0^2 + 3(1 - \alpha)\left(\frac{1}{b} - 1\right)\right\}x + 18(\eta^2 - \eta_0^2)\left(\eta^2 + \frac{1}{b} - 1\right)(1 - 2\alpha) \\ &\geq 0 \end{aligned}$$

for $x \in (a_1, \epsilon(\eta - \eta_0)) \cup (a_2, +\infty)$.

Let $\alpha = \frac{3}{4}$. Thus we get the following expression which is of degree 2 in x .

$$\begin{aligned} \Phi(x, \frac{3}{4}) &= 3\left[\{2\eta_0^2 + 3(\frac{1}{b} - 1)\}x^2 + \eta\{2(\eta_0^2 - \eta^2) + 3(\eta_0^2 + \frac{1}{b} - 1)\}x - 3(\eta^2 - \eta_0^2)(\eta^2 + \frac{1}{b} - 1)\right] \\ &= 3\Gamma(x). \end{aligned}$$

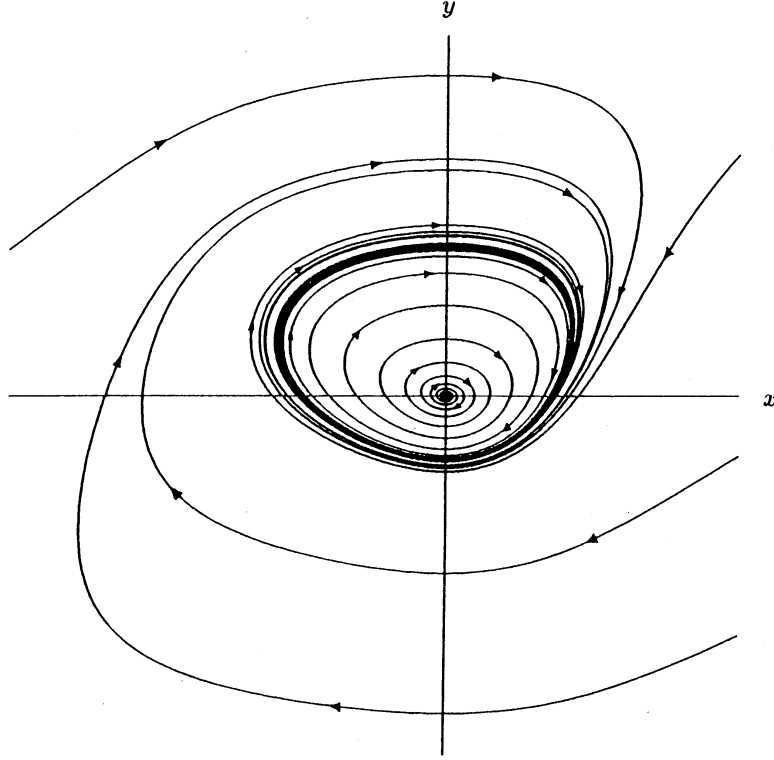
If $\eta^2 < \eta_0^2$, from the fact that Γ is a function of the degree 2, the inequality $\Gamma(0) > 0$ and Lemma 2, we conclude that $\Phi(x, \frac{3}{4}) \geq 0$ for $x \in (a_1, \epsilon(\eta - \eta_0)) \cup (a_2, +\infty)$. Therefore the condition [C6] in Lemma 1 is satisfied. $\square$

4. A numerical example

We shall present the phase portrait of the following system as an example illustrating the application of the Theorem. We consider the system (FNS) with $b = 1/2$, $\rho = 1$ and $\eta^2 = 3/8$:

$$\begin{cases} \dot{x} = y - \left(\frac{1}{3}x^3 + \frac{\sqrt{6}}{4}x^2 - \frac{1}{8}x\right) \\ \dot{y} = -\frac{1}{6}\left(x^3 + \frac{3\sqrt{6}}{4}x^2 + \frac{33}{8}x\right) \end{cases} \quad (2)$$

In this case, since $\eta_0^2 = 1 - \rho b = 1/2 > \eta^2$, the system (2) satisfies the condition in the Theorem. Thus we see that the system (2) has a unique non-trivial closed orbit (see the Figure below). We note that this system does not satisfy the condition in [H1] nor that of [K-S], either.



Figure

5. Appendix

Recently, in [H-T] the result that the system (FNS) has a unique non-trivial closed orbit if $\eta^2 = \eta_0^2 > \frac{1}{b} - 1$ was given by using some Lyapunov-type functions.

In [Su] it was shown that the system (FNS) has no non-trivial closed orbits if it satisfies the condition

$$\eta^2 \geq \eta_0^2 \text{ and } \eta^4 - 4\eta^2\eta_0^2 + \eta_0^4 + 2\left(\frac{1}{2} - 1\right)\eta^2 - 4\left(\frac{1}{b} - 1\right)\eta_0^2 + 4\left(\frac{1}{b} - 1\right)^2 \geq 0$$

or

$$2\left\{\eta_0^2 + \left(\frac{1}{b} - 1\right)\right\}^3 < \eta^2\left\{\eta^2 + 3\left(\frac{1}{b} - 1\right)\right\}^2.$$

We do not know yet what happens in the region in the (η, η_0) -plane in which $\eta^2 > \eta_0^2$, but the condition of [Su] is not satisfied. But some numerical experiments

tell us that the system may have two non-trivial closed orbits if (η, η_0) is in the above mentioned region. Thus we have a **conjecture**:

‘The system (FNS) has either exactly two non-trivial closed orbits or no non-trivial closed orbits in the above mentioned region.’

REFERENCES

- [Fi] R. FitzHugh, *Impulses and physiological states in theoretical models of nerve membrane*, Biophys. J., **1** (1961), 445–466.
- [H1] M. Hayashi, *On the uniqueness of the closed orbit of FitzHugh–Nagumo system*, Math. Japon., **46** (2) (1997), 331–336.
- [H–T] M. Hayashi and M. Tsukada, *A uniqueness theorem on the Liénard system with a non-hyperbolic equilibrium point*, submitted.
- [K–S] E. Kaumann and U. Staude, *Uniqueness and nonexistence of limit cycles for the FitzHugh equation*, Lecture Notes in Math. **1017**, Springer–Verlag, New York, (1983), p. 313–321.
- [Na] J. Nagumo, S. Arimoto and S. Yoshizawa, *An active pulse transmission line simulating nerve axon*, Proc. IRE., **50** (1962), 2061–2070.
- [Su] J. Sugie, *Non-existence of periodic solutions for the FitzHugh nerve system*, Quart. Appl. Math., **49** (1991), 543–554.
- [Ze] X. Zeng and others, *On the uniqueness of the limit cycle of the generalized Liénard equation*, Bull. London Math. Soc., **26** (1994), 213–247.

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