

Dynamics on character varieties

Serge Cantat
Université de Rennes I

Complex Dynamics and Related Topics

Research Institute for Mathematical Sciences, Kyoto University

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①

GOAL

- Study an action of the group

$$\Gamma_2^+ = \left\{ M \in \text{PGL}(2, \mathbb{Z}) ; M \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod{2} \right\}$$

on the family of surfaces

$$(S_{A,B,C,D}) \quad x^2 + y^2 + z^2 + xyz = Ax + By + Cz + D$$

by polynomial diffeomorphisms.

- Painlevé Equations #VI, monodromy of PVI.

Iwasaki and Uehara, Inaba, Iwasaki, Saito, ...

- Quasi-Fuchsian Groups, Character Varieties

Goldman, Benedetto, Brown, Neumann, Stantchev,
Pickrell, Previte, Xia, Souto, Storm, Tan, Wong, Zhang,
Yamashita, ...

- Holomorphic Dynamics.

Bedford, Diller, Dinh, Dujardin, Formaers, Lyubich,
Sibony, Smillie, ...

- Certain kind of "discrete Schrödinger Operators"

Bellissard, Roberto, Casdagli, Mackay, ...

Thanks to Frank Loray (partly a joint work)
with him

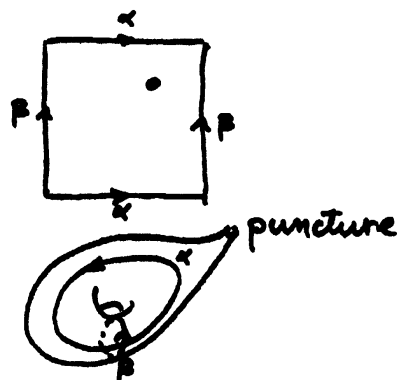
② The Torus and The Sphere.

- T_1 : the once punctured torus.

$$\pi_1(T_1) = \langle \alpha, \beta \mid \emptyset \rangle \cong F_2$$

(free group of rank 2)

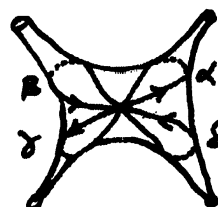
$[\alpha, \beta] = \alpha\beta\alpha^{-1}\beta^{-1}$ makes one turn around the puncture.



- S_4 : the four punctured sphere

$$\pi_1(S_4) = \langle \alpha, \beta, \gamma, \delta \mid \alpha\beta\gamma\delta = 1 \rangle \cong F_3$$

(free group of rank 3)



- If $X = T_1$ or S_4 then $\text{euler}(X) = -1$ or $-2 < 0$.

$$\Rightarrow \exists \rho : \pi_1(X) \longrightarrow \text{PSL}(2, \mathbb{R}) = \text{Isom}^+(\mathbb{D})$$

such that $\rho(\pi_1(X))$ is a discrete subgroup of $\text{PSL}(2, \mathbb{R})$ and $\mathbb{D} / \rho(\pi_1(X)) \cong X$.

Moreover, the Teichmüller space of X has real dimension 2.

- Since $\pi_1(X)$ is free, representations $\rho : \pi_1(X) \rightarrow \text{PSL}(2, \mathbb{R})$ can be lifted to $\text{SL}(2, \mathbb{R})$.

- The Mapping Class Group of X coincides with $\text{Aut}(\pi_1(X)) / \text{Inn}$ where $\text{Inn} =$ inner automorphisms (= conjugations).
It acts on the space of representations $\{\rho : \pi_1(X) \rightarrow \text{SL}(2, \mathbb{C})\}$ modulo $\text{SL}(2, \mathbb{C})$ -conjugations.

Goal : STUDY THIS ACTION !

②

Character Varieties.

$$\begin{aligned} \bullet \operatorname{Rep}(\pi_1(X), \operatorname{SL}(2, \mathbb{C})) &= \{ \rho: \pi_1(X) \rightarrow \operatorname{SL}(2, \mathbb{C}); \rho \text{ morphism} \} \\ &= \begin{cases} \{ (\rho(\alpha), \rho(\beta)) \in \operatorname{SL}(2, \mathbb{C})^2 \} = \operatorname{SL}(2, \mathbb{C})^2 \\ \text{or} \\ \operatorname{SL}(2, \mathbb{C})^3 \text{ if } X \text{ is } \mathbb{B}_4. \end{cases} \end{aligned}$$

$$\bullet X(X) = \operatorname{Rep}(\pi_1(X), \operatorname{SL}(2, \mathbb{C})) // \underbrace{\operatorname{SL}(2, \mathbb{C})}_{\substack{\text{Quotient in the} \\ \text{sense of Geometric} \\ \text{Invariant Theory}}} \swarrow \begin{array}{l} \operatorname{SL}(2, \mathbb{C}) \text{ acts by} \\ \text{conjugation:} \\ (\rho, A) \mapsto A \cdot \rho \cdot A^{-1}. \end{array}$$

• The Torus Π_1 :

- $\operatorname{tr}(\rho(\alpha)), \operatorname{tr}(\rho(\beta)), \operatorname{tr}(\rho(\alpha\beta))$ are invariant functions
- they generate the algebra of invariant functions
- there are no relations between these functions.

$$\Rightarrow [X(\Pi_1) = \mathbb{C}^3, (x, y, z) = (\operatorname{tr}(\rho(\alpha)), \dots)]$$

Remark: $\operatorname{tr}(\rho[\alpha, \beta]) = x^2 + y^2 + z^2 - xyz - 2$

• The Sphere $\mathbb{S}_4$:

$$\begin{aligned} \bullet \quad a &= \operatorname{tr}(\alpha) & b &= \operatorname{tr}(\beta) & c &= \operatorname{tr}(\gamma) & d &= \operatorname{tr}(\delta) \\ x &= \operatorname{tr}(\alpha\beta) & y &= \operatorname{tr}(\beta\gamma) & z &= \operatorname{tr}(\gamma\alpha) \end{aligned}$$

generate the algebra of invariant functions.

- They satisfy the equation

$$[x^2 + y^2 + z^2 + xyz = Ax + By + Cz + D]$$

$$\text{with } A = ab + cd \quad B = bc + ad$$

$$[C = ac + bd \text{ and } D = 4 - a^2 - b^2 - c^2 - d^2 - abcd]$$

$$\Rightarrow [X(\mathbb{S}_4^2) \text{ is a 6-dimensional complex quartic hypersurface in } \mathbb{C}^7.]$$

④

Action of the Mapping Class Group

- The group $\text{Aut}(\pi_1(X))$ acts on $\text{Rep}(\pi_1(X), \text{SL}(2, \mathbb{C}))$ by composition:

$$\rho \in \text{Rep}(\pi_1(X), \text{SL}(2, \mathbb{C})), \Phi \in \text{Aut}(\pi_1(X)) \mapsto \rho \circ \Phi.$$

- $\text{Inn}(\pi_1(X)) = \text{Inner automorphisms} = \{\gamma \mapsto \alpha \gamma \alpha^{-1}, \alpha \in \pi_1(X)\}$
The group $\text{Inn}(\pi_1(X))$ does not act on $\chi(X)$.

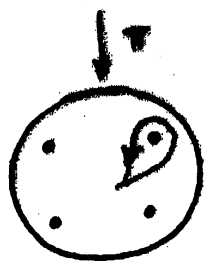
$$\Rightarrow \text{Out}(\pi_1(X)) := \text{Aut}(\pi_1(X)) / \text{Inn}(\pi_1(X)) \text{ acts on } \chi(X).$$

- The group $\text{Out}(\pi_1(X))$ coincides with the mapping class group of X .

Example: The 4-punctured sphere $\mathbb{S}_4$.



$$T = \mathbb{R}^2 / \mathbb{Z}^2$$



$$S = T / \sigma$$

$$\text{where } \sigma(x, y) = (-x, -y)$$

$\text{GL}(2, \mathbb{Z})$ acts on T and commutes with σ



$\text{PGL}(2, \mathbb{Z})$ acts on the sphere.

$H = \{(0, 0), (0, \frac{1}{2}), (\frac{1}{2}, 0), (\frac{1}{2}, \frac{1}{2})\}$ = 2-torsion of π
also acts $\Rightarrow \text{PGL}(2, \mathbb{Z}) \rtimes H$ acts on $\mathbb{S}_4$

Fact: This is $\text{MCG}^*(\mathbb{S}_4)$.

Remark: $\Gamma_2^* = \{M \in \text{PGL}(2, \mathbb{Z}) \mid M \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod{2}\}$

This group acts on $\mathbb{S}_4$ and preserves the punctures.

$\Rightarrow$ Acts on $\chi(\mathbb{S}_4)$ and preserves a, b, c, d , i.e. A, B, C , and D .

⑤

Automorphisms of $S_{A,B,C,D}$

• Summary:

The group Γ_2^* acts on the family of cubic surfaces $(S_{A,B,C,D})$ $x^2 + y^2 + z^2 + xyz = Ax + By + Cz + D$

where A, B, C , and D are parameters (complex or real).

One wants to describe this dynamical system.

→ Tools from holomorphic dynamics are useful for that!!

Automorphisms (= polynomial diffeomorphisms)

$$s_x : (x, y, z) \in S_{A,B,C,D} \mapsto (-x - yz + A, y, z)$$

$$s_y : (x, y, z) \in S_{A,B,C,D} \mapsto (x, -y - zx + B, z)$$

$$s_z : (x, y, z) \in S_{A,B,C,D} \mapsto (x, y, -z - xy + C)$$

THM (Él'-Huti, 1974)

• There are no relations between s_x, s_y, s_z :

$$\langle s_x, s_y, s_z \rangle \cong \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z} \subset \text{Aut}(S_{A,B,C,D})$$

• The index of $\langle s_x, s_y, s_z \rangle$ in $\text{Aut}(X)$ is ≤ 24

• For generic A, B, C, D , $\text{Aut}(X) = \langle s_x, s_y, s_z \rangle$.

• Fact (easy computation): The group Γ_2^* acts on $S_{A,B,C,D}$.

Its image in $\text{Aut}(X)$ coincides with $\langle s_x, s_y, s_z \rangle$.

$$\left. \begin{array}{ll} \cdot s_x & \text{corresponds to } \begin{pmatrix} -1 & 2 \\ 0 & 1 \end{pmatrix} \\ \cdot s_y & \text{" " } \begin{pmatrix} 1 & 0 \\ 2 & -1 \end{pmatrix} \\ \cdot s_z & \text{" " } \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \end{array} \right\} \text{These 3 matrices generate } \Gamma_2^*.$$

Example: $s_x \circ s_y \circ s_z$ corresponds to $\begin{pmatrix} 3 & 2 \\ 2 & 1 \end{pmatrix}$ and is given by

$$(x, y, z) \mapsto \begin{pmatrix} -x - (-y + xz + x^2y - C_x) \cdot (-z - xy + C) + A, \\ -y + xz + x^2y - C_x, -z - xy + C \end{pmatrix}$$

⑥

The Cayley Cubic.

- Choose $A, B, C, D = 0, 0, 0, 4$, then S is given by

$$x^2 + y^2 + z^2 + xyz = 4$$

- Consider $\eta: \mathbb{C}^* \times \mathbb{C}^* \rightarrow \mathbb{C}^* \times \mathbb{C}^*$, $\eta(u, v) = (\frac{1}{u}, \frac{1}{v})$

Then the map $\mathbb{C}^* \times \mathbb{C}^* \rightarrow S_{0,0,0,4}$
 $(u, v) \mapsto (-u - \frac{1}{u}, -v - \frac{1}{v}, -uv - \frac{1}{uv})$
 provides an isomorphism between $S_{0,0,0,4}$ and $\mathbb{C}^* \times \mathbb{C}^* / \eta$

- $S_{0,0,0,4}$ has 4 singularities corresponding to the 4 fixed points of η : $(1, -1) \in \mathbb{C}^* \times \mathbb{C}^* \mapsto (-2, 2, 2) \in \text{Sing}(S)$.

THM (Cayley, ~1880)

$S_{0,0,0,4}$ is the unique surface in the family $S_{A,B,C,D}$ with 4 singularities

We shall call it the Cayley cubic and denote it S_C

- The group $GL(2, \mathbb{Z})$ acts on $\mathbb{C}^* \times \mathbb{C}^*$ by monomial transformations:

$$\Pi = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, (u, v) \in \mathbb{C}^* \times \mathbb{C}^* \mapsto (u^a v^b, u^c v^d)$$

$\Rightarrow PGL(2, \mathbb{Z})$ acts on S_C by polynomial diffeomorphisms

$\Rightarrow \Gamma_2^*$ acts on S_C : this is the same action!

- Consequence: When $A, B, C, D = 0, 0, 0, 4$, the dynamics of Γ_2^* is "uniformized" by its usual linear action on $\mathbb{C} \times \mathbb{C}$:

$$\begin{array}{ccccc} \mathbb{C} \times \mathbb{C} & \xrightarrow{\quad} & \mathbb{C}^* \times \mathbb{C}^* & \xrightarrow{\quad} & S_C \\ s, t & \mapsto & \exp s, \exp t & \mapsto & (-\frac{1}{2} - u, -v - \frac{1}{v}, -uv - \frac{1}{uv}) \\ \text{Linear} & & \text{Monomial} & & \end{array}$$

② Action of Γ_2^* at infinity (II)

- Let $\gamma \in \Gamma_2^*$: γ corresponds to an isometry of $\mathbb{D}$
 γ corresponds to a 2×2 real matrix.

$\lambda(\gamma) :=$ Largest |eigenvalue| of γ .

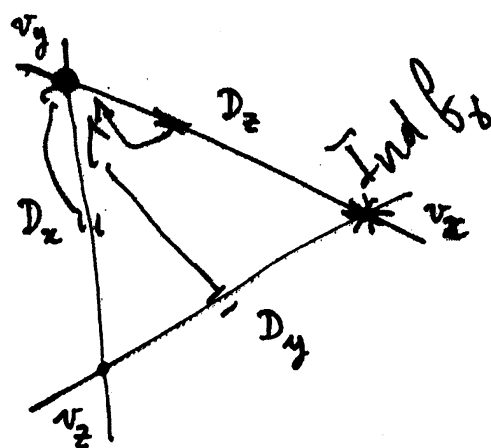
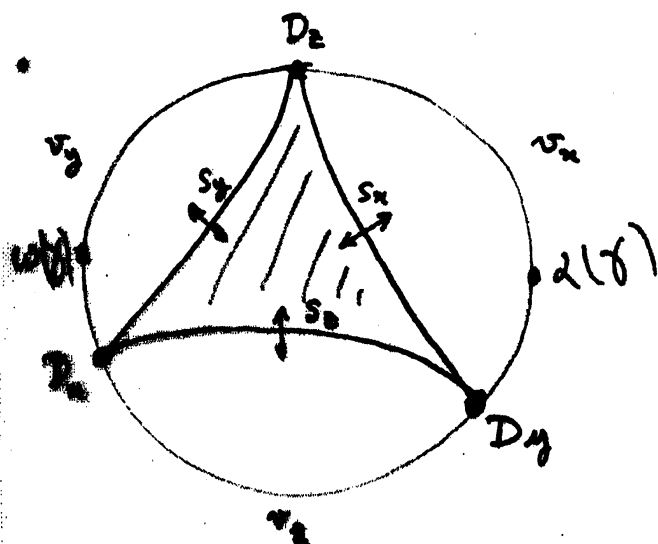
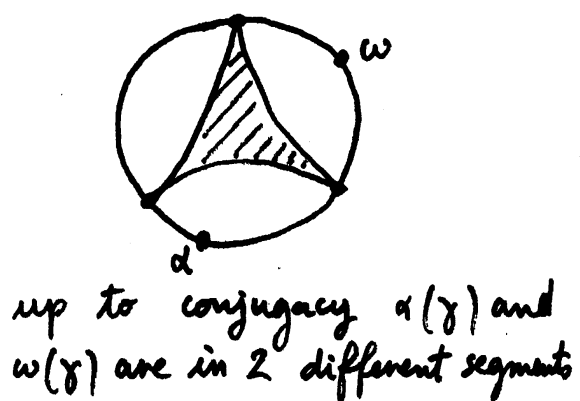
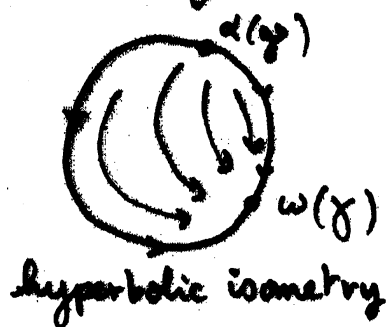
γ is said to be hyperbolic if $\lambda(\gamma) > 1$

γ is said to be parabolic if $\lambda(\gamma) = 1$ and $\gamma \neq \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}$

γ is said to be elliptic otherwise.

Fact: elliptic $\Leftrightarrow$ conjugated to s_x, s_y or s_z
 parabolic $\Leftrightarrow$ " " an iterate of
 $s_z \circ s_y$ or $s_y \circ s_x$ or $s_x \circ s_z$.

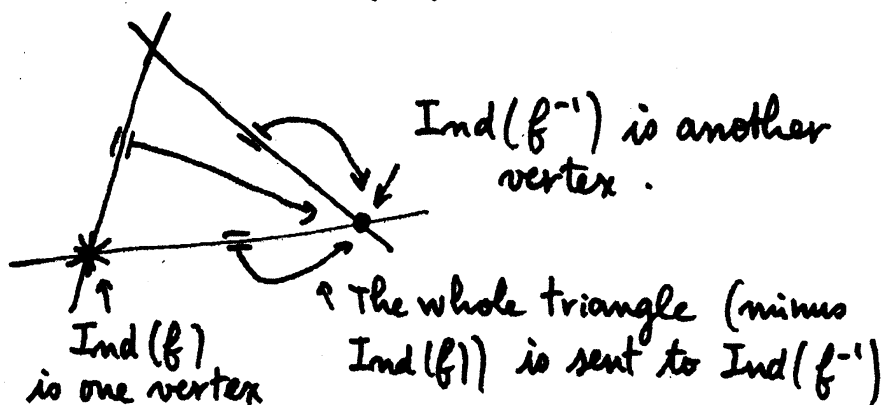
- If γ is hyperbolic then γ has two fixed points on $\partial\mathbb{D}$ and the dynamics is :



④

Topological Entropy.

- Summary: Let f be an automorphism of $S_{A,B,C,D}$. Assume that f is determined by a hyperbolic element of Γ_2^* . Then, after conjugacy in $\text{Aut}(S_{A,B,C,D})$ we have:



- Consequence: Up to conjugacy in $\text{Aut}(S_{A,B,C,D})$, f is algebraically stable.

THM (a new version of Iwasaki & Uehara)

For any set of parameters $A, B, C, D \in \mathbb{C}$
 For any ~~hyper~~ element f in $\text{Aut}(S_{A,B,C,D})$,
 The topological entropy of $f: S_{A,B,C,D}(\mathbb{C}) \rightarrow S_{A,B,C,D}(\mathbb{C})$
 is given by
$$h_{\text{top}}(f) = \log(\lambda(f))$$

Remark: $\lambda(f) := \lambda(\gamma)^{\frac{1}{k}}$ for any $k \geq 1$
 such that f^k is induced by $\gamma \in \Gamma_2^*$.

⑩

• proof 1 (Smillie, Bedford & Diller, Dujardin ; Dinh & Sibony)

• $f: S \rightarrow S$ a birational transformation of a complex projective surface.

• $\text{Ind}(f^{-1}) \cap \text{Ind}(f) = \emptyset$, $f^{-1}(\text{Ind} f) = \text{Ind}(f)$
 $f(\text{Ind} f^{-1}) = \text{Ind}(f^{-1})$

• $f^*: H^2(S, \mathbb{Z}) \rightarrow H^2(S, \mathbb{Z})$

$$\lambda(f^*) = \limsup_{n \rightarrow +\infty} \| (f^n)^* \|^{1/n}$$

$$\text{Then } h_{\text{top}}(f) = \log(\lambda(f^*)).$$

• Moreover: $H \subset S$ a hyperplane section, then

$$h_{\text{top}}(f) = \log \left(\limsup_{n \rightarrow +\infty} \| (f^n)^* [H] \|^{1/n} \right)$$

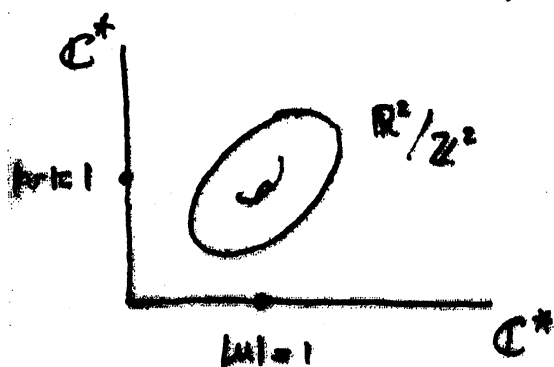
• proof 2: Assume that f is induced by $\gamma \in \Gamma_2^*$.

• The triangle at infinity is a hyperplane section of $\bar{S}_{A,B,C,D}$.

• The action of f^* on the triangle at infinity does not depend on A, B, C, D : $f^*: \text{Vect}([D_n], [D_y], [D_z]) \rightarrow$

• We compute $\lambda(f^*)$ in a specific case:
 The Cayley cubic case S_C .

• In this case, the dynamics is linear:



$$h_{\text{top}}(f) = \log(\lambda(f))$$

①

Normal forms at infinity (I)

- Germs of contracting holomorphic transformations (Dloussky, Favre).

$f: \mathbb{C}^2_0 \rightarrow \mathbb{C}^2_0$ a germ of holomorphic map near the origin.

Assume that f contracts both axes on $(0,0)$:

$$f(\{x=0\}) = f(\{y=0\}) = (0,0).$$

$$\text{Let } f_* : \pi_1(\mathbb{C}^* \times \mathbb{C}^*) \rightarrow \pi_1(\mathbb{C}^* \times \mathbb{C}^*) \\ \mathbb{Z}^2 \rightarrow \mathbb{Z}^2$$

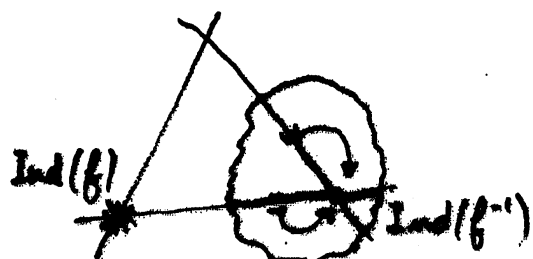
be the linear map induced by f :

$$f_* \begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix}, \det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \pm 1$$

THM (Dloussky, Favre): $\exists$ a germ of holomorphic diffeomorphism $\Psi: \mathbb{C}^2_0 \rightarrow \mathbb{C}^2_0$ such that

$$\left[\begin{array}{l} \Psi \left((x,y) \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right) = f(\Psi(x,y)) \\ \text{i.e. } \Psi \text{ conjugates } f \text{ to } (x,y) \mapsto (x^a y^b, x^c y^d) \end{array} \right.$$

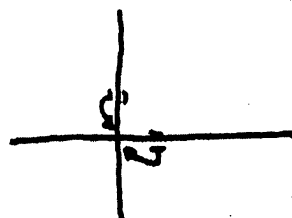
- Consequence (for $f \in \text{Aut}(S_{A,B,C,D})$)



f hyperbolic (after a good conjugacy in $\text{Aut}(S)$)



$$\exists N_f \in GL(2, \mathbb{Z})$$



$$(u,v) \mapsto (u,v)^{N_f}$$

⑫

Normal forms at infinity (II)

Proposition. Let $A, B, C, D \in \mathbb{C}$.

Let M be an element of Γ_2^* .

Let $f: S_{A,B,C,D} \rightarrow S_{A,B,C,D}$ be the automorphism corresponding to fM .

Assume that M is hyperbolic and $\text{Ind } f \neq \text{Ind } f^{-1}$.

Then

(i) $\exists N_f$ a 2×2 integer matrix with ≥ 0 entries which is conjugate to $\pm M$.

(ii) $\exists \Psi: (\mathbb{C}^2, 0) \rightarrow (\overline{S}_{A,B,C,D}, \text{Ind } f^{-1})$ a germ of holomorphic diffeomorphism such that

$$f(\Psi(u, v)) = \Psi((u, v)^{N_f})$$

Remark: $\forall M \in \text{PSL}(2, \mathbb{Z}) \quad \exists N$ with ≥ 0 entries such that M is conjugate to N in $\text{PSL}(2, \mathbb{Z})$.

Unbounded orbits:

Let $(x, y, z) \in S_{A,B,C,D}(\mathbb{C})$. Assume that the forward orbit of (x, y, z) is not bounded, then

$$f^n(x, y, z) \xrightarrow{n \rightarrow +\infty} \text{Ind}(f^{-1})$$

and the following limit is well defined:

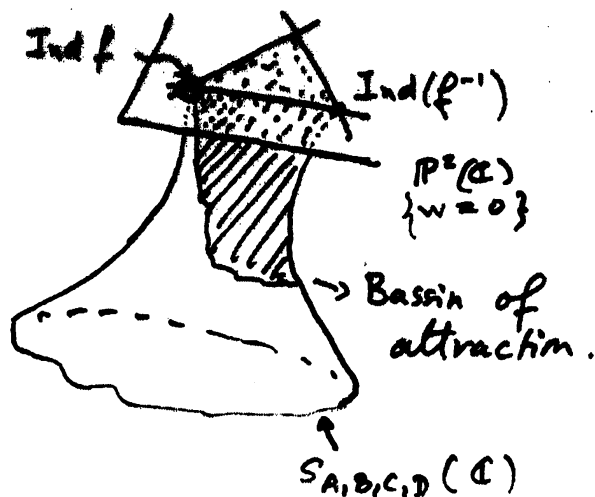
Green $G_f^+(x, y, z) = \lim_{n \rightarrow +\infty} \frac{1}{2(f)^n} \log \|f^n(x, y, z)\|$

(Here $\|(x, y, z)\| = |x|^2 + |y|^2 + |z|^2$.)

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Basin of attraction of $\text{Ind}(f^{-1})$

- Basin of attraction of $\text{Ind}(f^{-1})$:



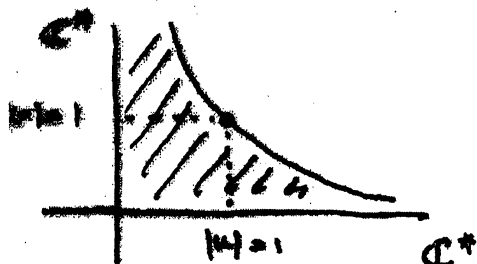
$$\Omega^*(\text{Ind}(f^{-1}))$$

$$= \{m \in S_{A,B,C,D}(\mathbb{C}) ; f^n(m) \xrightarrow{n \rightarrow +\infty} \text{Ind}(f^{-1})\}$$

$$\Omega_*(\text{Ind}(f^{-1}))$$

$$= \{m \in \bar{S}_{A,B,C,D}(\mathbb{C}) ; f^n(m) \xrightarrow{n \rightarrow +\infty} \text{Ind}(f^{-1})\}$$

- Monomial Model:



$$\Omega^*(N_f) = \{(\mu, v) \in \mathbb{C}^* \times \mathbb{C}^* ; |v| < |\mu|^{s(f)}\}$$

$$\text{where } N_f(s(f)) = \lambda(f)(s(f))$$

(i.e. $s(f)$ is the slope of the eigenline of N_f corresponding to the eigenvalue $\lambda(f)$)

Proposition:

The conjugacy Ψ extends to a holomorphic diffeomorphism between $\Omega^*(N_f)$ and $\Omega^*(\text{Ind}(f^{-1}))$.

⑫

Julia Sets and Currents.

- If the orbit of a point $m \in S_{A,B,C,D}(\mathbb{C})$ is unbounded, then

$$\text{either } f^n(m) \xrightarrow{n \rightarrow +\infty} \text{Ind}(f^{-1}) \text{ and } m \in \Omega^*(\text{Ind } f^{-1})$$

$$\text{or } f^n(m) \xrightarrow{n \rightarrow -\infty} \text{Ind}(f) \text{ and } m \in \Omega^*(\text{Ind } f)$$

• Notations.

— Interesting sets —

- $K^+(f) = \{m \mid \text{the forward orbit of } m \text{ is bounded}\}$
 $= \text{complement of } \Omega^*(\text{Ind } f^{-1})$

$$K^-(f) = \{m \mid \text{the backward orbit is bounded}\}$$

$$K(f) = K^+(f) \cap K^-(f)$$

$$J^+(f) = \partial K^+(f) \quad J^-(f) = \partial K^-(f)$$

$$J(f) = J^+(f) \cap J^-(f) \subsetneq \partial K(f)$$

- $J^*(f) = \text{closure of the set of saddle periodic points of } f.$

— Eigen currents —

- $T_f^+ = dd^c G_f^+$ where $G_f^+(m) = \lim_{n \rightarrow +\infty} \frac{1}{2(f)^n} \log \|f^n(m)\|$
 $T_f^- = dd^c G_f^-$ where $G_f^-(m) = \lim_{n \rightarrow -\infty} \frac{1}{2(f)^n} \log \|f^n(m)\|$

$$\mu_f = T_f^+ \wedge T_f^-$$

If T_f^+ and T_f^- are normalized correctly, then

μ_f is an f -invariant probability measure.

⑬ Results from holomorphic dynamics.

(Bedford, Diller, Din, Dujardin, Fornæss, Lyubich, Sibony, Smillie, ...)

- G_f^+ and G_f^- are Hölder continuous.

$\Rightarrow \mu_f$ is well defined.

- μ_f is the unique f -invariant probability measure with maximal entropy:

$$h_\mu(f) = h_{\text{top}}(f) = \log \lambda(f)$$

- The number of periodic points of f of period N is finite (Iwasaki-Uehara: explicit formula) $\approx \lambda(f)^N$. Most of them are hyperbolic saddle points.

$$\frac{1}{\lambda(f)^N} \sum_{m \in \text{Per}(f, N)} \text{Strac}_m \xrightarrow{N \rightarrow +\infty} \mu_f$$

where $\text{Per}(f, N) = \begin{cases} \text{periodic points of period } N \\ \text{or} \\ \text{saddle periodic points} \end{cases}$.

- $J^*(f)$ coincides with the support of μ_f .

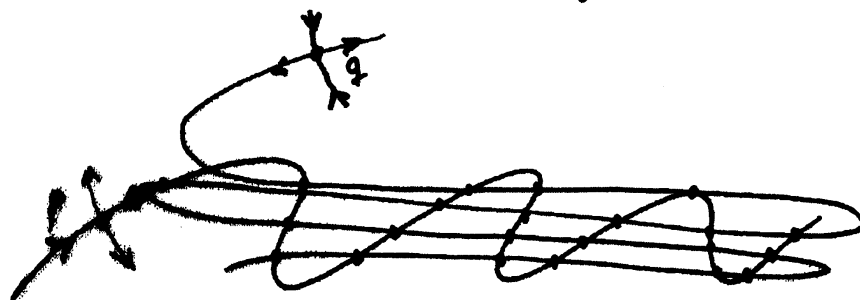
Any periodic saddle point is in the support of μ_f .

If p, q are periodic saddle points then

$$\overline{W^s(p) \cap W^u(q)} = J^*(f)$$

stable manifold
of p

unstable manifold
of q



⑩

- If p is a saddle periodic point of f , then $W^u(p)$ is parametrized by $\mathbb{C}$:

$$\exists \xi : \mathbb{C} \xrightarrow{\text{holo}} S_{A,B,C,D}(\mathbb{C})$$

with ξ injective, $\xi(0) = p$ and $\xi(\mathbb{C}) = W^u(p)$

Let $\mathbb{D} \subset \mathbb{C}$ be the unit disk, let χ be a smooth non negative function on $\xi(\mathbb{D})$ with $\chi(m) > 0$ and $\chi \equiv 0$ along $\partial \mathbb{D}$.

Let $[\xi(\mathbb{D})]$ be the current of integration on $\xi(\mathbb{D})$:

$$\langle [\xi(\mathbb{D})] \mid \alpha \text{ a 2-form} \rangle = \int_{\mathbb{D}} \xi^* \alpha.$$

Then

$$\frac{1}{\lambda(f)^n} f_*^{+n} (\chi \cdot [\xi(\mathbb{D})]) \xrightarrow{n \rightarrow +\infty} c^* T_f^-$$

- Since f is area preserving, we have

$$\begin{aligned} \text{Interior}(K(f)) &= \text{Interior}(K^+(f)) \\ &= \text{Interior}(K^-(f)) \\ &= \text{bounded open subset} \\ &\quad \text{of } S_{A,B,C,D}(\mathbb{C}). \end{aligned}$$

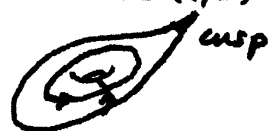
⑦

The Quasi-Fuchsian Space.

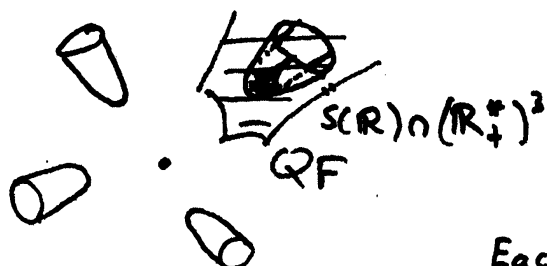
• Quasi Fuchsian Space. (for the once punctured torus).

• We consider $X(\pi_1) = \text{Rep}(\pi_1(T_1), SL(2, \mathbb{C})) //_{SL(2, \mathbb{C})}$
and we add the condition

$$\text{tr}(\rho[\alpha, \beta]) = -2.$$



• The real surface $S(\mathbb{R})$: $x^2 + y^2 + z^2 = xyz$



$$x = \text{tr}(\rho(\alpha))$$

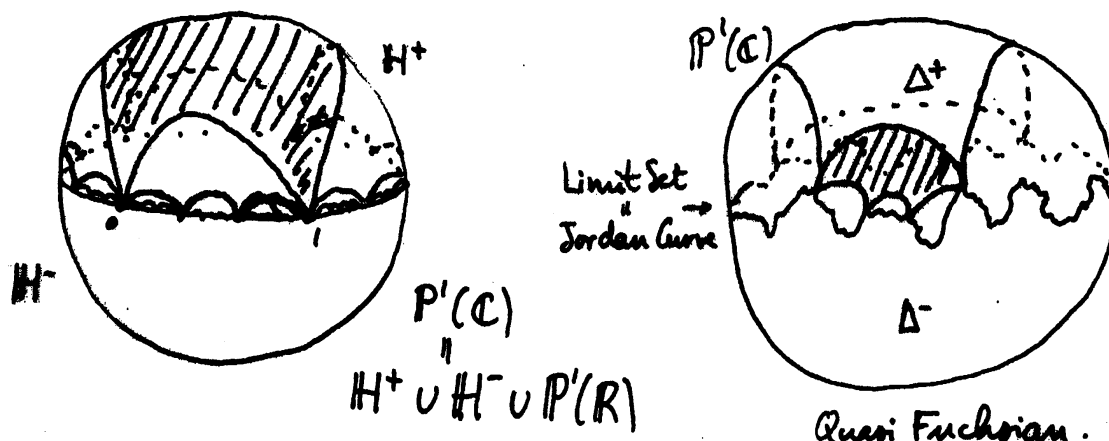
$$y = \text{tr}(\rho(\beta))$$

$$z = \text{tr}(\rho(\alpha\beta))$$

Each connected component $\neq \{(0,0,0)\}$
is homeomorphic to $\mathbb{D}$.

The action of $PGL(2, \mathbb{Z}) (\subset \Gamma_2^*)$ on $S(\mathbb{R}) \cap (\mathbb{R}_+^*)^3$
is conjugate to the action of $MCG^*(T_1)$ on
 $\text{Teich}(T_1)$, i.e. to the action of $PGL(2, \mathbb{Z})$ on
 $\mathbb{D}$: In particular, it is totally discontinuous.

• Quasi Fuchsian deformation.



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Bers Parametrization.

- Small deformations of fuchsian representations
→ quasi fuchsian representations:

$$QF \left\{ \begin{array}{l} \rho: F_2 = \langle \alpha, \beta \rangle \longrightarrow SL(2, \mathbb{C}) \\ \rho \text{ is faithful} \\ \rho(F_2) \text{ is discrete} \\ \rho(F_2) \text{ preserves a Jordan Curve } \Lambda \text{ and } \mathbb{P}^1(\mathbb{C}) \setminus \Lambda \\ \text{is the union of } 2\text{-invariant disks } \Delta^+ \text{ and } \Delta^-. \end{array} \right.$$

QF is an open subset of $S(\mathbb{C})$.

$$\overline{QF} = DF := \{[\rho]: F_2 \rightarrow SL(2, \mathbb{C}) \text{ discrete faithful}\}$$

- Bers Parametrization.

T_2' = the once punctured torus, with the opposite orientation.

$$\text{Teich}(T_2) \simeq \mathbb{H}^+, \quad \text{Teich}(T_2') \simeq \mathbb{H}^-.$$

$GL(2, \mathbb{Z})$ acts on $\mathbb{H}^+$ and $\mathbb{H}^{*-}$ simultaneously.

Thm (Bers) $\exists \text{ Bers} : \mathbb{H}^+ \times \mathbb{H}^- \longrightarrow QF$ a holomorphic

diffeomorphism such that

$$\left\{ \begin{array}{l} \text{Bers}(f(X), f(Y)) = f(\text{Bers}(X, Y)) \\ \forall (X, Y) \in \mathbb{H}^+ \times \mathbb{H}^- = \text{Teich}(T_1) \times \text{Teich}(T_1') \\ \forall f \in GL(2, \mathbb{Z}) = \text{MCG}(T_1) \end{array} \right.$$

This action also conjugates the action of $\text{MCG}(T_1)$ on $\{(z_1, z_2) \in \mathbb{H}^+ \times \mathbb{H}^- / z_1 = \bar{z}_2\}$

$$\overset{12}{\text{Teich}(T_1)}$$

to the action of $PGL(2, \mathbb{Z})$ on $S(\mathbb{R}) \cap (\mathbb{R}_*^+)^2$.

Ⓜ

Dynamics on $\overline{QF}$

• THM (Minsky)

The Bers map extends up to

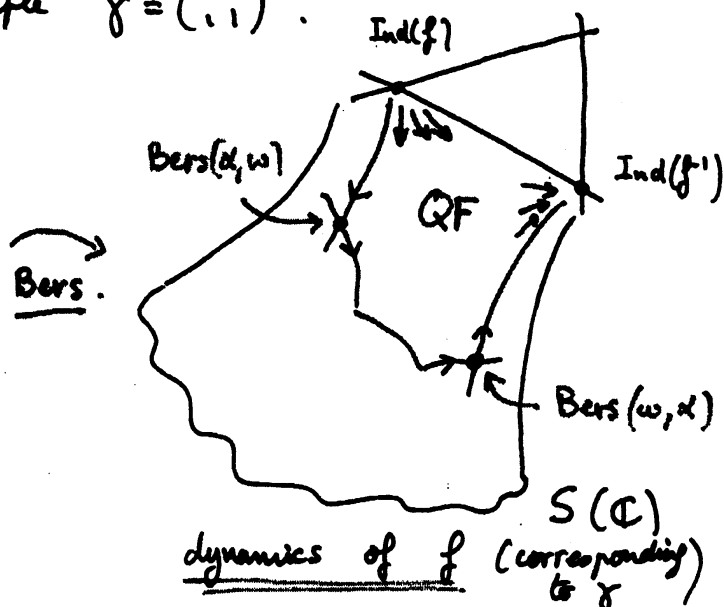
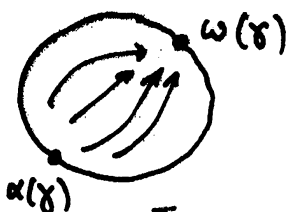
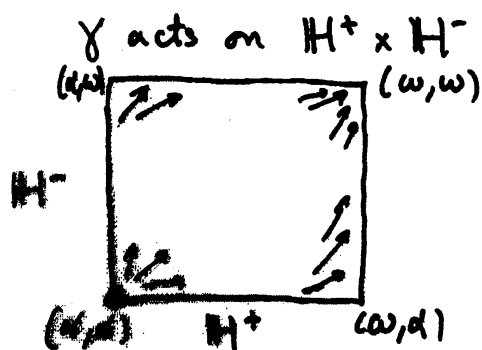
$$\partial^*(H^+ \times H^-) = \partial(\overline{H^+ \times H^-}) \setminus \{(x, x); x \in P'(\mathbb{R})\}$$

and provides a continuous bijection between

$$\overline{H^+ \times H^-} \setminus \{(x, x); x \in P'(\mathbb{R})\} \text{ and } DF \text{ .}$$

• Consequence: Take $\gamma \in PGL(2, \mathbb{Z})$, hyperbolic.

For example $\gamma = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$.



Fact :

$Bers(\alpha, \omega)$ and $Bers(\omega, \alpha)$ are two hyperbolic fixed points of f .

$$Bers(\alpha, H^-) \subset W^u(Bers(\alpha, \omega))$$

$$Bers(H^+, \omega) \subset W^s(Bers(\omega, \alpha))$$

⑩

Nice Orbits.

- The origin $(0,0,0)$

The point $(0,0,0) \in S$ is a singular point
 $(S) \quad x^2 + y^2 + z^2 = xyz.$

It corresponds to the finite representation $\rho: F_2 \rightarrow SL(2, \mathbb{C})$ defined by:

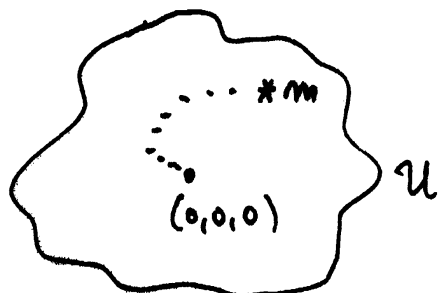
$$\rho(\alpha) = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad \rho(\beta) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

- THM: Let $\gamma \in PGL(2, \mathbb{Z})$ be any hyperbolic element
 Let f be the automorphism of S determined by γ .
 Let q be one of the 2 fixed points of f on ∂QF .
 Then exists $[p] \in S(\mathbb{C})$ such that the closure of the orbit $MCG(T_1) \cdot [p]$ contains both q and the origin $(0,0,0) = [p_0]$.

Proof:

Step 1 (Bowditch): $\exists$ a neighborhood $\mathcal{U}$ of the origin $((0,0,0) \in \mathcal{U} \subset S(\mathbb{C}))$ such that

$$\forall m \in \mathbb{N} \quad \overline{MCG(T_1) \cdot m} \ni (0,0,0).$$



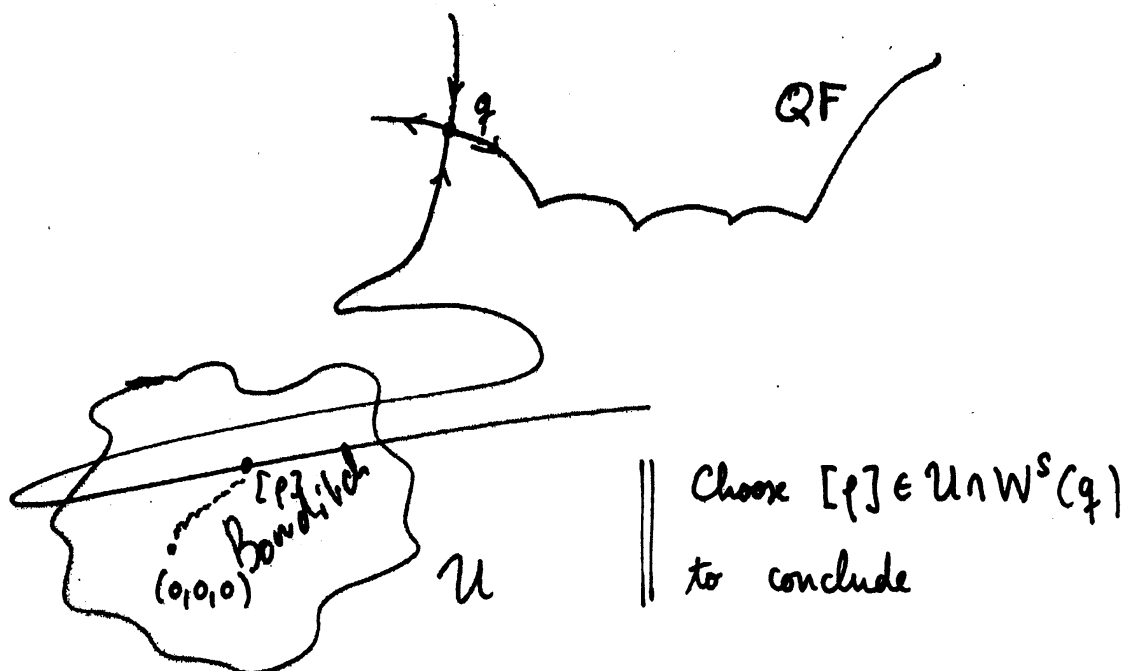
2)

• Step 2:

- $(0,0,0) \in K(f)$ because this is a fixed point.
 - If $(0,0,0) \in \text{Int}(K^-(f)) = \text{Int}(K(f))$, then f is linearizable at the origin
 - but $Df|_{(0,0,0)}$ has finite order and f is not periodic, so $(0,0,0) \notin \text{Int}(K^-(f))$.
- $\Rightarrow (0,0,0) \in \partial K^-(f)$.

• Conclusion:

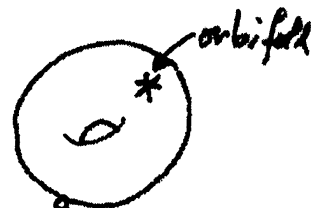
Since $W^s(q)$ is dense in $\partial K^-(f)$, $W^s(q)$ intersects the open set U .



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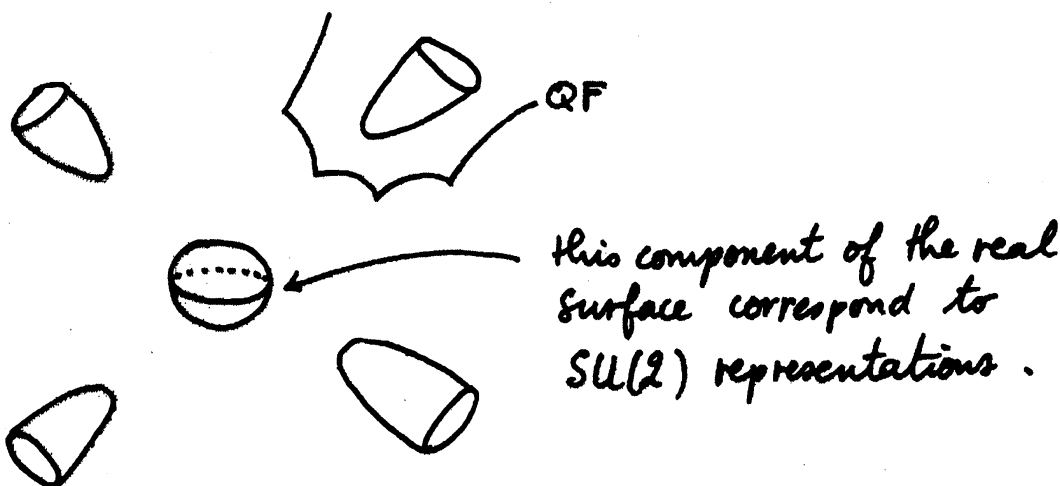
Another Example (Orbifold Structure on $\mathbb{T}_1$)

- Impose the condition $\text{tr}(\rho[\alpha, \beta]) = 0$.
i.e. $\rho[\alpha, \beta]^4 = \text{Id}$



The surface is now $x^2 + y^2 + z^2 - xyz = 2$.

- We can use Teichmüller theory + quasi fuchsian deformations in the orbifold category.
- New feature: The topology of $x^2 + y^2 + z^2 - xyz = 2$.



THM: $\forall \gamma \in \text{PGL}(2, \mathbb{Z})$ hyperbolic
 $\forall q$ one of the 2 fixed points of f on ∂QF

If $f: \ominus \rightarrow \ominus$ has a periodic saddle point then
 $\exists m \in \{x^2 + y^2 + z^2 - xyz = 2\}$ such that

$$f^n(m) \xrightarrow{n \rightarrow +\infty} \ominus$$

$$f^n(m) \xrightarrow{n \rightarrow -\infty} q$$

Moreover, if $\gamma = \begin{pmatrix} a & 1 \\ 1 & 1 \end{pmatrix}$, this works and $\overline{\text{HCG}(\mathbb{T}_2)} \cdot m$
 contains the whole bounded component $\ominus$

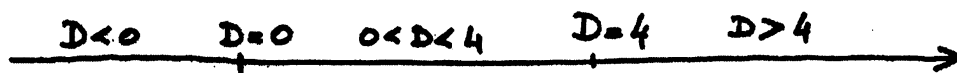
①

REAL versus COMPLEX Dynamics.

- Now we focus on the one parameter family

$$x^2 + y^2 + z^2 = xyz + D \quad (S_D)$$

- Topology of $S_D(\mathbb{R})$, $D \in \mathbb{R}$ (Benedetto, Goldman)



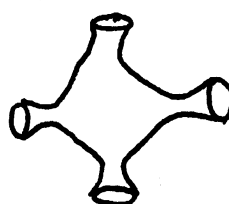
4 connected
components,
all unbounded



A sphere
appears



Cayley



Only one
connected component.

- Description of the real dynamics. (for $f \in \text{Aut}(S_D)$, hyper.)

TMT

$D < 0$	$D = 0$	$0 < D < 4$	$D > 4$
All periodic points of f are complex: $\text{Per}(f) \subset S_D(\mathbb{C}) \setminus S_D(\mathbb{R})$	The origine is the unique real periodic point	There are always complex (=non real) periodic points.	All periodic points are real.
$\text{Supp}(\mu_f) \cap S_D(\mathbb{R}) = \emptyset$		$\text{Supp}(\mu_f)$ may intersects $S_D(\mathbb{R})$ but is not contained in $S_D(\mathbb{R})$	$\text{Supp}(\mu_f)$ is contained in $S_D(\mathbb{R})$
$h_{\text{top}}(f _{\mathbb{R}}) = 0$	$h_{\text{top}}(f _{\mathbb{R}}) = 0$	$h_{\text{top}} < \frac{3}{2} \log(\lambda(f))$	$h_{\text{top}}(f _{\mathbb{R}}) = \log(\lambda(f))$
Totally disconnected	"	Totally discont. on the 4 disks	Uniformly hyperbolic on the Julia Set

⑫

Corollary:

Assume that A, B, C, D are real parameters.

Let $\gamma \in \Gamma_2^*$ be hyperbolic.

Let f be the automorphism of $S_{A,B,C,D}$ induced by γ .

If $S_{A,B,C,D}(\mathbb{R})$ is connected then the measure μ_f is singular with respect to the Lebesgue measure of $S_{A,B,C,D}(\mathbb{R})$; $\text{Haus-Dim}(\text{Supp } \mu_f) < 2$.

Sketch of the proof. (When $A, B, C, D = 0, 0, 0, D$)

Since the surface is connected, $D \geq 4$ and by the previous theorem the dynamics is uniformly hyperbolic.

If the Hausdorff dimension of $\text{Supp}(\mu_f) = 2$, then a result of Bowen and Ruelle implies that

$K(f) \cap S_D(\mathbb{R})$ is an attractor for $f: S_D(\mathbb{R}) \rightarrow$.

This contradicts the fact that $K(f)$ is compact and that f is area preserving. ■

Consequence (Answer to a question by Iwasaki).

There are parameters $\kappa_1, \kappa_2, \kappa_3, \kappa_4$ of the sixth Painlevé equation such that the monodromy along any loop with $\lambda(\gamma) > 1$ has a singular measure of maximal entropy.

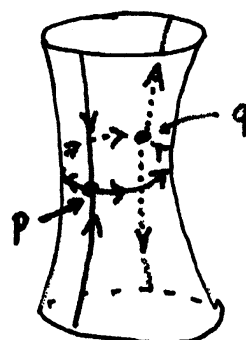
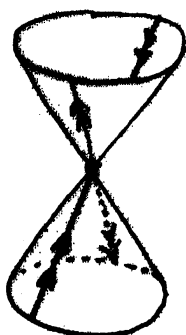
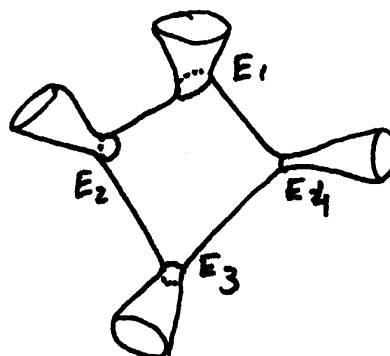
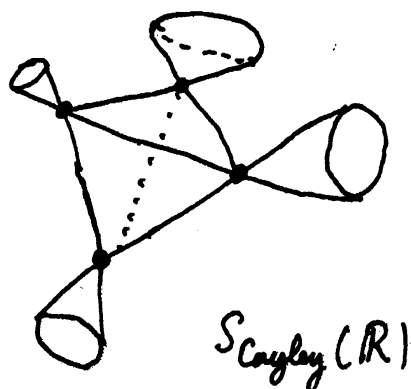
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Sketch of the proof of the theorem I.

- Goal : [Prove that the dynamics is uniformly hyperbolic if $D > 4$, and that $h_{top}(f|_R) = \log(\lambda(f))$ (if $D > 4$)]

- The Cayley Cubic

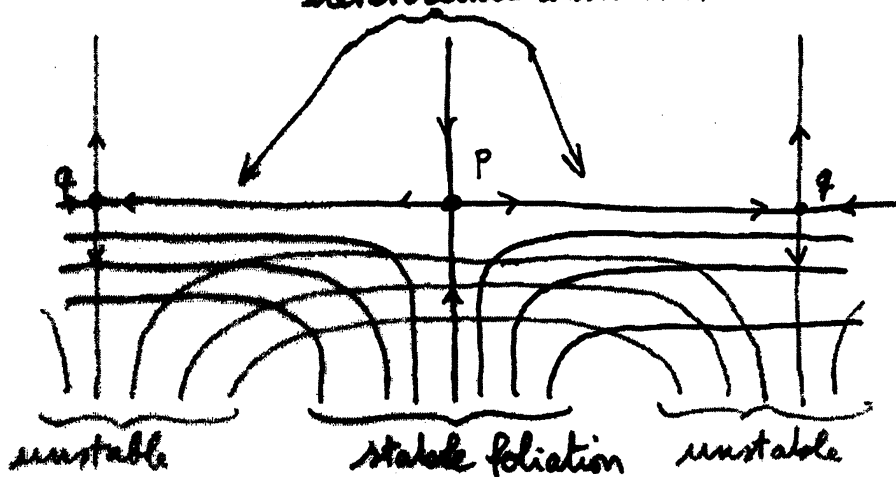
Blow Up Singularities.



Cut along the green unstable manifold:
heteroclinic connection

Wandering dynamics

Julia set



①

Sketch of the proof of the theorem II

Entropy.

- To Compute the entropy we know

$$h_{\text{top}}(f_R) \leq h_{\text{top}}(f_{\mathbb{C}}) = \log(\lambda(f))$$

New Version
of Iwasaki-Uehara.

- The estimate from below comes from Bowen's inequality:



$(x, y) \sim (-x, -y)$



Sphere $\setminus$ 4 points

In the Cayley Case, we remark that if you take a generic loop $l \in \pi_1(\text{Sphere} \setminus 4 \text{ pts})$ then

$$\text{length } f_{\#}^N[l] \sim \lambda(f)^N$$

word metric in $\pi_1(\mathbb{S}_4)$

Bowen's inequality says $h_{\text{top}}(f_R) \geq \log(\lambda(f))$.

Since the action of f on $\pi_1(S_D(\mathbb{R}))$ does not depend on $D > 4$ and is the same as the action of $\pi_1(\mathbb{S}_4)$, we get

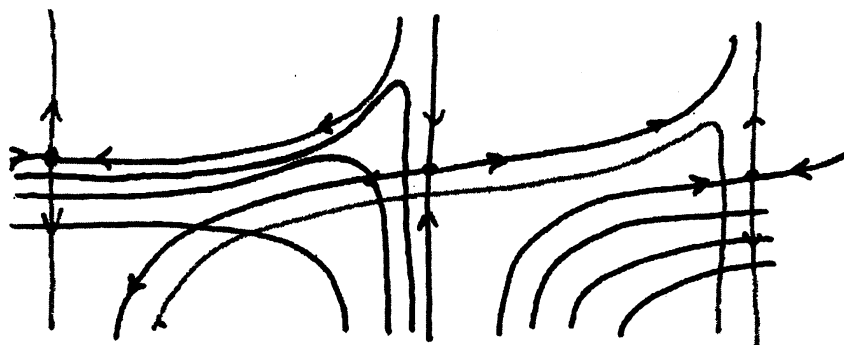
$$\forall D > 4 \quad h_{\text{top}}(f_R) \geq \log(\lambda(f)).$$

- In particular, $\left. \begin{array}{l} K(f) \subset S_D(\mathbb{R}) \\ \text{Per}(f) \subset S_D(\mathbb{R}) \\ W^s \cap W^u \subset S_D(\mathbb{R}) \end{array} \right\} \forall D \gg 4$

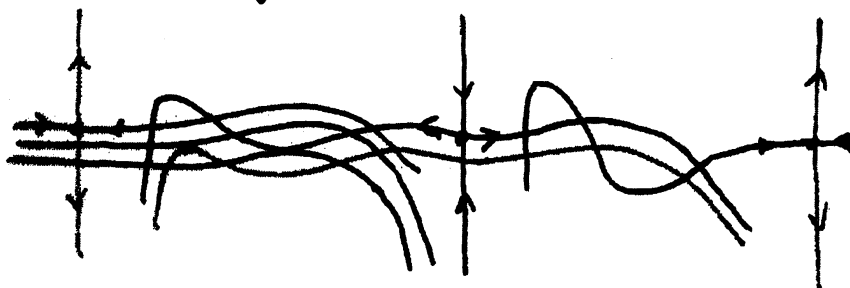


Sketch of the proof of the theorem III.

- What we want to show is that the bifurcation after a small perturbation, or even a large perturbation, with $D > 4$, gives rise to the following local picture:



and not something like

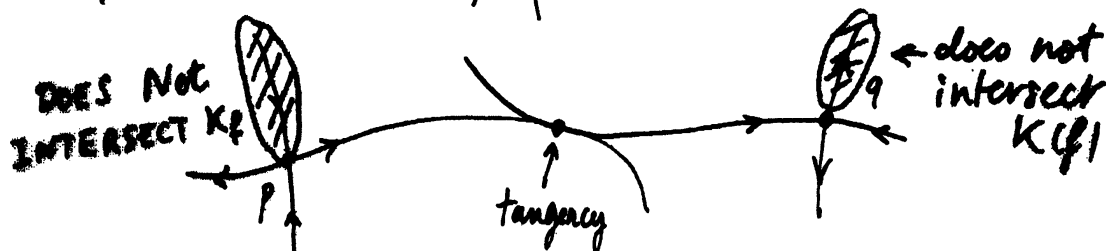


Theorem (Bedford, Smillie)

- Assume $D > 4$. If the dynamics of f on $K(f)$ is not uniformly hyperbolic then

$\exists p, q$ saddle fixed points such that

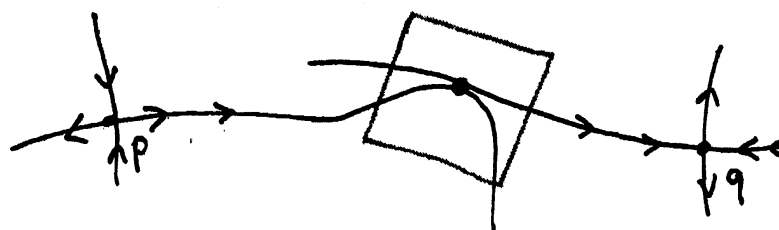
- (i) $W^u(p)$ intersects $W^s(q)$ tangentially (with order 2)
- (ii) p is s -one sided, q is u -one sided.



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Sketch of the proof of the theorem IV.

- Assume $D_0 > 4$, not uniformly hyperbolic



- Deform D_0 :



this "typical deformation" is not possible because for $D = D_0 + \epsilon$, $W^u(p) \cap W^s(q) \neq S_D(R)$

- Consequence: $\left[\begin{array}{l} \text{The tangency persists when one} \\ \text{deforms } D \text{ between } D_0 \text{ and } 4, \\ \text{up to } D=4 \end{array} \right.$

- Conclusion: Get a contradiction at $D=4$!

(Not so easy but
it does work)

