

Logarithmic truth-table reductions and minimum sizes of forcing conditions (preliminary draft)

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Abstract

In our former works, for a given concept of reduction, we study the following hypothesis: “For a random oracle A , with probability one, the degree of the one-query tautologies with respect to A is strictly higher than the degree of A .” In our former works, the following three results are shown: (1) the hypothesis for polynomial-time Turing reduction is equivalent to the assertion that the probabilistic complexity class R is not equal to NP , (2) the hypothesis for polynomial-time truth-table reduction implies that P is not NP , (3) (to appear in Arch. Math. Logic) the hypothesis holds for polynomial-time bounded-truth-table reduction. In this note, we show that the hypothesis holds for $(\log n)^{O(1)}$ -question truth-table-reduction (without polynomial-time bound). As applications of this result, we show a lower bound and an upper bound of forcing complexity (i.e., the minimum size of forcing condition that forces a given formula) of the one-query tautologies with respect to a random oracle. We show that if A is a random oracle then with probability one, the forcing complexity of the one-query tautology with respect to A is greater than polynomial of $\log |F|$, and it is at most $O(|F|^2)$, where $|F|$ denotes the length of a formula.

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1 Preface

In our former works [Su98, Su99, Su00, Su01, Su02, Su05], by extending the work of Ambos-Spies [Am86] and related works, we consider the relationships with the canonical product measure of Cantor space and complexity of one-query tautologies. A formula F of the relativized propositional calculus is called a *one-query formula* if F has exactly one occurrence of a query symbol. For example,

$$(q_0 \Leftrightarrow \xi^3(q_1, q_2, q_3)) \Rightarrow (q_1 \Rightarrow q_0)$$

is a one-query formula, where q_0, q_1, q_2, q_3 are usual propositional variables. We assume that each propositional variable takes the value 0 or 1 (0 denotes false and 1 denotes true). And, ξ^3 in the above formula is a query symbol. For a given oracle A , a function A^3 is defined as follows, where λ is the empty string, and the query symbol ξ^3 is interpreted as the function A^3 .

$$\begin{aligned} A^3(000) &= A(\lambda), & A^3(001) &= A(0), & A^3(010) &= A(1), & A^3(011) &= A(00), \\ A^3(100) &= A(01), & A^3(101) &= A(10), & A^3(110) &= A(11), & A^3(111) &= A(000). \end{aligned}$$

Thus, more informally, the following holds for each $j = 0, 1, \dots, 2^3 - 1$, where the order of strings is defined as the canonical length-lexicographic order.

$$A^3(\text{ the } (j+1)\text{st 3-bit string}) = A(\text{ the } (j+1)\text{st string}).$$

For each n , an n -ary Boolean function A^n is defined in the same way, and an interpretation of the query symbol ξ^n is defined in the same way. For an oracle A , the concept of a *tautology with respect to A* is defined in a natural way. If a one-query formula F is a tautology with respect to A , then we say F is a *one-query tautology with respect to A* . The set of all one-query tautologies with respect to A is denoted by 1TAUT^A .

In [Su02], for a given concept $\leq_\alpha$ of reduction, we study the following hypothesis, where 1TAUT^X denotes the set of all one-query tautologies with respect to an oracle X .

One-query-jump hypothesis for $\leq_\alpha$: The class $\{X : 1\text{TAUT}^X \leq_\alpha X\}$ has measure zero.

For a given reduction $\leq_\alpha$, we denote the corresponding one-query-jump hypothesis by $[\leq_\alpha]$.

In [Su98], it is shown that the one query-jump hypothesis for p-T reduction is equivalent to “ $R \neq \text{NP}$.”

And, in [Su02], it is shown that the one query-jump hypothesis for p-tt reduction implies “ $P \neq \text{NP}$.”

In [Su05], we show that the one query-jump hypothesis for p-btt reduction holds, where p-btt denotes polynomial-time bounded-truth-table reduction. The

anonymous referee of [Su05] noticed that the one query-jump hypothesis holds for bounded-truth-table reduction without polynomial-time bound, and Kumabe independently noticed the same result. The referee's proof, which may be found in [Su05], uses some concepts of resource-bounded generic oracles in [AM97]. Kumabe's proof is more simple.

In §3 of this note, we introduce Kumabe's proof of the above result. In §4, we extend the result, and show that the one query-jump hypothesis holds for $(\log n)^{O(1)}$ -question tt-reduction (without polynomial-time bound). In §5, as applications of the result in §4, we show a lower bound and an upper bound of forcing complexity (i.e., the minimum size of forcing condition that forces a given formula) of the one-query tautologies with respect to a random oracle. We show that if A is a random oracle then with probability one, the forcing complexity of the one-query tautologies with respect to A is greater than $(\log |F|)^{O(1)}$, and it is at most $O(|F|^2)$.

The three of authors had a meeting at July 22–23, 2004, at the office of T.S. in Osaka Prefecture University. This note is a research memo on the meeting, and is an extension of [Su05].

2 Notation

Most of our notation is the same as that of [Su02] and [Su05], and almost all undefined notions may be found in these papers. An article by Kawanishi and Suzuki [KS05] in this volume of *Sūrikaiseikikenkyūsho Kōkyūroku* contains basic definitions on the relativized propositional calculus and Dowd-type generic oracles. The journal version of [Su02] may be purchased at Science Direct.

<http://www.sciencedirect.com/science/journals>

ω stands for $\{0, 1, 2, 3, \dots\}$, while $\mathbb{N}$ stands for $\{1, 2, 3, \dots\}$. In some textbooks, the complexity class R is denoted by RP . For the detail of the class R , see for example [BDG88].

The definition of polynomial-time truth-table reduction and its variant may be found in [LLS75].

3 Bounded truth table reduction

In this section, we show the following.

Proposition 1 *The Lebesgue measure of the set*

$$\{X : 1\text{TAUT}^X \leq_{\text{btt}} X\}$$

is zero. In other words, one-query jump hypothesis [Su02, Su05] for btt-reduction (without polynomial-time bound) holds.

Sketch of proof (due to Kumabe):

For each oracle X , let $L^X := \bigcup_n \{(u, v, w) \in \{0, 1\}^n : |u| = |v| = |w| = n \text{ and } X^n(u) = X^n(v) = X^n(w)\}$. It is easy to see that $L^X \leq_m^p 1\text{TAUT}^X$.

Suppose that f is a recursive function such that for each string x , it holds that $f(x)$ is of the form $(\varphi_x, s_{x,1}, s_{x,2})$, where φ_x is a function from $\{0, 1\}^2$ to $\{0, 1\}$, and $s_{x,1}, s_{x,2}$ are strings.

It is enough to show the following class has measure zero.

$$\{X : L^X \text{ is 2tt-reducible to } X \text{ via } f\}$$

For each forcing condition S , there exists strings $x^{(1)}, x^{(2)}, x^{(3)}, x^{(4)}, x^{(5)}$ and a forcing condition T such that

(1) $\text{dom } T = \text{dom } S \cup \{x^{(1)}, x^{(2)}, x^{(3)}, x^{(4)}, x^{(5)}\}$, and

(2) for any oracle X extending T , it holds that L^X is not 2tt-reducible to X via f .

Therefore, the class $\{X : L^X \text{ is 2tt-reducible to } X \text{ via } f\}$ has measure zero. $\square$

4 $(\log n)^{O(1)}$ -question tt-reduction

Theorem 2 *The Lebesgue measure of the following set is zero.*

$$\{X : 1\text{TAUT}^X \leq_{(\log n)^{O(1)}\text{-tt}} X\}$$

In other words, one-query jump hypothesis for $(\log n)^{O(1)}$ -tt-reduction (without polynomial-time bound) holds.

5 Lower and upper bounds to forcing complexity

Theorem 3 *Let $\mathcal{D}_{\log}$ be the class of all oracles D such that there exists a positive integer c (c may depend on D) of the following property. For any $F \in 1\text{TAUT}^D$, there exists a forcing condition $S \sqsubseteq D$ such that S forces F to be a tautology and*

$$|\text{dom } S| \leq (\log |F|)^c.$$

Then $\mathcal{D}_{\log}$ has measure zero.

Question: Is $\mathcal{D}_{\log}$ empty?

Theorem 4 *Let $\mathcal{D}_{\text{quad}}$ be the class of all oracles D such that there exists a positive integer c (c may depend on D) of the following property. For any $F \in 1\text{TAUT}^D$, there exists a forcing condition $S \sqsubseteq D$ such that S forces F to be a tautology and*

$$|\text{dom } S| \leq c|F|^2 + c,$$

where $|F|$ denotes the length of the binary code of F .

Then $\mathcal{D}_{\text{quad}}$ has measure one.

Question: Let $\mathcal{D}_{\text{linear}}$ be the class defined similarly to $\mathcal{D}_{\text{quad}}$ by using a linear formula $c|F| + c$ instead of a quadratic $c|F|^2 + c$. Then, is $\mathcal{D}_{\text{linear}}$ empty? If non-empty, does it have positive measure?

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