

UNIVALENCE OF CERTAIN INTEGRAL OPERATORS

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Abstract

In this work some integral operators are studied and the authors determine conditions for the univalence of these integral operators.

1 INTRODUCTION

Let A be the class of the functions f which are analytic in the unit disc $U = \{z \in \mathbb{C}; |z| < 1\}$ and $f(0) = f'(0) - 1 = 0$.

We denote by S the class of the function $f \in A$ which are univalent in U .

Many authors studied the problem of integral operators which preserve the class S . In this sense an important result is due to J. Pfaltzgraff [4].

THEOREM A [4]. If f is univalent in U , α a complex number and $|\alpha| \leq \frac{1}{4}$, then the function

$$G_{\alpha}(z) = \int_0^z [f'(\xi)]^{\alpha} d\xi \quad (1)$$

is univalent in U .

THEOREM B[3]. If the function $g \in S$ and α is a complex number, $|\alpha| \leq \frac{1}{4n}$, then the function defined by

$$G_{\alpha,n}(z) = \int_0^z [g'(u^n)]^{\alpha} du \quad (2)$$

is univalent in U for all positive integer n .

2 PRELIMINARY RESULTS

We will need the following theorems in this work.

THEOREM C [2]. Let α be a complex number, $\operatorname{Re} \alpha > 0$ and $f \in A$. If

$$\frac{1 - |z|^{2\operatorname{Re}\alpha}}{\operatorname{Re}\alpha} \left| \frac{zf''(z)}{f'(z)} \right| \leq 1 \quad (3)$$

for all $z \in U$, then for any complex number $\beta, \operatorname{Re}\beta \geq \operatorname{Re}\alpha$ the function

$$F_\beta(z) = \left[\beta \int_0^z u^{\beta-1} f'(u) du \right]^{\frac{1}{\beta}} \quad (4)$$

is in the class S.

THEOREM D [1]. If the function g is regular in U and $|g(z)| < 1$ in U , then for all $\xi \in U$ and $z \in U$ the following inequalities hold

$$\left| \frac{g(\xi) - g(z)}{1 - \overline{g(z)}g(\xi)} \right| \leq \left| \frac{\xi - z}{1 - \overline{z}\xi} \right| \quad (5)$$

and

$$|g'(z)| \leq \frac{1 - |g(z)|^2}{1 - |z|^2}, \quad (6)$$

the equalities hold only in the case $g(z) = \epsilon \frac{z+u}{1+\overline{u}z}$ where $|\epsilon| = 1$ and $|u| < 1$.

REMARK [1]. For $z = 0$, from inequality (5)

$$\left| \frac{g(\xi) - g(0)}{1 - \overline{g(0)}g(\xi)} \right| \leq |\xi| \quad (7)$$

and, hence

$$|g(\xi)| \leq \frac{|\xi| + |g(0)|}{1 + |g(0)||\xi|}. \quad (8)$$

Considering $g(0) = a$ and $\xi = z$,

$$|g(z)| \leq \frac{|z| + |a|}{1 + |a||z|}. \quad (9)$$

for all $z \in U$.

LEMMA SCHWARZ [1]. If the function g is regular in U , $g(0) = 0$ and $|g(z)| \leq 1$ for all $z \in U$, then the following inequalities hold

$$|g(z)| \leq |z| \quad (10)$$

for all $z \in U$, and $|g'(0)| \leq 1$, the equalities (in inequality (10) for $z \neq 0$) hold only in the case $g(z) = \epsilon z$, where $|\epsilon| = 1$.

3 MAIN RESULTS

THEOREM 1. Let α, γ be complex numbers, $\operatorname{Re}\alpha = a > 0$ and $g \in A$.

If

$$\left| \frac{g''(z)}{g'(z)} \right| \leq \frac{1}{n} \quad (11)$$

for all $z \in U$ and

$$|\gamma| \leq \frac{n+2a}{2} \left(\frac{n+2a}{n} \right)^{\frac{n}{2a}} \quad (12)$$

then for any complex number β , $\operatorname{Re} \beta \geq a$, the function

$$G_{\beta, \gamma, n}(z) = \left\{ \beta \int_0^z u^{\beta-1} [g'(u^n)]^\gamma du \right\}^{\frac{1}{\beta}} \quad (13)$$

is in the class S for all $n \in N^* - \{1\}$.

PROOF. Let us consider the function

$$f(z) = \int_0^z [g'(u^n)]^\gamma du. \quad (14)$$

The function

$$p(z) = \frac{1}{|\gamma|} \frac{f''(z)}{f'(z)}, \quad (15)$$

where the constant $|\gamma|$ satisfies the inequality (12), is regular in U.

From (15) and (14) we obtain

$$p(z) = \frac{\gamma}{|\gamma|} \left[\frac{nz^{n-1}g''(z^n)}{g'(z^n)} \right]. \quad (16)$$

Using (16) and (11) we obtain

$$|p(z)| < 1 \quad (17)$$

for all $z \in U$. For $z = 0$ we have $p(0) = 0$.

From (16) and Schwarz-Lemma it results that

$$\frac{1}{|\gamma|} \left| \frac{f''(z)}{f'(z)} \right| \leq |z|^{n-1} \leq |z| \quad (18)$$

for all $z \in U$, and hence

$$\left(\frac{1-|z|^{2a}}{a} \right) \left| \frac{zf''(z)}{f'(z)} \right| \leq |\gamma| \left(\frac{1-|z|^{2a}}{a} \right) |z|^n. \quad (19)$$

Let us consider $Q: [0,1] \rightarrow R$, $Q(x) = \frac{(1-x^{2a})}{a} x^n$,
 $x = |z|$. We have

$$Q(x) \leq \frac{2}{n+2a} \left(\frac{n}{n+2a} \right)^{\frac{n}{2a}} \quad (20)$$

for all $x \in [0,1]$. From (20), (19) and (12) we obtain

$$\left(\frac{1-|z|^{2a}}{a} \right) \left| \frac{zf''(z)}{f'(z)} \right| \leq 1 \quad (21)$$

for all $z \in U$. Then, from (21) and Theorem C it follows that the function $G_{\beta, \gamma, n}$, is in the class S.

THEOREM 2. Let α, γ be complex numbers, $\operatorname{Re} \alpha = b > 0$ and the function $g \in A$,
 $g(z) = z + a_2 z^2 + \dots$. If

$$\left| \frac{g''(z)}{g'(z)} \right| < 1 \quad (22)$$

for all $z \in U$ and the constant $|\gamma|$ satisfies the condition

$$|\gamma| \leq \frac{1}{\max_{|z| \leq 1} \left[\frac{1-|z|^{2b}}{b} |z| \frac{|z|+2|a_2|}{1+2|a_2||z|} \right]} \quad (23)$$

then for any complex number β , $\operatorname{Re} \beta \geq b$ the function

$$G_{\beta, \gamma}(z) = \left\{ \beta \int_0^z u^{\beta-1} [g'(u)]^\gamma du \right\}^{\frac{1}{\beta}} \quad (24)$$

is in the class S .

Proof. Let us consider the function

$$f(z) = \int_0^z [g'(u)]^\gamma du. \quad (25)$$

The function

$$h(z) = \frac{1}{|\gamma|} \frac{f''(z)}{f'(z)}, \quad (26)$$

where the constant $|\gamma|$ satisfies the inequality (23), is regular in U .

From (26) and (25) we have

$$h(z) = \frac{\gamma}{|\gamma|} \frac{g''(z)}{g'(z)}. \quad (27)$$

Using (27) and (22) we obtain

$$|h(z)| < 1, \quad (28)$$

for all $z \in U$ and $|h(0)| = 2|a_2|$.

The above Remark applied to the function h gives

$$\frac{1}{|\gamma|} \left| \frac{f''(z)}{f'(z)} \right| \leq \frac{|z| + 2|a_2|}{1 + 2|a_2||z|} \quad (29)$$

for all $z \in U$.

From (29) we obtain

$$\frac{1-|z|^{2b}}{b} \left| \frac{zf''(z)}{f'(z)} \right| \leq |\gamma| \frac{1-|z|^{2b}}{b} |z| \frac{|z| + 2|a_2|}{1 + 2|a_2||z|} \quad (30)$$

for all $z \in U$. Hence, we have

$$\frac{1-|z|^{2b}}{b} \left| \frac{zf''(z)}{f'(z)} \right| \leq |\gamma| \max_{|z| \leq 1} \left[\frac{1-|z|^{2b}}{b} |z| \frac{|z| + 2|a_2|}{1 + 2|a_2||z|} \right]. \quad (31)$$

From (31) and (23) we obtain

$$\frac{1 - |z|^{2b}}{b} \left| \frac{zf''(z)}{f'(z)} \right| \leq 1 \quad (32)$$

for all $z \in U$. From Theorem C, it follows that the function $G_{\beta, \gamma}$ defined by (24) is in the class S.

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