

Strong Convergence Theorems with Compact Domains

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ABSTRACT. In this paper, we prove a nonlinear strong ergodic theorem for nonexpansive mappings of a compact convex subset of a strictly convex Banach space into itself. Further, we prove a nonlinear strong ergodic theorem for a one-parameter nonexpansive semigroup.

1. INTRODUCTION

Let C be a nonempty closed convex subset of a real Banach space E . Then, a mapping $T : C \rightarrow C$ is called nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for $x, y \in C$. We denote by $F(T)$ the set of fixed points of T . Let $\mathcal{S} = \{T(s) : 0 \leq s < \infty\}$ be a family of nonexpansive mappings of C into itself such that $T(s + t) = T(s)T(t)$ for $s, t \in \mathbb{R}^+$, $t \mapsto T(t)x$ is continuous for each $x \in C$ and $T(0) = I$, where I is the identity mapping, which is called a one-parameter nonexpansive semigroup on C . Let $x \in C$. Then, for a nonexpansive mapping $T : C \rightarrow C$, the ω -limit set of x is defined by

$$\omega(x) = \{z \in C : z = \lim_{i \rightarrow \infty} T^{n_i}x \text{ with } n_i \rightarrow \infty \text{ as } i \rightarrow \infty\}.$$

Similarly, the ω -limit set of x for a one-parameter semigroup $\mathcal{S}$ on C is defined by

$$\omega(\mathcal{S}, x) = \{z \in C : z = \lim_{i \rightarrow \infty} T(s_i)x \text{ with } s_i \rightarrow \infty \text{ as } i \rightarrow \infty\}.$$

Edelstein [10] obtained the following nonlinear ergodic theorem for nonexpansive mappings with compact domains in a Banach space: Let C be a nonempty compact convex subset of a strictly convex Banach space and let T be a nonexpansive mapping of C into itself. Let $x \in C$. Then, for any $\xi \in \overline{\text{co}}\omega(x)$, the Cesàro mean $S_n(\xi) = (1/n) \sum_{k=0}^{n-1} T^k \xi$ converges strongly to some $y \in F(T)$, where $\overline{\text{co}}A$ is the closure of the convex hull of A . Dafermos and Slemrod [9] obtained the following theorem: Let C be a nonempty compact convex subset of a strictly convex Banach space and let $\mathcal{S} = \{T(t) : 0 \leq t < \infty\}$ be a one-parameter nonexpansive semigroup on C . Let $x \in C$. Then, for any $\xi \in \overline{\text{co}}\omega(\mathcal{S}, x)$, $(1/t) \int_0^t T(s)\xi ds$ converges strongly to some $y \in \bigcap_{0 \leq t < \infty} F(T(t))$. On the other hand, the

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first nonlinear ergodic theorem for nonexpansive mappings with bounded domains was established in the framework of a Hilbert space by Baillon [5]: Let C be a nonempty bounded closed convex subset of a Hilbert space and let T be a nonexpansive mapping of C into itself. Then, for any $x \in C$, the Cesàro mean $S_n(x) = (1/n) \sum_{k=0}^{n-1} T^k x$ converges weakly to some $y \in F(T)$. Bruck [7] extended Baillon's theorem to a uniformly convex Banach space whose norm is Fréchet differentiable. Brézis and Browder [6] also proved a nonlinear strong ergodic theorem for nonexpansive mappings of odd-type in a Hilbert space (see also Reich [11]). In view of Edelstein's theorem, it is natural to ask the following question: For any $x \in C$, do the Cesàro mean $S_n(x)$ converges strongly to some $z \in F(T)$?

In this paper, we give an affirmative answer to the problem, that is, using Bruck [7, 8] and Atsushiba and Takahashi [1], we prove a nonlinear strong ergodic theorem for nonexpansive mappings of a compact convex subset of a strictly convex Banach space into itself. Further, we prove a nonlinear strong ergodic theorem for a one-parameter nonexpansive semigroup.

2. STRONG ERGODIC THEOREM FOR NONEXPANSIVE MAPPINGS

Throughout the rest of this paper, we assume that a Banach space E is real and we denote by E^* the dual space of E . In addition, we denote by $\mathbb{R}^+$ and $\mathbb{N}$ the sets of all nonnegative real numbers and all positive integers, respectively. For a subset A of E , we denote by $\text{co}A$ the convex hull of A .

A Banach space E is said to be strictly convex if $\|x + y\|/2 < 1$ for $x, y \in E$ with $\|x\| = \|y\| = 1$ and $x \neq y$. In a strictly convex Banach space, we have that if

$$\|x\| = \|y\| = \|(1 - \lambda)x + \lambda y\|$$

for $x, y \in E$ and $\lambda \in (0, 1)$, then $x = y$. Throughout the rest of this paper, we assume that E is a strictly convex Banach space.

In this section, we shall give a nonlinear strong ergodic theorem for nonexpansive mappings. First, we give two lemmas which play an important role in the proof (see also [3, 4, 7, 8]).

Lemma 2.1. Let C be a nonempty compact convex subset of E . Then,

$$\lim_{n \rightarrow \infty} \sup_{\substack{y \in C \\ T \in N(C)}} \left\| \frac{1}{n} \sum_{i=0}^{n-1} T^i y - T \left(\frac{1}{n} \sum_{i=0}^{n-1} T^i y \right) \right\| = 0,$$

where $N(C)$ denotes the set of all nonexpansive mappings of C into itself.

Lemma 2.2. Let C be a nonempty compact convex subset of E and let T be a nonexpansive mapping of C into itself. Let $x \in C$ and $n \in \mathbb{N}$. Then, for any $\varepsilon > 0$, there exists $l_0 = l_0(n, \varepsilon) \in \mathbb{N}$ such that

$$\sup_{k \in \mathbb{N}} \left\| \frac{1}{n} \sum_{l=0}^{n-1} T^{l+k+m} x - T^k \left(\frac{1}{n} \sum_{l=0}^{n-1} T^{l+m} x \right) \right\| < \varepsilon$$

for every $m \geq l_0$.

Using Lemma 2.2, we can prove the following lemma (see [3]).

Lemma 2.3. Let C be a nonempty compact convex subset of E and let T be a nonexpansive mapping of C into itself. Let $x \in C$. Then, there exists a sequence $\{i_n\}$ in $\mathbb{N}$ such that for each $z \in F(T)$,

$$\lim_{n \rightarrow \infty} \left\| \frac{1}{n} \sum_{j=0}^{n-1} T^{j+i_n} x - z \right\|$$

exists.

Sketch of the proof of Lemma 2.3. From [7], we have, for any $n, m \in \mathbb{N}$

$$\begin{aligned} & \frac{1}{m} \sum_{j=0}^{m-1} T^{j+i_m+i_n} x \\ &= \frac{1}{mn} \sum_{j=1}^{n-1} (n-j) (T^{j+i_m+i_n-1} x - T^{j+i_m+i_n+m-1} x) + \frac{1}{m} \sum_{j=0}^{m-1} \frac{1}{n} \sum_{h=0}^{n-1} T^{j+h+i_m+i_n} x. \end{aligned} \quad (1)$$

Fix $z \in F(T)$. From (1) and Lemma 2.2, we obtain

$$\begin{aligned} & \left\| \frac{1}{m} \sum_{j=0}^{m-1} T^{j+i_m+i_n} x - z \right\| \\ & \leq \left\| \frac{1}{mn} \sum_{j=1}^{n-1} (n-j) (T^{j+i_m+i_n-1} x - T^{j+i_m+i_n+m-1} x) \right\| \\ & \quad + \left\| \frac{1}{m} \sum_{j=0}^{m-1} \frac{1}{n} \sum_{h=0}^{n-1} T^{j+h+i_m+i_n} x - \frac{1}{m} \sum_{j=0}^{m-1} T^{j+i_m} \left(\frac{1}{n} \sum_{h=0}^{n-1} T^{h+i_n} x \right) \right\| \\ & \quad + \left\| \frac{1}{m} \sum_{j=0}^{m-1} T^{j+i_m} \left(\frac{1}{n} \sum_{h=0}^{n-1} T^{h+i_n} x \right) - z \right\| \\ & \leq \frac{1}{nm} \sum_{j=1}^{n-1} (n-j) \cdot 2M + \varepsilon + \left\| \frac{1}{n} \sum_{h=0}^{n-1} T^{h+i_n} x - z \right\| \leq \frac{Mn}{m} + \varepsilon + \left\| \frac{1}{n} \sum_{h=0}^{n-1} T^{h+i_n} x - z \right\|, \end{aligned}$$

where $M = \sup\{\|T^j x\| : j \in \mathbb{N} \cup \{0\}\}$. Therefore, we have

$$\overline{\lim}_{m \rightarrow \infty} \left\| \frac{1}{m} \sum_{j=0}^{m-1} T^{j+i_m} x - z \right\| = \overline{\lim}_{m \rightarrow \infty} \left\| \frac{1}{m} \sum_{j=0}^{m-1} T^{j+i_m+i_n} x - z \right\| \leq \varepsilon + \left\| \frac{1}{n} \sum_{h=0}^{n-1} T^{h+i_n} x - z \right\|.$$

Then, we can show that

$$\lim_{n \rightarrow \infty} \left\| \frac{1}{n} \sum_{j=0}^{n-1} T^{j+i_n} x - z \right\|$$

exists. □

Remark 2.4. In Lemma 2.3, take a sequence $\{i_n'\}$ in $\mathbb{N}$ such that $i_n' \geq i_n$ for each $n \in \mathbb{N}$. Then, we can see that

$$\lim_{n \rightarrow \infty} \left\| \frac{1}{n} \sum_{j=0}^{n-1} T^{j+i_n} x - z \right\| = \lim_{n \rightarrow \infty} \left\| \frac{1}{n} \sum_{j=0}^{n-1} T^{j+i_n'} x - z \right\|.$$

for every $z \in F(T)$.

Now, we can show a nonlinear strong ergodic theorem for nonexpansive mappings (see [3]).

Theorem 2.5. Let E be a strictly convex Banach space and let D be a nonempty closed convex subset of E . Let T be a nonexpansive mapping of D into itself such that $T(D) \subset K$ for some compact subset K of D and let $x \in D$. Then, $(1/n) \sum_{i=0}^{n-1} T^{i+h} x$ converges strongly to a fixed point of T uniformly in $h \in \mathbb{N} \cup \{0\}$. In this case, if $Qx = \lim_{n \rightarrow \infty} (1/n) \sum_{i=0}^{n-1} T^i x$ for each $x \in D$, then Q is a nonexpansive mapping of D onto $F(T)$ such that $QT^k = T^k Q = Q$ for every $k \in \mathbb{N}$ and $Qx \in \overline{\text{co}}\{T^k x : k \in \mathbb{N}\}$ for every $x \in D$.

Sketch of the proof of Theorem 2.5. From Mazur's theorem, $C = \overline{\text{co}}(\{x\} \cup T(D))$ is a compact subset of D . We see that $C = \overline{\text{co}}(\{x\} \cup T(D))$ is convex and invariant under T and contains $\overline{\text{co}}\{T^k x : k \in \mathbb{N} \cup \{0\}\}$. Thus, we may assume that T is a nonexpansive mapping of a compact convex subset of D into itself.

From Lemma 2.3, there exists a sequence $\{i_n\}$ in $\mathbb{N}$ such that for each $z \in F(T)$,

$$\lim_{n \rightarrow \infty} \left\| \frac{1}{n} \sum_{j=0}^{n-1} T^{j+i_n} x - z \right\| \tag{2}$$

exists. From Lemma 2.1, we have

$$\lim_{n \rightarrow \infty} \left\| \frac{1}{n} \sum_{j=0}^{n-1} T^{j+i_n} x - T \left(\frac{1}{n} \sum_{j=0}^{n-1} T^{j+i_n} x \right) \right\| = 0. \tag{3}$$

Let $\{\Phi_n\} = \left\{ (1/n) \sum_{j=0}^{n-1} T^{j+i_n} x \right\}$. From the compactness, $\{\Phi_n\}$ must contain a subsequence which converges strongly to a point in C . So, let $\{\Phi_{n_k}\}$ be a subsequence of $\{\Phi_n\}$ such that $\lim_{k \rightarrow \infty} \Phi_{n_k} = y_0$. From (3), we see that y_0 is a fixed point of T . From (2), we have $\Phi_n \rightarrow y_0$. In the above argument, take a sequence $\{i_n'\}$ in $\mathbb{N}$ such that $i_n' \geq i_n$ for each $n \in \mathbb{N}$. Then, repeating the above argument, we see that $\Phi_{n'} = (1/n) \sum_{j=0}^{n-1} T^{j+i_n'} x$ converges strongly to some $y_1 \in F(T)$. From Remark 2.4, we can show $y_0 = y_1$. Since $\{i_n'\}$ is any sequence in $\mathbb{N}$ such that $i_n' \geq i_n$ for each $n \in \mathbb{N}$, we see that $(1/n) \sum_{j=0}^{n-1} T^{j+h+i_n} x$ converges strongly to y_0 uniformly in $h \in \mathbb{N} \cup \{0\}$. Then, using an idea of (1), we can prove that $(1/n) \sum_{j=0}^{n-1} T^{j+h} x$ converges strongly to y_0 uniformly in $h \in \mathbb{N} \cup \{0\}$. If $Qx = \lim_{n \rightarrow \infty} (1/n) \sum_{i=0}^{n-1} T^i x$ for each $x \in D$, then Q is a nonexpansive mapping of D onto $F(T)$ such that $QT^k = T^k Q = Q$ for every $k \in \mathbb{N}$ and $Qx \in \overline{\text{co}}\{T^k x : k \in \mathbb{N}\}$ for every $x \in D$ (for example, see [12, 13]). $\square$

We also obtain the following corollary.

Corollary 2.6. Let E, C, T and x be as in Theorem 2.5. Then, $\{T^n x : n \in \mathbb{N}\}$ is strongly convergent if and only if

$$T^{n+1}x - T^n x \rightarrow 0.$$

In this case, the limit point of $\{T^n x : n \in \mathbb{N}\}$ is a fixed point of T .

3. STRONG ERGODIC THEOREM FOR A ONE-PARAMETER NONEXPANSIVE SEMIGROUP

A family $\mathcal{S} = \{T(s) : 0 \leq s < \infty\}$ of mappings of C into itself is called a one-parameter nonexpansive semigroup on C if it satisfies the following conditions:

- (i) $T(0)x = x$ for all $x \in C$;
- (ii) $T(s+t) = T(s)T(t)$ for all $s, t \in \mathbb{R}^+$;
- (iii) $\|T(s)x - T(s)y\| \leq \|x - y\|$ for all $x, y \in C$ and $s \in \mathbb{R}^+$;
- (iv) for each $x \in C$, $s \mapsto T(s)x$ is continuous.

We denote by $F(\mathcal{S})$ the set of common fixed points of $T(t), t \in \mathbb{R}^+$, that is, $F(\mathcal{S}) = \bigcap_{0 \leq t < \infty} F(T(t))$.

In this section, we give a strong ergodic theorem for a one-parameter nonexpansive semigroup. For a compact subset of a strictly convex Banach space, we obtained the following two lemmas (see [3]):

Lemma 3.1. Let C be a nonempty compact convex subset of E and let $n \in \mathbb{N}$. Then, there exists a strictly increasing continuous, convex function $\gamma_n : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ such that

$\gamma_n(0) = 0$ and

$$\gamma_n \left(\left\| \sum_{i=1}^n \lambda_i T y_i - T \left(\sum_{i=1}^n \lambda_i y_i \right) \right\| \right) \leq \max_{1 \leq i, j \leq n} (\|y_i - y_j\| - \|T y_i - T y_j\|)$$

for every nonexpansive mapping T of C into itself, every sequence $\{\lambda_i\}_{i=1}^n$ in $\mathbb{R}^+$ with $\sum_{i=1}^n \lambda_i = 1$ and $\{y_i\}_{i=1}^n$ in C .

Lemma 3.2. Let C be a nonempty compact convex subset of E . For any $\varepsilon > 0$, there exists $\delta > 0$ such that for any nonexpansive mapping T of C into itself,

$$\overline{\text{co}}F_\delta(T) \subset F_\varepsilon(T).$$

Using Lemmas 2.1 and 3.2, we obtain the following lemma (see [2, 4]).

Lemma 3.3. Let C be a nonempty compact convex subset of E and let $\mathcal{S} = \{T(t) : 0 \leq t < \infty\}$ be a one-parameter nonexpansive semigroup on C . Then, for any $h \in \mathbb{R}^+$,

$$\limsup_{t \rightarrow \infty} \sup_{y \in C} \left\| \frac{1}{t} \int_0^t T(s)y ds - T(h) \left(\frac{1}{t} \int_0^t T(s)y ds \right) \right\| = 0.$$

Sketch of the proof of Lemma 3.3. Let $\varepsilon > 0$ and $h \in \mathbb{R}^+$. From Lemma 3.2, there exists $\delta > 0$ such that $\overline{\text{co}}F_\delta(T) \subset F_\varepsilon(T)$ for every nonexpansive mapping T of C into itself. From Lemma 2.1, there exists $n_1 \in \mathbb{N}$ such that

$$\sup_{\substack{y \in C \\ s \in \mathbb{R}^+}} \left\| \frac{1}{n} \sum_{i=0}^{n-1} T(hi + s)y - T(h) \left(\frac{1}{n} \sum_{i=0}^{n-1} T(hi + s)y \right) \right\| < \delta$$

for every $n \geq n_1$. Then, we obtain

$$\frac{1}{n} \sum_{i=0}^{n-1} T(hi + s)y \in F_\delta(T(h)) \subset \overline{\text{co}}F_\delta(T(h)) \quad (4)$$

for every $s \in \mathbb{R}^+, n \geq n_1$ and $y \in C$. Let $n \geq n_1$. Then, we have that for any $t \in \mathbb{R}^+$ with $t > h(n-1)$ and $y \in C$,

$$\begin{aligned} & \left\| \frac{1}{t} \int_0^t T(s)y ds - T(h) \left(\frac{1}{t} \int_0^t T(s)y ds \right) \right\| \\ & \leq \frac{2}{n} \sum_{i=0}^{n-1} \left\| \frac{1}{t} \int_0^t T(s)y ds - \frac{1}{t} \int_0^t T(hi + s)y ds \right\| \\ & \quad + \left\| \frac{1}{n} \sum_{i=0}^{n-1} \frac{1}{t} \int_0^t T(hi + s)y ds - T(h) \left(\frac{1}{n} \sum_{i=0}^{n-1} \frac{1}{t} \int_0^t T(hi + s)y ds \right) \right\| \end{aligned}$$

and

$$\frac{1}{n} \sum_{i=0}^{n-1} \left\| \frac{1}{t} \int_0^t T(s)y ds - \frac{1}{t} \int_{hi}^{t+hi} T(s)y ds \right\| \leq \frac{M_0 h(n-1)}{t},$$

where $M_0 = \sup_{z \in C} \|z\|$. Using (4) and the separation theorem, we can prove that there exists $t_0 \in \mathbb{R}^+$ with $t_0 > h(n-1)$ such that $\frac{1}{n} \sum_{i=0}^{n-1} \frac{1}{t} \int_0^t T(hi+s)y ds \in \overline{\text{co}}F_\delta(T(h))$ for all $y \in C$ and $t \geq t_0$. From $\overline{\text{co}}F_\delta(T(h)) \subset F_\varepsilon(T(h))$, we have

$$\left\| \frac{1}{t} \int_0^t T(s)y ds - T(h) \left(\frac{1}{t} \int_0^t T(s)y ds \right) \right\| \leq \frac{2M_0 h(n-1)}{t} + \varepsilon$$

for $t \geq t_0$. Since $y \in C$ is arbitrary, we have

$$\lim_{t \rightarrow \infty} \sup_{y \in C} \left\| \frac{1}{t} \int_0^t T(s)y ds - T(h) \left(\frac{1}{t} \int_0^t T(s)y ds \right) \right\| = 0. \quad \square$$

Lemma 3.4. Let C be a nonempty compact convex subset of E and let $\mathcal{S} = \{T(s) : 0 \leq s < \infty\}$ be a one-parameter nonexpansive semigroup on C . Let $x \in C$ and $t > 0$. Then, for any $\varepsilon > 0$, there exists $p_t = p_t(\varepsilon) \in \mathbb{R}^+$ such that

$$\sup_{h \in \mathbb{R}^+} \left\| \frac{1}{t} \int_0^t T(h+p+\tau)x d\tau - T(h) \left(\frac{1}{t} \int_0^t T(p+\tau)x d\tau \right) \right\| < \varepsilon$$

for every $p \geq p_t$.

Sketch of the proof of Lemma 3.4. Let $t > 0$ and $\varepsilon > 0$. We know that there exists $\delta_1 = \delta_1(\varepsilon) > 0$ such that $\|T(s_1)x - T(s_2)x\| < \varepsilon/3$ if $|s_1 - s_2| \leq \delta_1$. Choose $N = N(t, \varepsilon) \in \mathbb{N}$ such that $N > t/\delta_1$ and $\left\| \frac{1}{t} \int_0^t T(\tau)x d\tau - \frac{1}{t} \sum_{i=1}^N T\left(\frac{it}{N}\right)x \right\| < \frac{\varepsilon}{3}$. Then, we can show, for each $h, p \in \mathbb{R}^+$,

$$\left\| \frac{1}{t} \int_0^t T(h+p+\tau)x d\tau - \frac{1}{t} \sum_{i=1}^N T\left(h+p+\frac{it}{N}\right)x \right\| < \frac{\varepsilon}{3}. \quad (5)$$

Hence, for each $p \in \mathbb{R}^+$,

$$\left\| \frac{1}{t} \int_0^t T(p+\tau)x d\tau - \frac{1}{t} \sum_{i=1}^N T\left(p+\frac{it}{N}\right)x \right\| < \frac{\varepsilon}{3}. \quad (6)$$

We see that for each $i, j \in \{1, 2, \dots, N\}$,

$$\lim_{s \rightarrow \infty} \left\| T\left(s + \frac{it}{N}\right)x - T\left(s + \frac{jt}{N}\right)x \right\| = \lim_{s \rightarrow \infty} \left\| T(s)T\left(\frac{t}{N}\right)^i x - T(s)T\left(\frac{t}{N}\right)^j x \right\|$$

exists. Let γ_N be as in Lemma 3.1. Since γ_N^{-1} is continuous and $\gamma_N^{-1}(0) = 0$, there exists $\delta_2 = \delta_2(\varepsilon) > 0$ such that $\gamma_N^{-1}(\delta) < \varepsilon/3$ for every δ with $0 \leq \delta \leq \delta_2$. Then, there exists $p_1 = p_1(\varepsilon, i, j, t) \in \mathbb{R}^+$ such that

$$0 \leq \left\| T\left(s + \frac{it}{N}\right)x - T\left(s + \frac{jt}{N}\right)x \right\| - \left\| T\left(q + s + \frac{it}{N}\right)x - T\left(q + s + \frac{jt}{N}\right)x \right\| < \delta_2$$

for every $s \geq p_1$ and $q \in \mathbb{R}^+$. Let $p_t = \max\{p_1(\varepsilon, i, j, t) : 1 \leq i, j \leq N\}$. It follows from Lemma 3.1 that

$$\begin{aligned} & \left\| \frac{1}{N} \sum_{i=1}^N T(h)T\left(p + \frac{it}{N}\right)x - T(h) \left(\frac{1}{N} \sum_{i=1}^N T\left(p + \frac{it}{N}\right)x \right) \right\| \\ & \leq \gamma_N^{-1} \left(\max_{1 \leq i, j \leq N} \left(\left\| T\left(p + \frac{it}{N}\right)x - T\left(p + \frac{jt}{N}\right)x \right\| - \left\| T\left(h + p + \frac{it}{N}\right)x - T\left(h + p + \frac{jt}{N}\right)x \right\| \right) \right) \\ & < \gamma_N^{-1}(\delta_2) < \frac{\varepsilon}{3} \end{aligned} \quad (7)$$

for every $i, j \in \{1, 2, \dots, N\}$, $h \in \mathbb{R}^+$ and $p \geq p_t$. Therefore, from (5), (6) and (7), we have

$$\left\| \frac{1}{t} \int_0^t T(h + p + \tau)x d\tau - T(h) \left(\frac{1}{t} \int_0^t T(p + \tau)x d\tau \right) \right\| < 3 \cdot \frac{\varepsilon}{3} = \varepsilon$$

for every $h \in \mathbb{R}^+$ and $p \geq p_t$. □

Using Lemma 3.4, we can show the following lemma (see [4]).

Lemma 3.5. Let C be a nonempty compact convex subset of E and let $\mathcal{S} = \{T(s) : 0 \leq s < \infty\}$ be a one-parameter nonexpansive semigroup on C . Let $x \in C$. Then, there exists a net $\{p_t\}$ in $\mathbb{R}^+$ such that for each $z \in F(\mathcal{S})$,

$$\lim_{t \rightarrow \infty} \left\| \frac{1}{t} \int_0^t T(\tau + p_t)x d\tau - z \right\|$$

exists.

Sketch of the proof of Lemma 3.5. Let $\varepsilon > 0$. From Lemma 3.4, for any $t > 0$, there exists $p_t \in \mathbb{R}^+$ such that

$$\left\| T(h) \left(\frac{1}{t} \int_0^t T(p + \tau)x d\tau \right) - \frac{1}{t} \int_0^t T(h + p + \tau)x d\tau \right\| < \varepsilon \quad (8)$$

for every $p \geq p_t$ and $h \in \mathbb{R}^+$. From an idea of [7], we have, for any $t, s > 0$,

$$\begin{aligned} & \frac{1}{t} \int_0^t T(\tau + p_t + p_s)x d\tau \\ &= \frac{1}{st} \int_0^s (s - \eta) [T(\eta + p_t + p_s)x - T(\eta + p_t + p_s + t)x] d\eta + \frac{1}{t} \int_0^t \left(\frac{1}{s} \int_0^s T(\tau + \eta + p_t + p_s)x d\eta \right) d\tau. \end{aligned} \quad (9)$$

Fix $z \in F(\mathcal{S})$ and $t, s > 0$. Put $M_0 = \sup\{\|v\| : v \in C\}$. Then, we have

$$\left\| \frac{1}{st} \int_0^s (s - \eta) [T(\eta + p_t + p_s)x - T(\eta + p_t + p_s + t)x] d\eta \right\| \leq \frac{2M_0}{st} \int_0^s ((s - \eta)d\eta) \leq \frac{M_0 s}{t}. \quad (10)$$

From (8), we have, for $t > 0$ with $t \geq p_s$,

$$\begin{aligned} & \left\| \frac{1}{t} \int_0^t \left(\frac{1}{s} \int_0^s T(\tau + \eta + p_t + p_s)x d\eta - z \right) d\tau \right\| \\ & \leq \left\| \frac{1}{t} \int_0^t \left(\frac{1}{s} \int_0^s T(\tau + p_t + \eta + p_s)x d\eta \right) d\tau - \frac{1}{t} \int_0^t T(\tau + p_t) \left(\frac{1}{s} \int_0^s T(\eta + p_s)x d\eta \right) d\tau \right\| \\ & \quad + \left\| \frac{1}{t} \int_0^t \left(T(\tau + p_t) \left(\frac{1}{s} \int_0^s T(\eta + p_s)x d\eta \right) - z \right) d\tau \right\| \\ & < \varepsilon + \left\| \frac{1}{s} \int_0^s T(\eta + p_s)x d\eta - z \right\|. \end{aligned} \quad (11)$$

Hence, from (9), (10) and (11), we have

$$\begin{aligned} \overline{\lim}_{t \rightarrow \infty} \left\| \frac{1}{t} \int_0^t T(\tau + p_t)x d\tau - z \right\| &= \overline{\lim}_{t \rightarrow \infty} \left\| \frac{1}{t} \int_0^t T(\tau + p_t + p_s)x d\tau - z \right\| \\ &\leq \varepsilon + \left\| \frac{1}{s} \int_0^s T(\eta + p_s)x d\eta - z \right\|. \end{aligned}$$

Then, we can show that

$$\lim_{t \rightarrow \infty} \left\| \frac{1}{t} \int_0^t T(\tau + p_t)x d\tau - z \right\|$$

exists for each $z \in F(\mathcal{S})$. $\square$

Remark 3.6. In Lemma 2.3, take a net $\{p_t'\}$ in $\mathbb{R}^+$ such that $p_t' \geq p_t$ for each $t > 0$. Then, we can see

$$\lim_{t \rightarrow \infty} \left\| \frac{1}{t} \int_0^t T(\tau + p_t)x d\tau - z \right\| = \lim_{t \rightarrow \infty} \left\| \frac{1}{t} \int_0^t T(\tau + p_t')x d\tau - z \right\|$$

for every $z \in F(\mathcal{S})$.

Now, we can show a nonlinear strong ergodic theorem for a one-parameter nonexpansive semigroup (see [4]).

Theorem 3.7. Let E be a strictly convex Banach space and let C be a nonempty compact convex subset of E . Let $S = \{T(t) : 0 \leq t < \infty\}$ be a one-parameter nonexpansive semigroup on C and let $x \in C$. Then, $(1/t) \int_0^t T(\tau + h)x d\tau$ converges strongly to a common fixed point of $T(t), t \in \mathbb{R}^+$ uniformly in $h \in \mathbb{R}^+$. In this case, if $Qx = \lim_{t \rightarrow \infty} (1/t) \int_0^t T(\tau)x d\tau$ for each $x \in C$, then Q is a nonexpansive mapping of C onto $F(S)$ such that $QT(q) = T(q)Q = Q$ for every $q \in \mathbb{R}^+$ and $Qx \in \overline{\text{co}}\{T(s)x : 0 \leq s < \infty\}$ for every $x \in C$.

Sketch of the proof of Theorem 3.7. From Lemma 3.5, there exists a net $\{p_t\}$ in $\mathbb{R}^+$ such that for each $z \in F(S)$,

$$\lim_{t \rightarrow \infty} \left\| \frac{1}{t} \int_0^t T(\tau + p_t)x d\tau - z \right\| \quad (12)$$

exists. From Lemma 3.3, we have, for any $q \in \mathbb{R}^+$,

$$\limsup_{t \rightarrow \infty} \sup_{y \in C} \left\| \frac{1}{t} \int_0^t T(\tau + p_t)y d\tau - T(q) \left(\int_0^t T(\tau + p_t)y d\tau \right) \right\| = 0. \quad (13)$$

Let $\{\Phi_t\} = \{(1/t) \int_0^t T(\tau + p_t)x d\tau\}$. From compactness of C , $\{\Phi_t\}$ must contain a subnet which converges strongly to a point in C . So, let $\{\Phi_{t_\alpha}\}$ be a subnet of $\{\Phi_t\}$ such that $\lim_\alpha \Phi_{t_\alpha} = y_0 \in C$. From (13), we can show that y_0 is a common fixed point of $T(t), t \in \mathbb{R}^+$. From (12), we can prove that $\Phi_t \rightarrow y_0 \in F(S)$. In the above argument, take a net $\{p'_t\}$ in $\mathbb{R}^+$ such that $p'_t \geq p_t$ for each $t > 0$. Then, repeating the above argument, we see that $\Phi'_t = (1/t) \int_0^t T(\tau + p'_t)x d\tau$ converges strongly to some $y_1 \in F(S)$. Using Remark 3.6, we can show $y_0 = y_1$. Since $\{p'_t\}$ is any net in $\mathbb{R}^+$ such that $p'_t \geq p_t$ for each $t > 0$, we see that $(1/t) \int_0^t T(\tau + p_t + h)x d\tau$ converges strongly to y_0 uniformly in $h \in \mathbb{R}^+$. Then, using an idea of (9), we can prove that $(1/t) \int_0^t T(\tau + h)x d\tau$ converges strongly to y_0 uniformly in $h \in \mathbb{R}^+$. If $Qx = \lim_{t \rightarrow \infty} (1/t) \int_0^t T(\tau)x d\tau$ for each $x \in C$, then Q is a nonexpansive mapping of C onto $F(S)$ such that $QT(q) = T(q)Q = Q$ for every $q \in \mathbb{R}^+$ and $Qx \in \overline{\text{co}}\{T(s)x : 0 \leq s < \infty\}$ for every $x \in C$. $\square$

We also obtain the following corollary.

Corollary 3.8. Let E, C, x and $S = \{T(t) : 0 \leq t < \infty\}$ be as in Theorem 3.7. Then, $\{T(t)x : 0 \leq t < \infty\}$ is strongly convergent if and only if

$$T(s+t)x - T(t)x \rightarrow 0 \quad \text{for every } s \in \mathbb{R}^+.$$

In this case, the limit point of $\{T(t)x : 0 \leq t < \infty\}$ is a common fixed point of $T(t), t \in \mathbb{R}^+$.

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