

ON THE GROUP OF HOLOMORPHIC AND ANTI-HOLOMORPHIC AUTOMORPHISMS OF A COMPACT HERMITIAN SYMMETRIC SPACE

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ABSTRACT. Let f be a complex function on a domain in the complex plane $\mathbb{C}$. Then f is holomorphic or anti-holomorphic, if and only if f is a conformal map. we are interested in generalizing this to higher dimensional cases. In this paper, for a compact irreducible Hermitian symmetric space M , we determine the group $H^\pm(M)$ of all holomorphic and anti-holomorphic automorphisms of M , and we characterize the group $H^\pm(M)$ as the automorphism group of a certain G -structure on M , called the *generalized conformal structure*. This paper is a short-cut version; the detailed one will appear elsewhere.

1. SIMPLE GRADED LIE ALGEBRAS AND COMPACT HERMITIAN SYMMETRIC SPACES

1.1.

- Let

$$\tilde{\mathfrak{g}} = \tilde{\mathfrak{g}}_{-1} + \tilde{\mathfrak{g}}_0 + \tilde{\mathfrak{g}}_1. \quad (1.1)$$

be a complex simple graded Lie algebra (abbrev. GLA).

- $Z \in \tilde{\mathfrak{g}}_0$ is the characteristic element of $\tilde{\mathfrak{g}}$, that is, $\text{ad } Z = k1$ on $\tilde{\mathfrak{g}}_k$, $k = 0, \pm 1$.
- τ is the grade-reversing Cartan involution of $\tilde{\mathfrak{g}}$, that is, $\tau(\tilde{\mathfrak{g}}_k) = \tilde{\mathfrak{g}}_{-k}$ $k = 0, \pm 1$, which is equivalent to $\tau(Z) = -Z$. Note that τ is a conjugation of $\tilde{\mathfrak{g}}$ with respect to a compact real form $\mathfrak{k}$ of $\tilde{\mathfrak{g}}$.
- $\text{Aut } \tilde{\mathfrak{g}}(\subset \text{GL}(\tilde{\mathfrak{g}}))$: the automorphism group of the complex Lie algebra $\tilde{\mathfrak{g}}$.
- $\tilde{G}_0 := \text{Aut}_{\text{gr}} \tilde{\mathfrak{g}} := \{g \in \text{Aut } \tilde{\mathfrak{g}} : g(\tilde{\mathfrak{g}}_k) = \tilde{\mathfrak{g}}_k, k = 0, \pm 1\}$: the group of grade-preserving automorphisms of $\tilde{\mathfrak{g}}$.

$\tilde{G}_0$ coincides with the centralizer $C_{\text{Aut } \tilde{\mathfrak{g}}}(Z)$ of Z in $\text{Aut } \tilde{\mathfrak{g}}$.

Note that $\text{Lie } \tilde{G}_0 = \tilde{\mathfrak{g}}_0$.

- $\tilde{U} := \tilde{G}_0 \exp \tilde{\mathfrak{g}}_{-1}$.
- $\tilde{G} := \tilde{G}_0 \text{Int } \tilde{\mathfrak{g}}$: an open subgroup of $\text{Aut } \tilde{\mathfrak{g}}$.
- $\tilde{U}$ is a parabolic subgroup of $\tilde{G}$, and $\tilde{G}_0$ is the Levi subgroup of $\tilde{U}$.
- We have the (complex) flag manifold $M = \tilde{G}/\tilde{U}$. It can be shown that $\tilde{G}$ acts on M effectively.
- The symmetric space expression of M .

$\tilde{\tau}$: the Cartan involution of $\tilde{G}$ defined by $\tilde{\tau}(g) = \tau g \tau$, $g \in \tilde{G}$.

Then the set K of all $\tilde{\tau}$ -fixed elements in $\tilde{G}$ is a compact real form of $\tilde{G}$. Note that $\text{Lie } K = \mathfrak{k}$. M is expressed as

$$M = \tilde{G}/\tilde{U} = K/K_0,$$

where $K_0 = K \cap \tilde{U}$. Here K/K_0 is a compact irreducible Hermitian symmetric space. K/K_0 has a K -invariant Kähler-Einstein metric (cf. [5]).

- The identity component of K coincides with that of the isometry group $I(M)$.

Theorem 1.1. *Let $\text{Hol}^+(M)$ be the group of all holomorphic automorphisms of $M = \tilde{G}/\tilde{U}$. Then we have*

$$\text{Hol}^+(M) = \tilde{G}.$$

Proof. (Sketch)

There are four steps. First of all, $\text{Hol}^+(M)$ is a complex Lie group by a theorem of Bochner-Montgomery ([1, 2]).

(1) As was noted before, $\tilde{G}$ acts on M effectively and holomorphically. Hence $\tilde{G} \subset \text{Hol}^+(M)$.

(2) The existence of the K -invariant Kähler-Einstein metric on M implies that

$$\text{Lie } \text{Hol}^+(M) = (\text{Lie } I(M))^{\mathbb{C}} = \mathfrak{k}^{\mathbb{C}} = \tilde{\mathfrak{g}},$$

by Matsushima [6]. Thus $\tilde{G}$ is an open subgroup of $\text{Hol}^+(M)$.

(3) One can show that the center of $\text{Hol}^+(M)$ reduces to the identity. Therefore $\text{Hol}^+(M)$ is realized as an open subgroup of $\text{Aut } \tilde{\mathfrak{g}}$ by taking the adjoint representation of $\text{Hol}^+(M)$ on $\tilde{\mathfrak{g}}$.

(4) M has the coset space expression in two ways:

$$M = \tilde{G}/\tilde{U} = \text{Hol}^+(M)/\hat{U},$$

where $\hat{U} \supset \tilde{U}$. It is easy to see that $\hat{U} = \tilde{U}$, which shows the coincidence of the numerators. $\square$

1.2. Here we consider the scalar restrictions of the objects in 1.1 to $\mathbb{R}$.

- Let

$$\mathfrak{g} = \mathfrak{g}_{-1} + \mathfrak{g}_0 + \mathfrak{g}_1$$

be the real simple GLA, which is the scalar restriction of the complex GLA (1.1) to $\mathbb{R}$.

Let I be the complex structure on $\mathfrak{g}$ corresponding to the i -multiplication on $\tilde{\mathfrak{g}}$.

$\tilde{\mathfrak{g}}$ can be expressed as the pair $(\mathfrak{g}, I)$.

- $Z \in \mathfrak{g}$ and τ are the same as those for $\tilde{\mathfrak{g}}$.
- $\text{Aut } \mathfrak{g}(\subset \text{GL}(\mathfrak{g}))$: the automorphism group of the real Lie algebra $\mathfrak{g}$.
Note that $\text{Aut } \tilde{\mathfrak{g}} \subset \text{Aut } \mathfrak{g}$.
- $G_0 := \text{Aut}_{\text{gr}} \mathfrak{g}$. Note that the inclusion $\tilde{G}_0 \subset G_0$ and $\text{Lie } G_0 = \mathfrak{g}_0$ are valid.
- $U := G_0 \exp \mathfrak{g}_{-1} \supset \tilde{U}$.
- The open subgroup G of $\text{Aut } \mathfrak{g}$:
 $\text{Aut } \mathfrak{g} \supset G := G_0 \text{Int } \mathfrak{g} \supset \tilde{G}$.
 U is a parabolic subgroup of G , and G_0 is the Levi subgroup of U .
- As a real manifold, M is expressed as a (real) flag manifold G/U .

This is non-trivial, and will be proved in Corollary 2.4.

The following theorem will be proved in the section 3.

Theorem 1.2. *Let $\text{Hol}^\pm(M)$ be the group of all holomorphic or anti-holomorphic automorphisms of M . Then we have*

$$\text{Hol}^\pm(M) = G.$$

2. THE RELATION BETWEEN THE GROUPS $\tilde{G}$ AND G

Lemma 2.1. *Let $\tilde{\mathfrak{g}} = \tilde{\mathfrak{g}}_{-1} + \tilde{\mathfrak{g}}_0 + \tilde{\mathfrak{g}}_1$ be a complex simple GLA and let Z and τ be as before. Then there exists a unique normal real form $\mathfrak{g}^N$ of $\tilde{\mathfrak{g}}$ such that $Z \in \mathfrak{g}^N$ and that $\tau(\mathfrak{g}^N) \subset \mathfrak{g}^N$.*

$\mathfrak{g}^N$ can be expressed as a GLA

$$\mathfrak{g}^N = \mathfrak{g}_{-1}^N + \mathfrak{g}_0^N + \mathfrak{g}_1^N,$$

where $\mathfrak{g}_k^N = \mathfrak{g}^N \cap \tilde{\mathfrak{g}}_k$ ($k = 0, \pm 1$).

Now let ν be the conjugation of $(\mathfrak{g}, I)$ with respect to $\mathfrak{g}^N$. Then ν satisfies the following equalities:

$$\nu^2 = 1, \quad \nu I = -I\nu.$$

Since $\nu(Z) = Z$, ν is grade-preserving on $\mathfrak{g}$. Hence we have

$$\nu \in G_0 \setminus \tilde{G}_0, \quad \nu \in \text{Aut } \mathfrak{g} \setminus \text{Aut } \tilde{\mathfrak{g}}.$$

Let $\bar{\mathfrak{g}}$ be the complexification of $\mathfrak{g}$. We extend ν $\mathbb{C}$ -linearly to $\bar{\mathfrak{g}}$.

Proposition 2.2.

$$\text{Aut } \mathfrak{g} = (\text{Aut } \tilde{\mathfrak{g}}) \cdot \langle \nu \rangle. \quad (2.1)$$

Proof. Let Π be the Dynkin diagram of the complex simple Lie algebra $\tilde{\mathfrak{g}}$. Then it is well-known that

$$\text{Aut } \tilde{\mathfrak{g}} / \text{Int } \tilde{\mathfrak{g}} = \text{Aut}(\Pi). \quad (2.2)$$

The Satake diagram of the real simple Lie algebra $\mathfrak{g}$ is given by the pair $(\bar{\Pi}, \nu)$, where $\bar{\Pi}$ is the Dynkin diagram of $\bar{\mathfrak{g}}$ which is the pair of two copies of Π . ν acts on $\bar{\Pi}$ as the Satake involution. Now let us denote by $(\text{Aut } \mathfrak{g})^z$ the Zariski connected component of $\text{Aut } \mathfrak{g}$. Then we see that $(\text{Aut } \mathfrak{g})^z = \text{Int } \tilde{\mathfrak{g}}$. Applying a result of H. Matsumoto ([7]) we conclude that

$$\text{Aut } \mathfrak{g} / \text{Int } \tilde{\mathfrak{g}} = \text{Aut } \mathfrak{g} / (\text{Aut } \mathfrak{g})^z = \text{Aut}(\bar{\Pi}, \nu) = \langle \nu \rangle (\text{Aut}(\Pi)). \quad (2.3)$$

(2.1) follows from (2.2) and (2.3). $\square$

From Proposition 2.2 we have

Theorem 2.3. (1) $G_0 = \tilde{G}_0 \cdot \langle \nu \rangle$,
 (2) $U = \tilde{U} \cdot \langle \nu \rangle$,
 (3) $G = \tilde{G} \cdot \langle \nu \rangle$. In particular, $\tilde{G}$ is a normal subgroup of G .

Corollary 2.4. *The complex flag manifold M is expressed as the real flag manifold*

$$M = \tilde{G} / \tilde{U} = G / U.$$

Proof. By Theorem 2.3, we have $G = \tilde{G}U$. Consequently we get

$$G/U = \tilde{G}U/U = \tilde{G}/\tilde{G} \cap U = \tilde{G}/\tilde{U} = M.$$

□

3. THE PROOF OF THEOREM 1.2

Definition 3.1. Let X be a smooth manifold, I a complex structure on X and let $\sigma : X \rightarrow X$ be a diffeomorphism. Then σ is said to be an *anti-holomorphic involution*, if the following conditions are satisfied on X

$$\sigma^2 = 1, \quad \sigma_* I = -I \sigma_*,$$

where σ_* is the differential of σ . The pair (σ, I) is called an *anti-holomorphic pair* (shortly, AHP).

3.1. The AHP $(\tilde{\nu}, \tilde{I})$ on $\tilde{G}$

We identify the Lie algebra $(\mathfrak{g}, I)$ with the Lie algebra of left-invariant vector fields on $\tilde{G}$. The complex structure I on $\mathfrak{g}$ and the left-invariant complex structure $\tilde{I}$ on $\tilde{G}$ are related with each other by the equality

$$\tilde{I}_p X_p = (IX)_p, \quad p \in \tilde{G}, X \in \mathfrak{g},$$

which is also expressed as

$$\tilde{I}X = IX, \tag{3.1}$$

where both sides are vector fields on $\tilde{G}$.

Next, noting that $\nu\tilde{G}\nu^{-1} \subset \tilde{G}$, we define the automorphism $\tilde{\nu} : \tilde{G} \rightarrow \tilde{G}$ as

$$\tilde{\nu}(a) = \nu a \nu^{-1}, \quad a \in \tilde{G}. \tag{3.2}$$

Then $\tilde{\nu}$ is naturally extended to the whole G .

Lemma 3.2. $(\tilde{\nu}, \tilde{I})$ is an AHP on $\tilde{G}$

Proof. Note that $\tilde{\nu}_* = \nu$. By using this equality, (3.1) and the anti-linearity of ν , we can conclude the equality $\tilde{\nu}_* \tilde{I} = -\tilde{I} \tilde{\nu}_*$. □

3.2. The AHP (ν_M, J) on M

First of all, note that

$$\tilde{\nu}(\tilde{U}) = \nu \tilde{U} \nu^{-1} = \tilde{U}. \tag{3.3}$$

The left action of ν on G/U at a point gU ($g \in G$) can be expressed as

$$\nu(gU) = \nu g U = \nu g \nu^{-1} \nu U \nu^{-1} = \nu g \nu^{-1} U = \tilde{\nu}(g)U.$$

Restricting this equality to $\tilde{G}/\tilde{U}$, we have the following action of ν on $\tilde{G}/\tilde{U}$:

$$\nu(a\tilde{U}) = \tilde{\nu}(a)\tilde{U}, \quad a \in \tilde{G}. \tag{3.4}$$

In the following, the ν acting on $\tilde{G}/\tilde{U}$ will be denoted by ν_M .

Let $\pi : \tilde{G} \rightarrow M = \tilde{G}/\tilde{U}$ be the natural projection. Then the following commutativity follows from (3.4):

$$\pi\tilde{\nu} = \nu_M\pi. \quad (3.5)$$

Next we will define the invariant complex structure J on $M = \tilde{G}/\tilde{U}$, which is π -related to $\tilde{I}$. We consider the following identification for the complex tangent space of M at the origin o :

$$T_o(M) = \text{Lie } \tilde{G} / \text{Lie } \tilde{U} = \tilde{\mathfrak{g}}_1 = \mathfrak{g}_1^N + I\mathfrak{g}_1^N.$$

The complex structure J_o on $\tilde{\mathfrak{g}}_1$ is given by

$$J_o = I|_{\tilde{\mathfrak{g}}_1} = \text{ad}_{\tilde{\mathfrak{g}}_1}(iZ).$$

J_o commutes with the linear isotropy representation of $\tilde{U}$, that is,

$$[\text{Ad}_{\tilde{\mathfrak{g}}_1} \tilde{G}_0, J_o] = 0.$$

Therefore J_o extends uniquely to a $\tilde{G}$ -invariant almost complex structure J on M . It can be seen from the construction that $\tilde{I}$ and J are π -related, that is,

$$\pi_*\tilde{I} = J\pi_*. \quad (3.6)$$

It follows from (3.6) that the almost complex structure J is integrable.

Proposition 3.3. (ν_M, J) is an AHP on M .

Proof. In view of (3.5), (3.6) and Lemma 3.2, we have

$$\nu_{M*}J\pi_* = \nu_{M*}\pi_*\tilde{I} = \pi_*\tilde{\nu}_*\tilde{I} = -\pi_*\tilde{I}\tilde{\nu}_* = -J\pi_*\tilde{\nu}_* = -J\nu_{M*}\pi_*.$$

Therefore we have the equality $\nu_{M*}J = -J\nu_{M*}$. □

Proof of Theorem 1.2

We denote by $\text{Hol}^-(M)$ the totality of anti-holomorphic automorphisms of M . Since ν_M interchanges $\text{Hol}^+(M)$ with $\text{Hol}^-(M)$, we have the expression

$$\text{Hol}^\pm(M) = \text{Hol}^+(M) \amalg \nu_M \text{Hol}^+(M). \quad (3.7)$$

As is seen in the proof of Theorem 1.1, $\text{Hol}^+(M)$, realized as a subgroup of $\text{Aut } \tilde{\mathfrak{g}}$, coincides with $\tilde{G}$. Also ν is the realization of ν_M as an element of G . Therefore, considering (3.7) and Theorem 1.1, we have

$$\text{Hol}^\pm(M) = \tilde{G} \amalg \nu\tilde{G} = \tilde{G} \cdot \langle \nu \rangle = G.$$

4. RELATION TO THE GENERALIZED CONFORMAL STRUCTURE ON M

First of all, let us remind the basic facts on the generalized conformal structure (simply, GCS) on the real flag manifold $M = G/U$ (cf. [3]). Let r be the rank of the symmetric space M , and let o be the origin of the coset space $M = G/U$. As for the case of the complex tangent space $T_o(M)$, the real tangent space at the origin $o \in M$ can be identified with $\mathfrak{g}_1$. Let ρ be the linear isotropy representation of U on $\mathfrak{g}_1$. Then we have $\rho(U) = G_0$. The G_0 -orbit decomposition of $\mathfrak{g}_1$ is given by

$$\mathfrak{g}_1 = V_r \amalg V_{r-1} \amalg \dots \amalg V_0,$$

where V_r is a single open orbit and $V_0 = (0)$. Since G_0 contains $\mathbb{C}^*$, all orbits are cones. The union of singular orbits, denoted by C_o , is an algebraic cone. The automorphism group $\text{Aut } C_o$ is defined as the subgroup of $GL(\mathfrak{g}_1)$ consisting of all elements leaving C_o stable.

Lemma 4.1. ([3]) *Suppose that $r \geq 2$. Then we have*

$$\text{Aut } C_o = G_0.$$

By this lemma, one can translate the cone C_o to each point of M by the action of G . Thus we have the cone field $\mathcal{C} = \{C_p\}_{p \in M}$ on M , which is called the generalized conformal structure (simply GCS) on M . Now we are going to define the automorphism group $\text{Aut}(M, \mathcal{C})$ of the GCS $\mathcal{C}$. $\text{Aut}(M, \mathcal{C})$ is defined to be the group of all smooth diffeomorphisms f of M leaving $\mathcal{C}$ invariant, in other words, for $\mathcal{C} = \{C_p\}_{p \in M}$, f satisfies

$$f_{*p}C_p = C_{f(p)}, \quad p \in M.$$

We can characterize the group G as the automorphism group of the GCS, namely,

Theorem 4.2. ([3]) *Let G be as above. Suppose that $r \geq 2$. Then*

$$\text{Aut}(M, \mathcal{C}) = G.$$

Combining the above theorem with Theorem 1.2, we have

Theorem 4.3. *Let M be a compact irreducible Hermitian symmetric space of rank ≥ 2 . Then we have*

$$\text{Hol}^\pm(M) = \text{Aut}(M, \mathcal{C}).$$

The following theorem gives a necessary and sufficient condition for the global extension of a local holomorphic or local anti-holomorphic transformation on M . The proof is similar to the case of the causal structure (cf. [4]).

Theorem 4.4. *Let D be a domain in M and let f be a local holomorphic or local anti-holomorphic transformation of M defined on D . Suppose that $\text{rank } M \geq 2$. Then f extends uniquely to an element of $\text{Hol}^\pm(M)$ if and only if f is a local $\mathcal{C}$ -conformal transformation on D .*

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HOLOMORPHIC AND ANTI-HOLOMORPHIC AUTOMORPHISMS

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