

Gevrey Class での Fuchs 双曲型方程式

(Fuchsian Hyperbolic Equations in Gevrey Classes)

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In this note, I want to report my recent results on Fuchsian hyperbolic equations in Gevrey classes. Fuchsian hyperbolic equations were introduced by the author in [5] and they have been investigated in various spaces; e.g., by [5] in analytic-hyperfunction spaces, by [6] in C^∞ spaces (or in Sobolev spaces), by [1] in distribution spaces, and [7,8,9] in Gevrey classes etc.

§1. Review of C^∞ Results.

First, let us consider the following operator:

$$\begin{aligned} L(t, x, t\partial_t, t^{\kappa}\partial_x) & (=L) \\ & = (t\partial_t)^m + \sum_{\substack{j+|\alpha|\leq m \\ j < m}} a_{j,\alpha}(t, x) (t\partial_t)^j (t^{\kappa}\partial_x)^{\alpha}, \end{aligned}$$

where $(t, x) = (t, x_1, \dots, x_n) \in [0, T] \times \mathbb{R}^n$ ($T > 0$), $m \in \mathbb{N}$ ($= \{1, 2, 3, \dots\}$), $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{Z}_+^n$ ($= \{0, 1, 2, \dots\}^n$), $|\alpha| = \alpha_1 + \dots + \alpha_n$, $a_{j,\alpha}(t, x) \in C^\infty([0, T] \times \mathbb{R}^n)$ ($j + |\alpha| \leq m$ and $j < m$), $\kappa = (\kappa_1, \dots, \kappa_n) \in \mathbb{N}^n$; and

$$\begin{aligned}\partial_t &= \frac{\partial}{\partial t}, \quad t^{\kappa} \partial_x = (t^{\kappa_1} \frac{\partial}{\partial x_1}, \dots, t^{\kappa_n} \frac{\partial}{\partial x_n}), \\ (t^{\kappa} \partial_x)^{\alpha} &= (t^{\kappa_1} \frac{\partial}{\partial x_1})^{\alpha_1} \dots (t^{\kappa_n} \frac{\partial}{\partial x_n})^{\alpha_n} \\ &= t^{\kappa_1 \alpha_1 + \dots + \kappa_n \alpha_n} (\frac{\partial}{\partial x_1})^{\alpha_1} \dots (\frac{\partial}{\partial x_n})^{\alpha_n}.\end{aligned}$$

In addition, we impose the following condition on L :

(A-1) All the roots $\lambda_i(t, x, \xi)$ ($1 \leq i \leq m$) of

$$\lambda^m + \sum_{\substack{j+|\alpha|=m \\ j < m}} a_{j, \alpha}(t, x) \lambda^j \xi^{\alpha} = 0$$

are real, simple and bounded on $\{(t, x, \xi); |\xi|=1\}$.

Then, L is one of the most elementary examples of Fuchsian hyperbolic operators (in [6]). Therefore, we can define the characteristic exponents $\rho_1(x), \dots, \rho_n(x)$ by the roots of

$$L(0, x, \rho, 0) = 0.$$

Let $\mathcal{S}(\mathbb{R}^n)$ be the Schwartz space on $\mathbb{R}^n$, and let $C^\infty([0, T], \mathcal{S}(\mathbb{R}^n))$ [resp. $C^\infty((0, T), \mathcal{S}(\mathbb{R}^n))$] be the space of all infinitely differentiable functions on $[0, T]$ [resp. $(0, T)$] with values in $\mathcal{S}(\mathbb{R}^n)$. Then, we have the following results.

Unique Solvability (Tahara [6-III]).

Assume that $\rho_j(x) \notin \mathbb{Z}_+$ holds for any $x \in \mathbb{R}^n$ and $1 \leq i \leq m$. Then, the equation

$$L(t, x, t \partial_t, t^{\kappa} \partial_x) u = f$$

is uniquely solvable in $C^\infty([0, T], \mathcal{S}(\mathbb{R}^n))$.

Asymptotic Expansions (Tahara [6-V]).

Assume that $\rho_i(x) - \rho_j(x) \notin \mathbb{Z}$ holds for any $x \in \mathbb{R}^n$ and $1 \leq i \neq j \leq m$.

Then, we have the following results on

$$(E) \quad L(t, x, t\partial_t, t^{\kappa}\partial_x)u = 0 \quad \text{in } C^\infty((0, T), \mathcal{E}(\mathbb{R}^n)).$$

(1) Any solution $u(t, x) \in C^\infty((0, T), \mathcal{E}(\mathbb{R}^n))$ of (E) can be expanded asymptotically into the form

$$u(t, x) \sim \sum_{i=1}^m \left(\varphi_i(x) t^{\rho_i(x)} + \sum_{k=1}^m \sum_{h=0}^{\infty} \varphi_{i,k,h}(x) t^{\rho_i(x)+k} (\log t)^h \right) \quad (*)$$

(as $t \rightarrow +0$) for some unique $\varphi_i(x), \varphi_{i,k,h}(x) \in \mathcal{E}(\mathbb{R}^n)$.

(2) Conversely, for any $\varphi_i(x), \dots, \varphi_n(x) \in \mathcal{E}(\mathbb{R}^n)$ there exist a unique solution $u(t, x) \in C^\infty((0, T), \mathcal{E}(\mathbb{R}^n))$ of (E) and unique coefficients $\varphi_{i,k,h}(x) \in \mathcal{E}(\mathbb{R}^n)$ such that the asymptotic relation in (1) holds.

Example. Let P be of the form

$$P = (t\partial_t)^2 - t^{2\kappa_1} \partial_{x_1}^2 - t^{2\kappa_2} \partial_{x_2}^2 \\ + t^{p_1} a_1(t, x) \partial_{x_1} + t^{p_2} a_2(t, x) \partial_{x_2} \\ + b(t, x)(t\partial_t) + c(t, x),$$

where $\kappa_1, \kappa_2, p_1, p_2 \in \mathbb{N}$. Then, if $p_1 \geq \kappa_1$ and $p_2 \geq \kappa_2$ hold, we can apply our C^∞ results to P .

In this example, the condition " $p_1 \geq \kappa_1$ & $p_2 \geq \kappa_2$ " seems to be essential to the C^∞ results quoted above. Therefore, if we want to consider the case without " $p_1 \geq \kappa_1$ & $p_2 \geq \kappa_2$ ", we must treat

P in suitable subclasses in $C^\infty([0, T], \mathcal{G}(\mathbb{R}^n))$ or $C^\infty((0, T), \mathcal{G}(\mathbb{R}^n))$.

§2. Results in Gevrey Classes.

In order to treat P in Example for the general case, let us consider here the following operator:

$$P(t, x, t\partial_t, \partial_x) = L(t, x, t\partial_t, t^{\chi}\partial_x) + \sum_{j+|\alpha| < m} t^{p(j, \alpha)} b_{j, \alpha}(t, x) (t\partial_t)^j \partial_x^\alpha,$$

where $L(t, x, t\partial_t, t^{\chi}\partial_x)$ is the same as in §1, $p(j, \alpha) \in \mathbb{N}$ ($j+|\alpha| < m$) and $b_{j, \alpha}(t, x) \in C^\infty([0, T] \times \mathbb{R}^n)$ ($j+|\alpha| < m$).

A function $f(x)$ ($\in C^\infty(\mathbb{R}^n)$) is said to belong to the Gevrey class $\mathcal{G}^{\{s\}}(\mathbb{R}^n)$, if $f(x)$ satisfies the following: for any compact subset K of $\mathbb{R}^n$ there are $C > 0$ and $h > 0$ such that

$$\sup_{x \in K} |\partial_x^\alpha f(x)| \leq C h^{|\alpha|} (|\alpha|!)^s \quad \text{for any } \alpha \in \mathbb{Z}_+^n. \quad (2.1)$$

We denote by $C^\infty([0, T], \mathcal{G}^{\{s\}}(\mathbb{R}^n))$ [resp. $C^\infty((0, T), \mathcal{G}^{\{s\}}(\mathbb{R}^n))$] the space of all infinitely differentiable functions on $[0, T]$ [resp. $(0, T)$] with values in $\mathcal{G}^{\{s\}}(\mathbb{R}^n)$ equipped with the locally convex topology in [4].

Let S_n denote the permutation group of n -numbers, put

$$M_{j, \alpha}(\tau, r) = \frac{\sum_{i=1}^r (\chi_{\tau(i)} - \chi_{\tau(r)})^{\alpha_{\tau(i)} + (m-j)\chi_{\tau(r)} - p(j, \alpha)}}{(m-j-|\alpha|)\chi_{\tau(r)}}$$

and define the irregularity σ (≥ 1) by

$$\sigma = \max \left[1, \max_{\substack{j+|\alpha| < m \\ j < m}} \left\{ \min_{\tau \in S_n} \left(\max_{1 \leq r \leq n} M_{j,\alpha}(\tau, r) \right) \right\} \right].$$

Using this irregularity, we impose the following conditions:

$$(A-2) \quad 1 < s < \sigma/(\sigma-1).$$

$$(A-3) \quad a_{j,\alpha}(t,x), b_{j,\alpha}(t,x) \in C^\infty([0,T], \mathcal{E}^{\{s\}}(\mathbb{R}^n)) \text{ for any } (j,\alpha).$$

When $\sigma=1$, (A-2) is read $1 < s < \infty$. Then, we have the following results.

Theorem 1 (Unique Solvability).

Assume that $\rho_i(x) \notin \mathbb{Z}_+$ holds for any $x \in \mathbb{R}^n$ and $1 \leq i \leq m$. Then, the equation

$$P(t,x,t\partial_t, \partial_x)u = f$$

is uniquely solvable in $C^\infty([0,T], \mathcal{E}^{\{s\}}(\mathbb{R}^n))$.

Theorem 2 (Asymptotic Expansions).

Assume that $\rho_i(x) - \rho_j(x) \notin \mathbb{Z}$ holds for any $x \in \mathbb{R}^n$ and $1 \leq i \neq j \leq m$.

Then, we have the following results on

$$(E_s) \quad P(t,x,t\partial_t, \partial_x)u = 0 \quad \text{in } C^\infty((0,T), \mathcal{E}^{\{s\}}(\mathbb{R}^n)).$$

(1) Any solution $u(t,x) \in C^\infty((0,T), \mathcal{E}^{\{s\}}(\mathbb{R}^n))$ of (E_s) can be expanded asymptotically into the form

$$u(t, x) \sim \sum_{i=1}^m \left(\varphi_i(x) t^{\rho_i(x)} + \sum_{k=1}^m \sum_{h=0}^{\infty} \varphi_{i,k,h}(x) t^{\rho_i(x)+k} (\log t)^h \right) \quad (**)$$

(as $t \rightarrow +0$) for some unique $\varphi_i(x), \varphi_{i,k,h}(x) \in \mathcal{E}^{\{s\}}(\mathbb{R}^n)$.

(2) Conversely, for any $\varphi_1(x), \dots, \varphi_n(x) \in \mathcal{E}^{\{s\}}(\mathbb{R}^n)$ there exist a unique solution $u(t, x) \in C^\infty((0, T), \mathcal{E}^{\{s\}}(\mathbb{R}^n))$ of (E_s) and unique coefficients $\varphi_{i,k,h}(x) \in \mathcal{E}^{\{s\}}(\mathbb{R}^n)$ such that the asymptotic relation in (1) holds.

Here, the meaning of the asymptotic relation $(**)$ [resp. $(*)$] is as follows: for any $a > 0$ and any compact subset K of $\mathbb{R}^n$ there is an $N_0 \in \mathbb{N}$ such that for any $N \geq N_0$

$$t^{-a} (t \partial_t)^j \left[u(t, x) - \sum_{i=1}^n \left(\varphi_i(x) t^{\rho_i(x)} + \sum_{k=1}^m \sum_{h=0}^{\infty} \varphi_{i,k,h}(x) t^{\rho_i(x)+k} (\log t)^h \right) \right] \Big|_K \longrightarrow +0 \quad \text{in } \mathcal{E}^{\{s\}}(K) \text{ [resp. } \mathcal{E}(K)]$$

as $t \rightarrow +0$ for any $j \in \mathbb{Z}_+^n$,

where $\mathcal{E}^{\{s\}}(K)$ is the locally convex space of all functions $f(x) \in C^\infty(K)$ satisfying (2.1) for some $C > 0$ and $h > 0$ (see [4]).

§3. Remark.

(1) $\sigma=1$ is equivalent to the following:

$$p(j, \alpha) \geq \chi_1 \alpha_1 + \dots + \chi_n \alpha_n \quad \text{for any } (j, \alpha).$$

Hence, when $\sigma=1$, we can apply our C^∞ results to $P(t, x, t\partial_t, \partial_x)$.

(2) When P is of the form

$$P = (t\partial_t)^2 - t^{2\kappa} \partial_x^2 + t^P a(t, x) \partial_x + b(t, x)(t\partial_t) + c(t, x),$$

σ is given by $\sigma = \max[1, (2\kappa - p)/\kappa]$. Therefore, if $p < \kappa$, (A-2) is given by $1 < s < (2\kappa - p)/(\kappa - p)$. This coincides with the example in [2, 3, 9].

(3) When P is of the form

$$P = (t\partial_t)^2 - t^{2\kappa_1} \partial_{x_1}^2 - \dots - t^{2\kappa_n} \partial_{x_n}^2 \\ + t^{P_1} a_1(t, x) \partial_{x_1} + \dots + t^{P_n} a_n(t, x) \partial_{x_n} \\ + b(t, x)(t\partial_t) + c(t, x),$$

σ is given by $\sigma = \max[1, (2\kappa_1 - P_1)/\kappa_1, \dots, (2\kappa_n - P_n)/\kappa_n]$.

(4) When $\kappa_1 = \dots = \kappa_n (=k)$, σ is given by

$$\sigma = \max \left[1, \max_{\substack{j+|\alpha| < m \\ j < m}} \left(\frac{m-j-p(j, \alpha)/k}{m-j-|\alpha|} \right) \right].$$

(5) When $\kappa_1 = \dots = \kappa_n$, Uruy [9] defines an index $\sigma_u (\geq 1)$ and obtain Theorem 1 under the condition $1 < s < (\sigma_u - 1)/\sigma_u$. The relation between σ and σ_u is as follows.

(i) $1 \leq \sigma \leq \sigma_u$ holds in general.

(ii) There are examples for which $1 < \sigma < \sigma_u$ holds.

For example, in the case $P = (t\partial_t)((t\partial_t)^2 - t^8 \partial_x^2) + t\partial_x + t^3 \partial_x^2$, we have $\sigma = 9/4$ and $\sigma_u = 5/2$. Hence, we can say that our condition (A-2) is better than that in [9].

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