

A solution of the equation $f'(x) = \lambda^2 f(\lambda x)$, $\lambda > 1$.

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1. INTRODUCTION

The purpose of this paper is to give solutions for the functional-differential equation of advanced type

$$(1.1) \quad \begin{cases} f'(x) = \lambda^2 f(\lambda x), & x \in \mathbb{R} = (-\infty, +\infty), \\ f(0) = 0, \end{cases}$$

where λ is a constant, $\lambda > 1$. Our solutions are infinitely differentiable on $\mathbb{R}$. Moreover, if $\lambda \geq 2$, then the solutions are bounded and have arbitrarily large zeros. Our methods give numerical data readily.

Frederickson [1, 2] (1971) investigated functional-differential equations of advanced type

$$(1.2) \quad f'(x) = af(\lambda x) + \lambda f(x),$$

here $\lambda > 1$, and proved several properties of solutions. Later, Kato and McLeod [5] (1971) and Kato [4] (1972) studied the asymptotic behaviour of solutions of (1.2).

Frederickson [1] provided a global existence theorem for equations

$$f'(x) = F(f(2x)), \quad x \in \mathbb{R},$$

where F is an odd, continuous function with $F(s) > 0$ for $s > 0$, by application of the Schauder fixed point theorem. He showed that the absolute value of the solution $|f(x)|$ is periodic for $x \geq 0$. Frederickson [2] also provided a constructive method for solutions for equations

$$f'(z) = af(\lambda z) + bf(z),$$

where $a, b \in \mathbb{C}$ and $\lambda > 1$. He further gave solutions in the form of a Diriclet series

$$\varphi(z, \beta) = \sum_{n \in \mathbb{Z}} c_n e^{\beta \lambda^n z}, \quad \Re(\beta z) \leq 0,$$

where β is allowed to vary as a parameter. In the case of $b = 0$ and $\beta = i$, the solution is analytic in the upper half plane $\Im z > 0$, continuous on $\Im z \geq 0$, and the line $\Im z = 0$ is a natural boundary. From his result it follows that our solutions of (1.1) cannot be real analytic.

Ivanov, Kitamura, Kusano and Shevelo [3] (1982) investigated the higher order functional-differential equations of the form

$$(1.3) \quad f^{(n)}(x) = p(x)F(f(g(x))),$$

where p, F and g satisfy appropriate conditions. Kusano [6] (1984) also investigated the functional differential equation

$$(1.4) \quad f^{(n)}(x) = p(x)f(g(x))$$

where n is even, $p : [0, \infty) \rightarrow \mathbb{R}$ and $g : [0, \infty) \rightarrow \mathbb{R}$ are continuous, $p(t) > 0$, $g(t)$ is nondecreasing and $\lim_{t \rightarrow \infty} g(t) = \infty$. They [3, 6] gave sufficient conditions that the solutions are oscillatory.

If f is a solution of (1.1), then f is also a solution of the equations

$$(1.5) \quad f''(x) = \lambda^4 \lambda f(\lambda^2 x), \quad x \in \mathbb{R},$$

and

$$(1.6) \quad f'''(x) = \lambda^6 \lambda^3 f(\lambda^3 x), \quad x \in \mathbb{R}.$$

However, (1.5) and (1.6) don't satisfy the sufficient conditions in [3, 6].

Recently, the author [9] constructed solutions of (1.1) with $\lambda = 2$ by using a little different method from this paper.

We state the main theorem (Theorem 2.3) and application in next section. We can easily apply the solution for the case $\lambda = 2$ to Friedrichs' molifier theorem and we can rewrite defferential operator. For the proof of main theorem, see [10]. In the third section, we give graphs of solutions for the case of $\lambda = 4, 3, 2, 31/16, 15/8, 7/4, 2/3, 5/4$.

In the last section, we will give Mathematica programs.

2. MAIN RESULTS

First, we state two lemmas. Let

$$\hat{f}(\xi) = \mathcal{F}[f](\xi) = \int_{\mathbb{R}} f(x) e^{-ix\xi} dx, \quad \mathcal{F}^{-1}[f](\xi) = \frac{1}{2\pi} \int_{\mathbb{R}} f(x) e^{ix\xi} dx,$$

and

$$\text{sinc } \xi = \begin{cases} \sin(\pi\xi)/(\pi\xi), & \xi \neq 0, \\ 1, & \xi = 0. \end{cases}$$

Lemma 2.1. *The product*

$$\prod_{k=1}^{\infty} \text{sinc} \left(\frac{\xi}{2\lambda^k \pi} \right), \quad \xi \in \mathbb{R}$$

is converges pointwise and in $L^1(\mathbb{R})$.

Lemma 2.2. *Let*

$$(2.1) \quad u = \mathcal{F}^{-1}[U], \quad U(\xi) = \exp \left(-\frac{i\xi}{2(\lambda-1)} \right) \prod_{k=1}^{\infty} \text{sinc} \left(\frac{\xi}{2\lambda^k \pi} \right).$$

Then u has the following properties:

$$\begin{aligned} u &\in C^\infty(\mathbb{R}), \\ u(x) &> 0 \text{ for } x \in \left(0, \frac{1}{\lambda-1} \right), \quad u(x) = 0 \text{ for } x \notin \left(0, \frac{1}{\lambda-1} \right), \\ u(x) &= u(1/(\lambda-1) - x), \\ \int_{\mathbb{R}} u(x) dx &= 1, \end{aligned}$$

and

$$(2.2) \quad u'(x) = \lambda^2 u(\lambda x) \quad \text{for } x \in \left[0, \min \left(\frac{1}{\lambda}, \frac{1}{\lambda(\lambda-1)} \right) \right].$$

Let we define the operator $T : L^1 \rightarrow L^1$ as follows.

$$(2.3) \quad Tf(x) = \lambda (\chi_{[0,1]} * f)(\lambda x), \quad f \in L^1.$$

Then the function u in Lemma 2.2 is given by the following equation.

$$(2.4) \quad u = \lim_k T^k \chi_{[0,1]}.$$

Secondly, we define sequences $\{n_k\}_{k=1}^{\infty}$ and $\{y_k\}_{k=1}^{\infty}$ as follows:

$$(2.5) \quad \begin{cases} n_1 = 0, & n_2 = 1, \\ n_{2k-1} = 1, & n_{2k} = 0, \quad \text{if } n_k = 1 \quad (k \geq 2), \\ n_{2k-1} = 0, & n_{2k} = 1, \quad \text{if } n_k = 0 \quad (k \geq 2), \end{cases}$$

and

$$(2.6) \quad y_k = \sum_{l=1}^{\infty} C_{k,l} \lambda^{l-1}, \quad k = 1, 2, 3, \dots,$$

where $C_{k,l} \in \{0, 1\}$ ($l = 1, 2, 3, \dots$) are coefficients of the binary system such that

$$k - 1 = \sum_{l=1}^{\infty} C_{k,l} 2^{l-1}, \quad k = 1, 2, 3, \dots$$

Then we have the following relations.

$$(2.7) \quad \begin{cases} (-1)^{n_{2k-1}} = (-1)^{n_k}, \\ (-1) \cdot (-1)^{n_{2k}} = (-1)^{n_k}, \end{cases} \quad k = 1, 2, 3, \dots,$$

$$(2.8) \quad \begin{cases} y_{2k-1}/\lambda = y_k, \\ y_{2k}/\lambda = y_k + 1/\lambda, \end{cases} \quad k = 1, 2, 3, \dots,$$

and

$$(2.9) \quad y_k \geq \lambda^j \quad \text{if } k - 1 \geq 2^j, \quad j = 0, 1, 2, \dots$$

Hence $\lim_{k \rightarrow \infty} y_k = \infty$. If $\lambda \geq 2$, then y_k is strictly increasing. For example,

$$\{n_k\}_{k=1}^{\infty} = \{0, 1, 1, 0, 1, 0, 0, 1, 1, 0, 0, 1, 0, 1, 1, 0, \dots\},$$

$$\{y_k\}_{k=1}^{\infty} = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, \dots\} \text{ for } \lambda = 2,$$

$$\{y_k\}_{k=1}^{\infty} = \{0, 1, 4, 5, 16, 17, 20, 21, 64, 65, 68, 69, 80, \dots\} \text{ for } \lambda = 4,$$

$$\{y_k\}_{k=1}^{\infty} = \left\{0, 1, \frac{3}{2}, \frac{5}{2}, \frac{9}{4}, \frac{13}{4}, \frac{15}{4}, \frac{19}{4}, \frac{27}{8}, \frac{35}{8}, \frac{39}{8}, \frac{47}{8}, \frac{45}{8}, \dots\right\} \text{ for } \lambda = 3/2.$$

Our main result is the following:

Theorem 2.3. *Let $\lambda > 1$. Then a solution f of (1.1) can be found as*

$$f(x) = \sum_{k=1}^{\infty} (-1)^{n_k} u(x - y_k),$$

where u , $\{n_k\}_{k=1}^{\infty}$ and $\{y_k\}_{k=1}^{\infty}$ are as in (2.1), (2.5) and (2.6), respectively. The solution f is in $C^{\infty}(\mathbb{R})$ and $f(x) = 0$ for $x \leq 0$. If $\lambda \geq 2$, then f is bounded.

Let we define a function space $L^{1,\nu}(\mathbb{R})$.

$$L^{1,\nu} = \{f \in L_{loc}^1; \|f\|_{L^{1,\nu}} < \infty\}$$

$$\|f\|_{L^{1,\nu}} = \sup_{r>0} \frac{1}{r^\nu} \int_{-r}^{+r} |f(x)| dx$$

Theorem 2.4. *The solution f of (1.1) is in $C^\infty \cap L^{1,1/\log_2 \lambda}$.*

Remark 2.1. The solution of (1.1) is tempered distribution.

Remark 2.2. A constant times f is also a solution.

Theorem 2.5. *Let f be the solution in Theorem 2.3 for $\lambda = 2$ and*

$$G_{k,\epsilon}(x) = \left(2^{k(k-1)/2} \epsilon^{k+1}\right)^{-1} (f \chi_{[0,2^k]})(x/\epsilon).$$

If $v \in C^k(\mathbb{R})$ or $v \in L_k^p(\mathbb{R})$ ($k \geq 0$, $1 \leq p < \infty$), then

$$\frac{d^k v}{dx^k} = \lim_{\epsilon \rightarrow 0} v * G_{k,\epsilon},$$

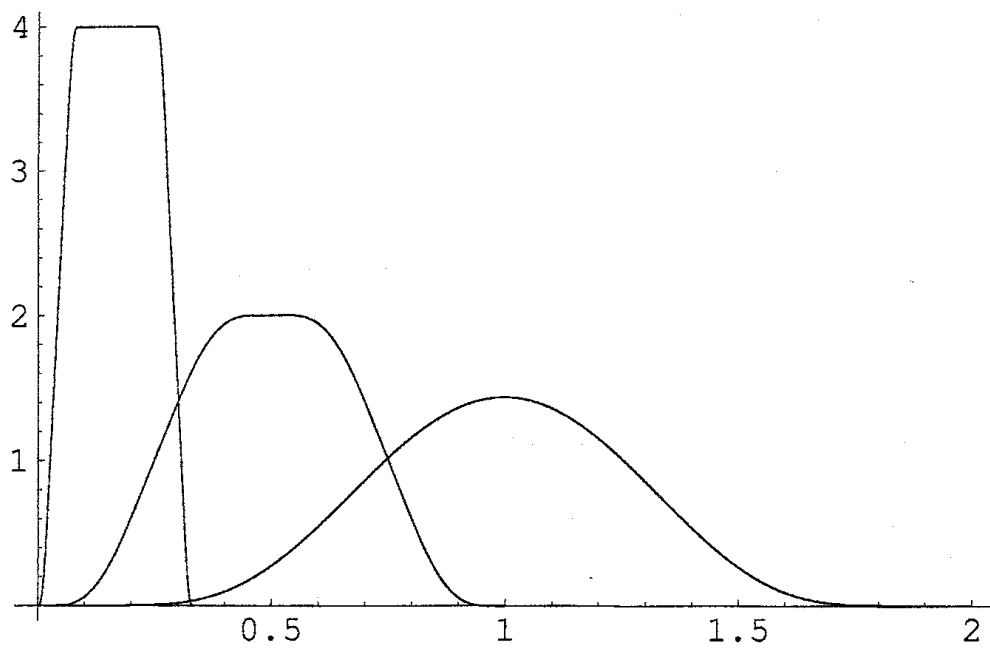
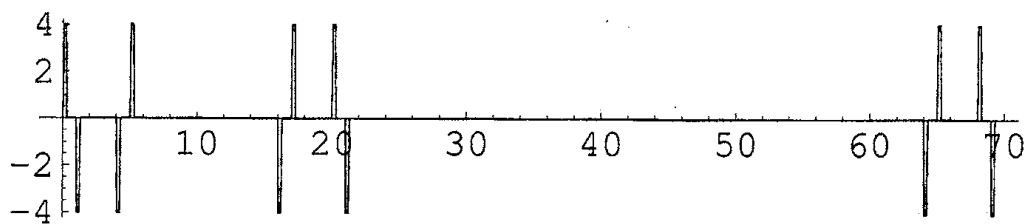
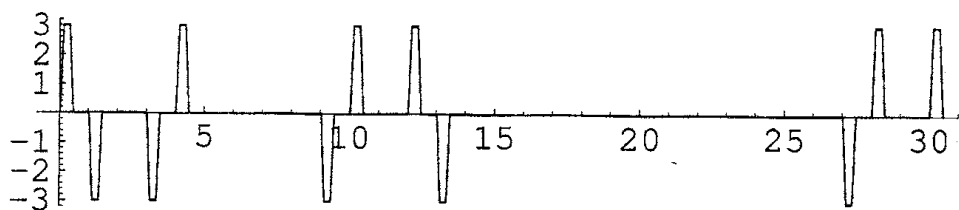
uniformly on each compact subset in $\mathbb{R}$ or in $L^p(\mathbb{R})$, respectively.

Remark 2.3. $G_{k,\epsilon}$ is in $C^\infty(\mathbb{R})$ with compact support. To prove the theorem we use Friedrichs' molifier $\frac{d^k v}{dx^k} * u_\delta = v * \frac{d^k u_\delta}{dx^k}$, where $u_\delta = u(x/\delta)/\delta$, $\delta > 0$, and u is the function in Lemma 2.2 ($\lambda = 2$).

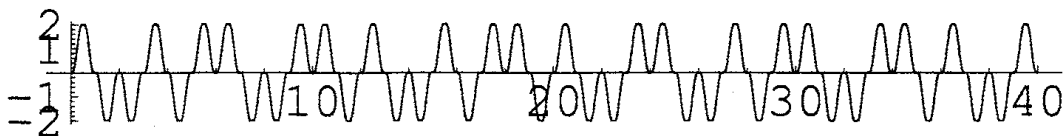
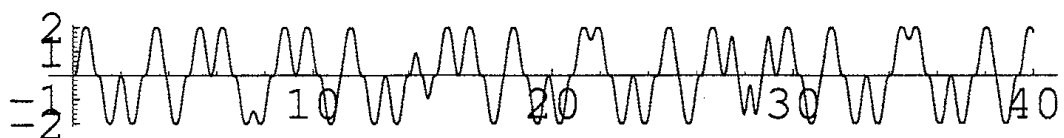
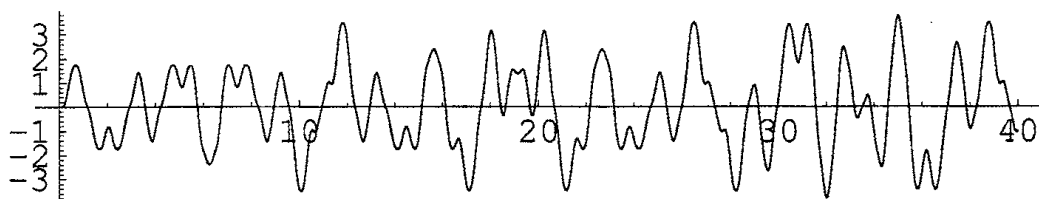
3. EXAMPLES

In this section we give graphs for $\lambda = 4, 3, 2, 31/16, 15/8, 7/4, 3/2, 5/4$.

If $\lambda = 2$, then $\{x > 0 : f(x) = 0\} = \{1, 2, 3, \dots\}$. If $\lambda > 2$, then $\{x > 0 : f(x) = 0\} = \cup_{k=1}^{\infty} [y_k + 1/(\lambda - 1), y_{k+1}]$ and its measure is infinity, since $1/(\lambda - 1) < 1 < y_{k+1} - y_k$, $k = 1, 2, 3, \dots$.

FIGURE 1. u ($\lambda = 4, 2, 3/2$)FIGURE 2. $f'(x) = 4^2 f(4x)$ FIGURE 3. $f'(x) = 3^2 f(3x)$ 4. PROGRAM OF $U(x)$

Mathematica program (Part 1): The function u [by using $u_{n+1}=T(u_n)$]

FIGURE 4. $f'(x) = 2^2 f(2x)$ FIGURE 5. $f'(x) = (31/16)^2 f(31x/16)$ FIGURE 6. $f'(x) = (15/8)^2 f(15x/8)$ FIGURE 7. $f'(x) = (7/4)^2 f(7x/4)$

* Setting lambda ($1 < \text{lam} < 9$)

```
In[1]: lam = 1.75;
```

* Calculation of the data ; $u_{\{0\}}, \dots, u_{\{50\}}$

```
In[2]: udata[0]=
```

```
Table[If[0 < i - 10000 ==< 1000, lam - 1, 0], {i, 1, 20000}];
```

```
In[3]: Timing[ Do[udata[k]= Table[If[1 ==< j - 10000 ==< 1000,
```

```
lam * Sum[udata[k = 1][[i + 10000]],
```

```
{i, Round[lam * (j - 10000)] - Round[(lam - 1) * 1000] + 1,
```

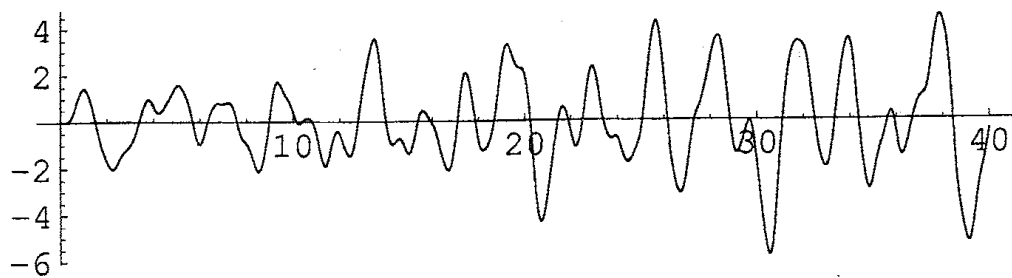


FIGURE 8. $f'(x) = (3/2)^2 f(3x/2)$

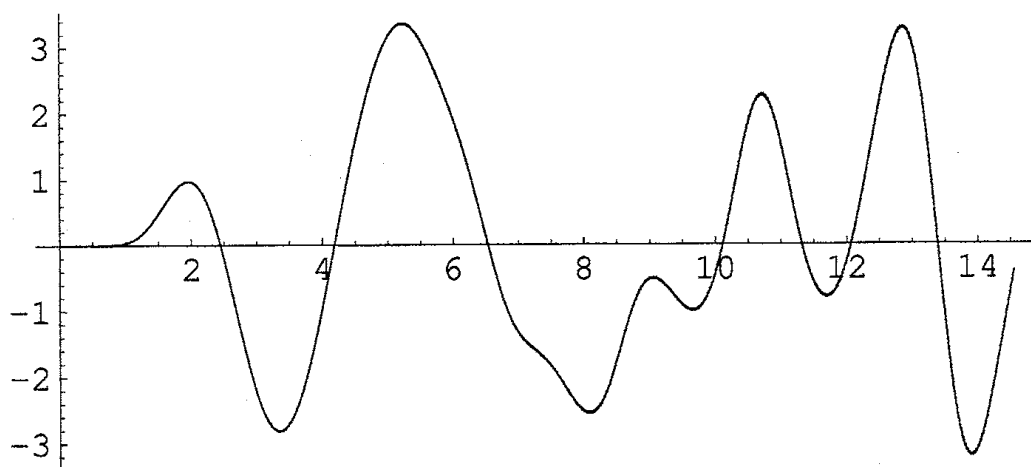


FIGURE 9. $f'(x) = (5/4)^2 f(5x/4)$

```
Round[lam * (j - 10000)] }]* 0.001/(lam - 1), 0],
  {j, 1, 20000}], {k, 1, 50}]]
```

```
Out[3]: {121.14 Second, Null}
```

```
* Graph of u_{50}
```

```
In[4]: ulist[k_] := Table[{(i - 10000)* 0.001/(lam - 1),
  Part[udata[k], i]}, {i, 10000, 11000}]
```

```
In[5]: ListPlot[ulist[50], PlotJoined -> True,
  PlotRange -> {0, 1.1 * lam}]
```

```
* Save the data
```



```

In[6]: udata[50]>> c:/mdata/udata7ov4-50
In[7]: ulist[50]>> c:/mdata/ulist7ov4-50
In[8]: Export["c:/mdata/u7ov4.eps",
             ListPlot[ulist[50], PlotJoined -> True,
             PlotRange -> {0, 1.1 * lam}]]

```

5. PROGRAM OF F(X)

Mathematica program (Part 2): The solution f on the interval [0,tau]

* Setting lambda ($1 < \text{lam} < 9$) and tau

```

In[1]: lam = 1.75; tau = 30;
In[2]: kk = Round[Log[lam, tau] + 0.5]
Out[2]: 7

```

* Load the data

```

In[3]: udata = << c:/mdata/udata7ov4-50;
In[4]: ud = Table[Part[udata, i], {i, 10000, 11000}];

```

* Sequences $m_{\{k\}}$ and $y_{\{k\}}$

```

In[5]: m[1]= 0; m[2]= 1;
          Do[m[k]= If[Mod[k, 2]==0, Mod[m[k /2] + 1, 2], m[(k + 1)/2]],
          {k, 3, 2^kk + 1}]
In[6]: Do[b[k, 1]= k - 1; Do[c[k, 1]= Mod[b[k, 1], 2];
          b[k, 1 + 1]= (b[k, 1] - c[k, 1])/2, {1, 1, kk + 1}],
          {k, 1, 2^kk + 1}]
In[7]: Do[y[k]=
          Sum[c[k, 1]* lam^(1 - 1), {1, 1, kk + 1}], {k, 1, 2^kk + 1}]

```

* Calculation of the solution

as the sum of $(-1)^{m_{\{k\}}}u(x-y_{\{k\}})$, $k=1, 2, \dots, 2^{\text{kk}}$.

```

In[8]: Do[yy[k]= Round[y[k]* 1000 * (lam - 1)], {k, 1, 2^kk + 1}]
In[9]: zz[1]= Table[0, {i, 1, yy[2^kk]}]; Do[z[k]=
      Table[0, {i, 1, yy[k]}], {k, 2, 2^kk}];
      Do[zz[k]= Table[0, {i, 1, yy[2^kk]- yy[k]}], {k, 2, 2^kk}];
In[10]: udy[1]= Join[ud, zz[1]]; Do[udy[k]=
      Join[z[k], ud* (-1)^m[k], zz[k]], {k, 2, 2^kk}]
In[11]: fd = Sum[udy[k], {k, 1, 2^kk}];

* Save the graph of the solution

In[12]: ii=tau*(lam-1)*1000;
In[13]: flist = Table[{i * 0.001/(lam - 1),
      Part[fd, i]}, {i, 1, ii}];
In[14]: Export["c:/mdata/f7ov4.eps",
      ListPlot[flist, PlotJoined -> True, AspectRatio -> Automatic]];

```

6. ACKNOWLEDGEMENT

The author would like to thank Professors Eiichi Nakai and Hiroyuki Usami for their useful comments and the author also would like to thank Professor Eiichi Nakai for their useful Mathematica program.

REFERENCES

- [1] P. O. Frederickson, Global solutions to certain nonlinear functional differential equations. J. Math. Anal. Appl. 33 (1971), 355-358.
- [2] P. O. Frederickson, Dirichlet series solutions for certain functional differential equations. Japan-United States Seminar on Ordinary Differential and Functional Equations (Kyoto, 1971), 249-254. Lecture Notes in Math., Vol. 243, Springer, Berlin, 1971.

- [3] A. F. Ivanov, Y. Kitamura, T. Kusano and V. N. Shevelo, *Oscillatory solutions of functional differential equations Generated by deviation of arguments of mixed type*, Hiroshima Math. J. 12 (1982), 645–655
- [4] T. Kato *Asymptotic behavior of solutions of the functional differential equation $y'(x) = ay(\lambda x) + by(x)$* , Delay and functional differential equations and their applications (Proc. Conf., Park City, Utah, 1972), Academic Press, New York, 1972, 197–217.
- [5] T. Kato and J. B. McLeod, *The functional-differential equation $y'(x) = ay(\lambda x) + by(x)$* , Bull. Amer. Math. Soc. 77 (1971), 891–937.
- [6] T. Kusano, *Oscillation of even order Linear Functional Differential Equations with Deviating Arguments of Mixed Type*, J. Math. Anal. Appl. 98 (1984), 341–347
- [7] Zhi-cheng Wang, Ionannis P. Stavroulakis, Xiang-zheng Qian
A Survey On the Oscillation Of Solutions Of First Order Linear Differential Equations With Deviating Arguments, P Applied Mathematics E-Notes, 2(2002), 171-191
- [8] Hiroshi Onose
Oscillatory Properties of the First Order Differential Inequalities with Deviating Argument, Funkcialaj Ekvacioj, 26(1983), 189-195
- [9] T. Yoneda, *On the functional-differential equation of advanced type $f'(x) = af(2x)$ with $f(0) = 0$* , preprint.
- [10] T. Yoneda, *On the functional-differential equation of advanced type $f'(x) = af(\lambda x)$, $\lambda > 1$ with $f(0) = 0$* , preprint.
- [11] T. Yoneda, *On the functional-differential equation of advanced type $f'(x) = \sum_j a_j f(\lambda x - b_j)$* , in preparation.

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