

ON FINITE GALOIS COVERINGS OF

COMPACT COMPLEX MANIFOLDS

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1. Introduction. A finite branched covering of an n -dimensional connected compact complex manifold M is, by definition, an irreducible normal complex space X together with a surjective proper finite holomorphic mapping $\pi : X \rightarrow M$. A point $p \in X$ is called an unramified point if π is locally biholomorphic around p . Otherwise, p is called a ramified point. The set of all ramified points forms a hypersurface R_π of X , called the ramification locus of π . The image $B_\pi = \pi(R_\pi)$ of R_π is called the branch locus of π , which is a hypersurface of M .

A morphism (resp. isomorphism) of $\pi : X \rightarrow M$ to $\pi' : X' \rightarrow M$ is a holomorphic (resp. biholomorphic) mapping $\phi : X \rightarrow X'$ such that $\pi'\phi = \pi$. An automorphism of π is

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called a covering transformation of π . The set of all covering transformations of π forms a group G_π under composition, called the covering transformation group of π . The covering $\pi : X \rightarrow M$ is called a Galois covering if G_π acts transitively on every fiber.

Let $\pi : X \rightarrow M$ be a finite Galois covering and $B_\pi = D_1 \cup \dots \cup D_s$ be the irreducible decomposition of the branch locus B_π . For each non-singular point q of B_π , every point $p \in \pi^{-1}(q)$ is a non-singular point of both X and $\pi^{-1}(B_\pi)$. Moreover, there are coordinate systems $(w_1, \dots, w_n)$ and $(z_1, \dots, z_n)$ around q and p , respectively, such that $q = (0, \dots, 0)$, $p = (0, \dots, 0)$, $B_\pi = \{w_1 = 0\}$, $\pi^{-1}(B_\pi) = \{z_1 = 0\}$ and

$$\pi : (z_1, \dots, z_n) \rightarrow (w_1, \dots, w_n) = (z_1^e, z_2, \dots, z_n),$$

locally. The positive integer $e = e_j$ (≥ 2) is constant as a function of q on $D_j - \text{Sing } B_\pi$ for every D_j and is called the ramification index of π around D_j . (Sing

B_π is the singular locus of B_π .) In this case, π is said to branch along the positive divisor $D = e_1 D_1 + \dots + e_s D_s$ on M .

The purpose of this note is to give a sufficient condition for the existence of a finite Galois covering $\pi : X \rightarrow M$ of M with non-singular X which branches along a given positive divisor $D = e_1 D_1 + \dots + e_s D_s$ ($e_j \geq 2$) on M . The existence of such coverings for some special cases were proved in the interesting papers Hirzebruch [3], Höfer [4] and Kato [6]. In this note, we make use of Selberg's theorem on the monodromy representation of a Fuchsian differential equation.

2. Kato's criterion. Let B be a hypersurface of M and $B = D_1 \cup \dots \cup D_s$ be its irreducible decomposition.

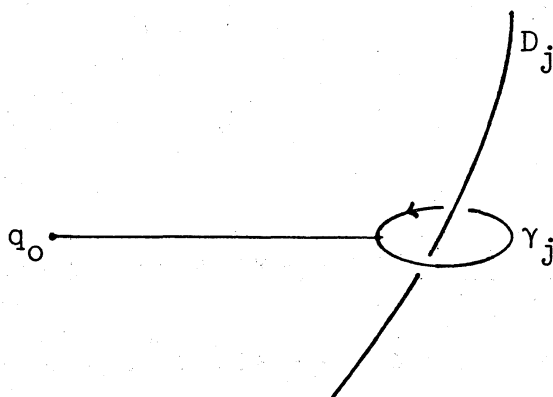


Figure 1

We fix a point $q_0 \in M - B$ once for all. Let γ_j be a loop in $M - B$ starting and terminating at q_0 encircling a point of $D_j - \text{Sing } B$ in the positive sense as in Figure 1. We identify γ_j with its homotopy class in the fundamental group $\pi_1(M - B, q_0)$. Let $e_1, \dots, e_s$ be integers greater than one. We denote by

$$J = \langle \gamma_1^{e_1}, \dots, \gamma_s^{e_s} \rangle^{\pi_1}$$

the smallest normal subgroup of $\pi_1(M - B, q_0)$ which contains $\gamma_1^{e_1}, \dots, \gamma_s^{e_s}$.

For a subgroup K of $\pi_1(M - B, q_0)$ with $J \subset K$, we consider the following condition:

Condition A. If γ_j^d belongs to K , then $d \equiv 0 \pmod{e_j}$ for every j ($1 \leq j \leq s$).

Then we have

Theorem 1 (Namba [8]). There exists a finite Galois covering $\pi : X \rightarrow M$ which branches along $D = e_1 D_1 + \dots + e_s D_s$ if and only if there exists a normal subgroup K of

$\pi_1(M - B, q_0)$ of finite index which contains J and satisfies Condition A. The correspondence

$$\pi \rightarrow K = K(\pi) = \pi_*(\pi_1(X - \pi^{-1}(B)))$$

between (isomorphism classes of) such π 's and such K 's is one-to-one. G_π is isomorphic to $\pi_1(M - B, q_0)/K$.

Henceforth, we suppose that B is simple normally crossing.

For any point $q \in B$, we take a local coordinate system

$(w_1, \dots, w_n)$ around q such that $q = (0, \dots, 0)$ and

$B = \{w_1 \dots w_t = 0\}$ locally. We may suppose $D_j = \{w_j = 0\}$

locally for $1 \leq j \leq t$.

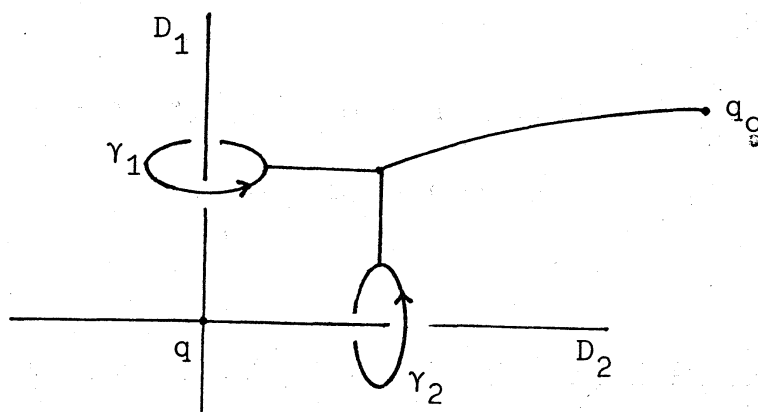


Figure 2

Let $\hat{\gamma}_j$ be a loop in $M - B$ starting and terminating at q_0 encircling a point of $D_j - \text{Sing } B$ near q in the positive sense as in Figure 2. $\hat{\gamma}_j$ is then conjugate to γ_j in $\pi_1(M - B, q_0)$. Note that $\hat{\gamma}_1, \dots, \hat{\gamma}_t$ are mutually commutative.

For a subgroup K of $\pi_1(M - B, q_0)$ with $J \subset K$, we consider the following condition:

Condition A'. If $\hat{\gamma}_1^{d_1} \dots \hat{\gamma}_t^{d_t}$ belongs to K , then $d_1 \equiv 0 \pmod{e_1}, \dots, d_t \equiv 0 \pmod{e_t}$ for every point $q \in B$.

Then, as a special case of Kato [5], we have

Theorem 2 (Kato). In Theorem 1, if $K = K(\pi)$ satisfies Condition A', then X is non-singular.

3. Fuchsian differential equations. Let $B = D_1 \cup \dots \cup D_s$ be as above simple normally crossing. Let Ω be an $(r \times r)$ -matrix-valued meromorphic 1-form on M such that $d\Omega + \Omega \wedge \Omega = 0$ and such that Ω is holomorphic on $M - B$.

For an unknown r -vector-valued function Y , the differential equation

$$dY = Y\Omega \quad (1)$$

(of order r) is called a Fuchsian differential equation with regular singularity along B , if, for every point $q \in B$, Ω can be locally written as

$$\Omega = A_1(w) \frac{dw_1}{w_1} + \dots + A_t(w) \frac{dw_t}{w_t} + A_{t+1}(w) dw_{t+1} + \dots + A_n(w) dw_n,$$

around q , under the notations in §2, where $A_j(w)$ are $(r \times r)$ -matrix-valued holomorphic functions around q . In this case,

$$\text{Res}_{D_j} \Omega = A_j(q) \quad (1 \leq j \leq t)$$

is a constant matrix on D_j , called the residue of Ω at D_j .

Theorem 3 (Gérard [2], Yoshida-Takano [10]). A fundamental matrix solution $F(w)$ around $q \in B$ of the Fuchsian differential equation (1) with regular singularity along B can be written as

$$F(w) = w_1^{C_1} \dots w_t^{C_t} w_1^{N_1} \dots w_t^{N_t} G(w),$$

where $C_1, \dots, C_t$ are mutually commutative constant matrices,

$N_1, \dots, N_t$ are diagonal matrices whose components are non-negative integers, and $G(w)$ is an $(r \times r)$ -matrix-valued holomorphic function around q with $\det G(w)$ nowhere vanishing. Moreover, if none of the differences of the eigenvalues of $\text{Res}_{D_j} \Omega$ are non-zero integers for $1 \leq j \leq t$, then C_j and N_j can be so chosen that $N_j = 0$ ($1 \leq j \leq t$) and C_j is conjugate to $\text{Res}_{D_j} \Omega$ ($1 \leq j \leq t$).

4. Existence of finite Galois coverings. Let $B = D_1 \cup \dots \cup D_s$ be as above simple normally crossing. Suppose (i) $\pi_1(M - B, q_0)$ is finitely generated. (This condition is satisfied if M is projective as Prof. M. Oka informed us.) Suppose that there is a Fuchsian differential equation (1) with regular singularity along B such that (ii) the order r of (1) satisfies $r \geq n$, (iii) every $\text{Res}_{D_j} \Omega$ ($1 \leq j \leq s$) is diagonalizable and (iv) every eigenvalue of $\text{Res}_{D_j} \Omega$ is a rational number.

We write the eigenvalue as

$$\frac{a_{j1}}{e_j}, \dots, \frac{a_{jr}}{e_j}, \quad (2)$$

where e_j (≥ 2) and a_{jv} are integers such that

$$(a_{j1}, \dots, a_{jr}, e_j) = 1,$$

where $(*, \dots, *)$ denotes the GCD of the components.

Suppose moreover (v) if $a_{jv} \equiv a_{j\mu} \pmod{e_j}$, then $a_{jv} = a_{j\mu}$.

For a point $q \in \text{Sing } B$, $\text{Res}_{D_j} \Omega$ ($1 \leq j \leq t$) under the notations in §2 are mutually commutative. Hence they can be simultaneously diagonalizable:

$$P(\text{Res}_{D_j} \Omega)P^{-1} = \begin{pmatrix} a_{j1}/e_j & & & 0 \\ & \ddots & & \\ & & \ddots & \\ 0 & & & a_{jr}/e_j \end{pmatrix}$$

for $1 \leq j \leq t$, where P is a non-singular matrix.

Let

$$\Delta_1, \dots, \Delta_N \quad (N = \binom{r}{t}).$$

be the $(t \times t)$ -minors of the $(t \times r)$ -matrix (a_{jv}) . Put

$$f_j = e_j \langle e_1, \dots, e_{j-1}, e_{j+1}, \dots, e_t \rangle / \langle e_1, \dots, e_t \rangle$$

for $1 \leq j \leq t$, where $\langle *, \dots, * \rangle$ denotes the LCM of the components.

Theorem 4. Under the above notations and assumptions

(i) $\sim$ (v), suppose moreover (vi) $(\Delta_1, \dots, \Delta_N, f_j) = 1$
 $1 \leq j \leq t$ for every point $q \in \text{Sing } B$. Then there exists a finite Galois covering $\pi : X \rightarrow M$ with non-singular X which branches along $D = e_1 D_1 + \dots + e_s D_s$.

For the proof of Theorem 4, we make use of the following theorem of Selberg [9] (see also Borel [1]).

Theorem 5 (Selberg). Let Γ be a finitely generated subgroup of $GL(r, \mathbb{C})$. Then there exists a normal subgroup Γ_0 of Γ of finite index and of torsion free. If $\Gamma \neq \{1\}$, then Γ_0 can be so chosen that $\Gamma_0 \neq \Gamma$.

Proof of Theorem 4. Let $R : \pi_1(M - B, q_0) \rightarrow GL(r, \mathbb{C})$ be the monodromy representation of the equation (1) and put $\Gamma = R(\pi_1(M - B, q_0))$. Let Γ_0 be a normal subgroup of Γ of finite index and of torsion free in Theorem 5. Put $K = R^{-1}(\Gamma_0)$.

Then K is a normal subgroup of $\pi_1(M - B, q_0)$ of finite index.

We show that K satisfies the conditions of Theorem 2.

We first show that K contains J . By Theorem 3 and by the assumption (v),

$$\begin{aligned} R(\gamma_j^{e_j}) &= R(\gamma_j)^{e_j} \sim (\exp 2\pi i \operatorname{Res}_{D_j} \Omega)^{e_j} \\ &= \exp 2\pi i e_j \operatorname{Res}_{D_j} \Omega \\ &\sim \exp 2\pi i \begin{pmatrix} a_{j1} & & & 0 \\ & \ddots & & \\ 0 & & & a_{jr} \end{pmatrix} = 1, \end{aligned}$$

where $\sim$ means the conjugacy relation. Hence

$$\gamma_j^{e_j} \in \ker(R) \subset K,$$

so

$$J \subset K.$$

Next, suppose $\gamma_j^d \in K$. Then $R(\gamma_j)^d \in \Gamma_0$. Note that

$$(R(\gamma_j)^d)^{e_j} = (R(\gamma_j)^{e_j})^d = 1.$$

Since Γ_0 is torsion free, we have $R(\gamma_j)^d = 1$. This means that

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$$da_{j1}/e_j, \dots, da_{jr}/e_j$$

are integers. Hence $d \equiv 0 \pmod{e_j}$. This holds for $1 \leq j \leq s$.

Finally, for a point $q \in \text{Sing } B$, suppose $\hat{\gamma}_1^{d_1} \dots \hat{\gamma}_t^{d_t} \in K$, under the notations in §2. Then $R(\hat{\gamma}_1)^{d_1} \dots R(\hat{\gamma}_t)^{d_t} \in \Gamma_0$. Note that

$$(R(\hat{\gamma}_1)^{d_1} \dots R(\hat{\gamma}_t)^{d_t})^{e_1 \dots e_t} = 1.$$

Since Γ_0 is torsion free, we have $R(\hat{\gamma}_1)^{d_1} \dots R(\hat{\gamma}_t)^{d_t} = 1$.

This means that

$$\begin{aligned} & d_1 a_{11}/e_1 + \dots + d_t a_{t1}/e_t \\ & \dots\dots\dots \\ & d_1 a_{1r}/e_1 + \dots + d_t a_{tr}/e_t \end{aligned}$$

are integers. Now the assumption (vi) implies easily that

$$d_1 \equiv 0 \pmod{e_1}, \dots, d_t \equiv 0 \pmod{e_t}.$$

Q.E.D.

5. Application to Appell's F_1 . Let $(Z_0 : Z_1 : Z_2)$ be a homogeneous coordinate system on the complex projective plane $\mathbb{P}^2$ and let $(x, y) = (Z_1/Z_0, Z_2/Z_0)$ be the affine coordinate system. Appell's hypergeometric differential equation F_1 can be written as

$$d(f, x \frac{\partial f}{\partial x}, y \frac{\partial f}{\partial y}) = (f, x \frac{\partial f}{\partial x}, y \frac{\partial f}{\partial y}) \Omega_0,$$

where

$$\Omega_0 = A \frac{dx}{x} + B \frac{dy}{y} + C \frac{dx}{x-1} + D \frac{dy}{y-1} + E \frac{d(x-y)}{x-y},$$

$$A = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 1-\gamma+\beta' & -\beta' \\ 0 & 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & -\beta & 1-\gamma+\beta \end{pmatrix}$$

$$C = \begin{pmatrix} 0 & -\alpha\beta & 0 \\ 0 & \gamma-\alpha-\beta-1 & 0 \\ 0 & -\beta & 0 \end{pmatrix}, \quad D = \begin{pmatrix} 0 & 0 & -\alpha\beta' \\ 0 & 0 & -\beta' \\ 0 & 0 & \gamma-\alpha-\beta'-1 \end{pmatrix}$$

$$E = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -\beta' & \beta' \\ 0 & \beta & -\beta \end{pmatrix},$$

where α, β, β' and γ are constants, (see Kimura [7]).

Ω_0 is a (3×3) -matrix-valued meromorphic 1-form on $\mathbb{P}^2$ which is holomorphic on $\mathbb{P}^2 - B_0$, where

$$B_0 = D_1 \cup D_2 \cup D_3 \cup D_4 \cup D_5 \cup D_6,$$

$$D_1 = \{x = 0\}, \quad D_2 = \{y = 0\},$$

$$D_3 = \{x = 1\}, \quad D_4 = \{y = 1\},$$

$$D_5 = \{x = y\}, \quad D_6 = \text{the line at infinity}.$$

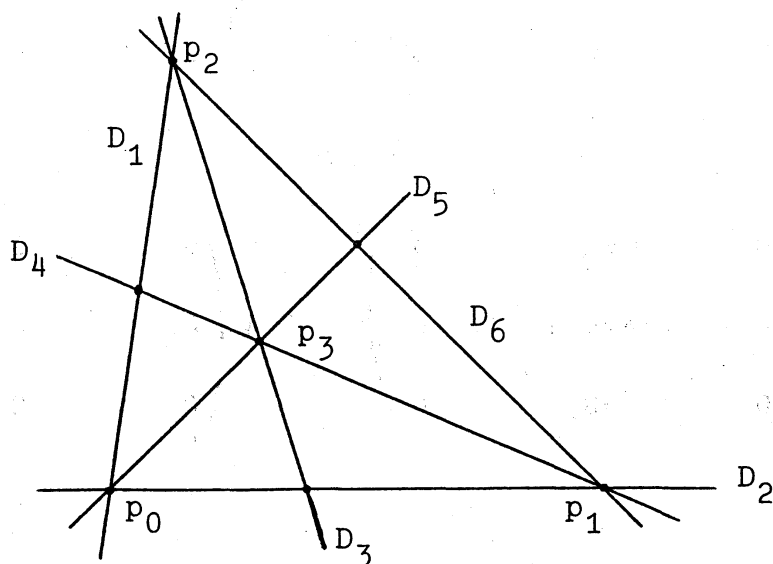


Figure 3

Consider the blowing up $\rho : M = \hat{\mathbb{P}}^2 \rightarrow \mathbb{P}^2$ at the four points

$$p_0 = (1 : 0 : 0), p_1 = (0 : 1 : 0), p_2 = (0 : 0 : 1), p_3 = (1 : 1 : 1)$$

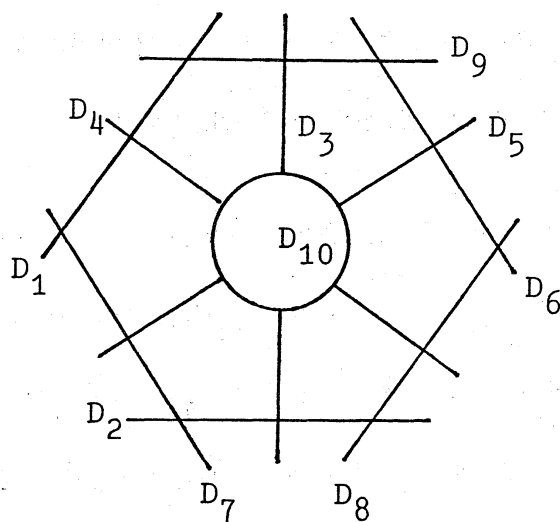


Figure 4

Then the differential equation

$$dY = Y\Omega \quad (3)$$

on $\hat{\mathbb{P}}^2$, where $\Omega = \rho^*\Omega_0$, is Fuchsian with regular singularity along

$$B = D_1 \cup \dots \cup D_{10},$$

where D_7, D_8, D_9, D_{10} are exceptional curves as in Figure 4.

(By abuse of notation, we write the strict transform of D_j as D_j again for $1 \leq j \leq 6$). Note that B is simple normally crossing.

We apply Theorem 4 to the equation (3). Suppose that α, β, β' and γ are rational numbers and suppose

$$1 - \gamma + \beta' = a_1/e_1, \text{ where } e_1 \geq 2 \text{ and } (a_1, e_1) = 1,$$

$$1 - \gamma + \beta = a_2/e_2, \text{ where } e_2 \geq 2 \text{ and } (a_2, e_2) = 1,$$

$$\gamma - \alpha - \beta - 1 = a_3/e_3, \text{ where } e_3 \geq 2 \text{ and } (a_3, e_3) = 1,$$

$$\gamma - \alpha - \beta' - 1 = a_4/e_4, \text{ where } e_4 \geq 2 \text{ and } (a_4, e_4) = 1,$$

$$-\beta - \beta' = a_5/e_5, \text{ where } e_5 \geq 2 \text{ and } (a_5, e_5) = 1,$$

$$1 - \gamma = a_7/e_7, \text{ where } e_7 \geq 2 \text{ and } (a_7, e_7) = 1,$$

$$\gamma - 1 - \alpha - \beta - \beta' = a_{10}/e_{10}, \text{ where } e_{10} \geq 2 \text{ and } (a_{10}, e_{10}) = 1,$$

$$\beta + \beta' - \alpha = a_6/e_6, \alpha = b_6/e_6, \text{ where } e_6 \geq 2 \text{ and } (a_6, b_6, e_6) = 1,$$

$$\beta - \alpha = a_8/e_8, \alpha = b_8/e_8, \text{ where } e_8 \geq 2 \text{ and } (a_8, b_8, e_8) = 1,$$

$$\beta' - \alpha = a_9/e_9, \alpha = b_9/e_9, \text{ where } e_9 \geq 2 \text{ and } (a_9, b_9, e_9) = 1.$$

Then, after some simple calculations, we have by

Theorem 4:

Theorem 6. Under the above notations and assumptions, suppose moreover that $a_j \not\equiv 0 \pmod{e_j}$ for $j = 6, 8, 9$, and $(a_6, e_6, a_8, e_8) = 1$ and $(a_6, e_6, a_9, e_9) = 1$. Then there exists a finite Galois covering $\pi : X \rightarrow \hat{\mathbb{P}}^2$ with non-singular X which branches along $D = e_1 D_1 + \dots + e_{10} D_{10}$.

There are a lot of examples in which the conditions of Theorem 6 are satisfied. We give three such examples. (The numbers attached to the curves in the figures below mean the ramification indices along the curves.)

Example 1. $\alpha = 3 - 1/a$, $\beta = \beta' = 1 - 1/4a$ and $\gamma = 3 - 3/4a$, where $a \geq 2$ and $(a, 3) = 1$.

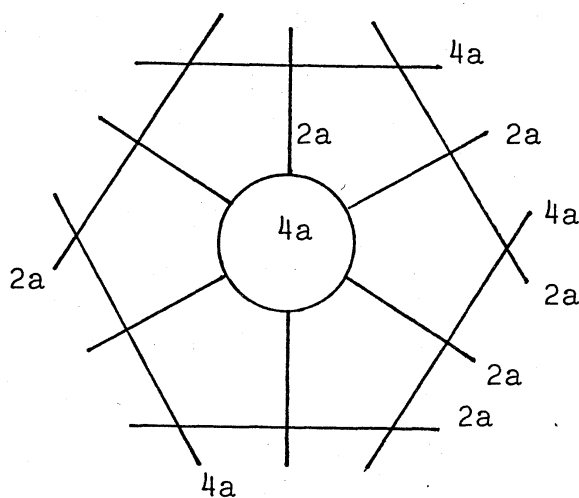


Figure 5

Example 2. $\alpha = 3 - 1/3a$, $\beta = \beta' = 1 - 1/12a$ and $\gamma = 3 - 1/4a$, where a is a positive integer.

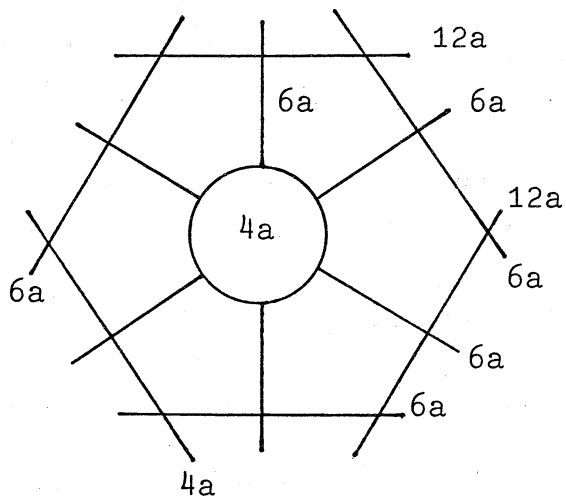


Figure 6

Example 3. $\alpha = 3 - 4/3a$, $\beta = \beta' = 1 - 1/3a$ and $\gamma = 3 - 1/a$, where $a \geq 3$ and $(a, 2) = 1$.

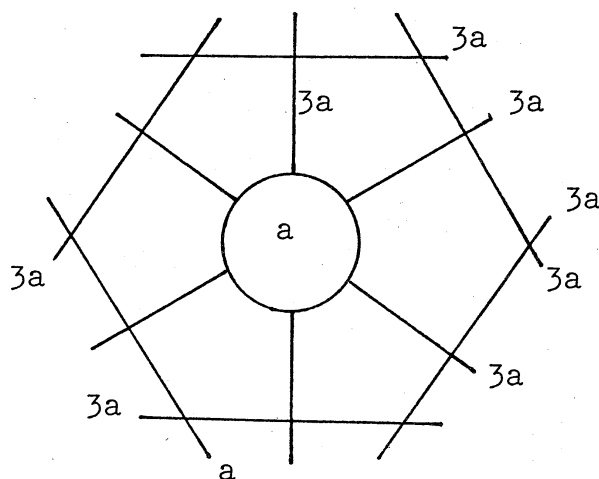


Figure 7

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