

SUPPLEMENTS TO NON-LC IDEAL SHEAVES

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ABSTRACT. We consider various definitions of non-lc ideal sheaves – generalizations of the multiplier ideal sheaf which define the non-lc (non-log canonical) locus. We introduce the maximal non-lc ideal sheaf and intermediate non-lc ideal sheaves and consider the restriction theorem for these ideal sheaves. We also begin the development of the theory of a characteristic $p > 0$ analog of maximal non-lc ideals, utilizing some recent work of Blickle.

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1. MOTIVATION

In this short section, we explain our motivation for the study of *non-lc ideal sheaves*.

Received April 29, 2010. Accepted November 30, 2010.

2000 *Mathematics Subject Classification*. Primary 14B05, 14E15; Secondary 14E30, 13A35.

1.1 (Motivation). Let X be a normal variety and Δ an effective $\mathbb{Q}$ -divisor on X such that $K_X + \Delta$ is $\mathbb{Q}$ -Cartier. In this situation, we want to define an ideal sheaf $I(X, \Delta)$ satisfying the following properties.

- (A) The pair (X, Δ) is log canonical if and only if $I(X, \Delta) = \mathcal{O}_X$.
- (B) (Kodaira type vanishing theorem). Assume that X is projective. Let D be a Cartier divisor on X such that $D - (K_X + \Delta)$ is ample. Then

$$H^i(X, I(X, \Delta) \otimes \mathcal{O}_X(D)) = 0$$

for every $i > 0$.

- (C) (Bertini type theorem). Let H be a general member of a free linear system Λ on X . Then

$$I(X, \Delta) = I(X, \Delta + H).$$

- (D) (Restriction theorem). Assume that $\Delta = S + B$ such that S is a normal prime Weil divisor on X , B is an effective $\mathbb{Q}$ -divisor, and that S and B have no common irreducible components. Then

$$I(X, S + B)|_S = I(S, B_S),$$

where $(K_X + S + B)|_S = K_S + B_S$.

We have already known that the (*minimal*) *non-lc ideal sheaf* $\mathcal{J}_{NLC}(X, \Delta)$ introduced in [F2] satisfies all the above properties. The *intermediate non-lc ideal sheaves* $\mathcal{J}_l'(X, \Delta)$ for every negative integer l , which will be defined in Section 8 below, satisfy (A), (B), and (C). However, in general, (D) does not always hold for $\mathcal{J}_l'(X, \Delta)$ with $l = -1, -2, \dots$. The *maximal non-lc ideal sheaf* $\mathcal{J}'(X, \Delta)$ also satisfies (A), (B), and (C), and we do not know if (D) holds for $\mathcal{J}'(X, \Delta)$ or not. However, we have some evidence that it is true. In particular, we will give partial answers to this question in Section 12 and in Section 13. In section 13, we also mention a link between $\mathcal{J}'(X, \Delta)$ and ideals that appear naturally in the study of the Hodge-theory of singular varieties. We conclude this paper by developing a characteristic $p > 0$ analog of $\mathcal{J}'(X, \Delta)$, relying heavily on some recent interesting work of Blickle, see [B].

Finally, it should be noted that, in various presentations, Sándor Kovács has recently been discussing how the ideal $\mathcal{J}'(X, \Delta)$ is a natural ideal to consider. His work in this direction is independent of the authors although certainly inspired by connections with Du Bois singularities; see Section 13.

2. INTRODUCTION

The main purpose of this paper is to consider variants of the *non-lc ideal sheaf* $\mathcal{J}_{NLC}$ introduced in [F2]. We also consider various *non-klt ideal sheaves*. We will explain our motivation, observations, and some attempts in the study of *non-lc ideal sheaves*.

Let D be an effective $\mathbb{R}$ -divisor on a smooth complex variety X . We put

$$\mathcal{J}'(X, D) := \mathcal{J}(X, (1 - \varepsilon)D)$$

for $0 < \varepsilon \ll 1$, where the right hand side is the *multiplier ideal sheaf* associated to $(1 - \varepsilon)D$ and it is independent of ε for sufficiently small $0 < \varepsilon \ll 1$. By the definition of $\mathcal{J}'(X, D)$, the pair (X, D) is log canonical if and only if $\mathcal{J}'(X, D) = \mathcal{O}_X$. We call

$\mathcal{J}'(X, D)$ the *maximal non-lc ideal sheaf* of (X, D) . We will discuss the definition and the basic properties of $\mathcal{J}'$ in Section 7. In general, $\mathcal{J}(X, D) \subsetneq \mathcal{J}(X, (1 - \varepsilon)D)$ for $0 < \varepsilon \ll 1$ and the relationship between $\mathcal{J}(X, D)$ and $\mathcal{J}'(X, D)$ is not clear. So, we need new ideas and techniques to handle $\mathcal{J}'(X, D)$. We believe that the Kawamata–Viehweg–Nadel vanishing theorem is not powerful enough for the study of $\mathcal{J}'(X, D)$. However, the new cohomological package of the first author, explained in Section 4, seems well suited for this task.

Let X be a smooth variety and S a smooth irreducible divisor on X . Let B be an effective $\mathbb{R}$ -divisor on X such that $S \not\subset \text{Supp } B$. Then we have the following equality

$$\mathcal{J}'(S, B|_S) = \mathcal{J}'(X, S + B)|_S.$$

See, for example, Theorem 12.7. We call it the *restriction theorem*. We will partially generalize it to the case of singular varieties in Section 13. In [F2], the first author introduced the notion of (*minimal*) *non-lc ideal sheaves* $\mathcal{J}_{NLC}$ and proved the restriction theorem

$$\mathcal{J}_{NLC}(S, B|_S) = \mathcal{J}_{NLC}(X, S + B)|_S.$$

Both of $\mathcal{J}'(X, D)$ and $\mathcal{J}_{NLC}(X, D)$ define the non-lc locus of the pair (X, D) . However, in general, $\mathcal{J}'(X, D)$ does not always coincide with $\mathcal{J}_{NLC}(X, D)$. We note that

$$\mathcal{J}(X, D) \subset \mathcal{J}_{NLC}(X, D) \subset \mathcal{J}'(X, D)$$

holds by the definitions of $\mathcal{J}$, $\mathcal{J}_{NLC}$, and $\mathcal{J}'$. Although $\mathcal{J}_{NLC}(X, D)$ seems to be the most natural ideal that defines the non-lc locus of (X, D) from the point of view of the minimal model program (cf. [F4]), $\mathcal{J}'(X, D)$ may be more suitable to the theory of multiplier ideal sheaves than $\mathcal{J}_{NLC}(X, D)$.

More generally, we consider a family of non-lc ideal sheaves. We define *intermediate non-lc ideal sheaves* $\mathcal{J}'_l(X, D)$ for every negative integer l . By the definition of $\mathcal{J}'_l(X, D)$ (which is a sheaf that varies with each negative integer l), $\mathcal{J}'_l(X, D)$ defines the non-lc locus of (X, D) and satisfies many of the same useful properties that the first author's original non-lc ideal $\mathcal{J}_{NLC}(X, D)$ enjoys.

Furthermore, there are natural inclusions (where again, the l vary of the *negative* integers)

$$\begin{aligned} \mathcal{J}_{NLC}(X, D) &\subset \cdots \subset \mathcal{J}'_{l-1}(X, D) \\ &\subset \mathcal{J}'_l(X, D) \subset \mathcal{J}'_{l+1}(X, D) \subset \cdots \subset \mathcal{J}'(X, D) \subset \mathcal{O}_X. \end{aligned}$$

Similarly, we also define a family of *non-klt ideal sheaves* $\mathcal{J}_l(X, D)$ for every non-positive integer l . These sheaves satisfy

$$\begin{aligned} \mathcal{J}(X, D) &\subset \cdots \subset \mathcal{J}_{l-1}(X, D) \\ &\subset \mathcal{J}_l(X, D) \subset \mathcal{J}_{l+1}(X, D) \subset \cdots \subset \mathcal{J}_0(X, D) \subset \mathcal{O}_X \end{aligned}$$

and put possibly different scheme structures on the non-klt locus of (X, D) . We have natural inclusions

$$\mathcal{J}_l(X, D) \subset \mathcal{J}'_l(X, D)$$

for every negative integer l , as well as the inclusions

$$\mathcal{J}(X, D) \subset \mathcal{J}_{NLC}(X, D) \quad \text{and} \quad \mathcal{J}_0(X, D) \subset \mathcal{J}'(X, D).$$

Let W be the union of all the lc centers of (X, D) (see our slightly non-standard definition of lc centers in Section 3) and $U = X \setminus W$. Then

$$\mathcal{J}_l(X, D)|_U = \mathcal{J}'_l(X, D)|_U$$

for every negative integer l ,

$$\mathcal{J}(X, D)|_U = \mathcal{J}_{NLC}(X, D)|_U \quad \text{and} \quad \mathcal{J}_0(X, D)|_U = \mathcal{J}'(X, D)|_U.$$

Because the multiplier ideal sheaf has emerged as such a fundamental tool in higher dimensional algebraic geometry, it is natural to desire a non-lc ideal sheaf which agrees with the multiplier ideal sheaf in as wide a setting as possible. If we assume that this is a desired property, then $\mathcal{J}_{NLC}(X, D)$ is the right generalization of $\mathcal{J}(X, D)$.

On the other hand, with regards to condition (B) from Section 1, one way to interpret the term $I(X, \Delta)$ is as a correction term which mitigates for the singularities of Δ . From this point of view, using the maximal non-lc ideal sheaf $\mathcal{J}'(X, \Delta)$ gives a heuristically stronger statement as it says one has to “adjust” to lesser extent.

The multiplier ideal $\mathcal{J}(X = \text{Spec } R, \Delta)$ is also very closely related to the test ideal $\tau_b(R, \Delta)$, a notion that appears in the theory of commutative algebra in positive characteristic, see for example [T1]. We conclude this paper with several sections which explore a positive characteristic analog of $\mathcal{J}'(X, \Delta)$ which we call the non- F -pure ideal and denote it by $\sigma(X, \Delta)$. In order to define this ideal, we rely heavily on some recent work of Blickle, see [B]. In section 15 we then relate the characteristic zero notion $\mathcal{J}'(X, \Delta)$ and the characteristic $p > 0$ notion $\sigma(R, \Delta)$. In the final section, we prove a restriction theorem for $\sigma(X, \Delta)$, including a proof that the formation of $\sigma(X, \Delta)$ commutes with the restriction to an arbitrary codimension normal F -pure center (which is a characteristic $p > 0$ analog of a log canonical center).

We summarize the contents of this paper. This paper is divided into two parts. Part I, consisting of Section 3–13, is devoted to the study of variants of non-lc ideal sheaves $\mathcal{J}_{NLC}$ on complex algebraic varieties. Part II, consisting of Section 14–16, is devoted to the study of a positive characteristic analog of the maximal non-lc ideal sheaves. These two parts are independent to each other, except for the definition of the maximal non-lc ideal sheaves.

In Section 3, we define *lc centers*, *non-klt centers*, and *non-lc centers*. It is very important to distinguish these three notions. In Section 4, we recall Ambro’s formulation of Kollár’s torsion-free and vanishing theorems for the reader’s convenience. In Section 5, we recall the notion of non-lc ideal sheaves introduced in [F2]. In Section 6, we discuss how to define certain non-lc ideal sheaves. We also discuss some properties which should be satisfied by these ideal sheaves. This section is an informal discussion. In Section 7, we will define the *maximal non-lc ideal sheaf* $\mathcal{J}'$ and investigate basic properties of $\mathcal{J}'$. In Section 8, we introduce the notion of *intermediate non-lc ideal sheaves*. Section 9 is a supplement to the fundamental theorems for the log minimal model program in [F4]. In Section 10, we discuss various *non-klt ideal sheaves*. In Section 11, we recall Shokurov’s *differents* for the restriction theorem discussed in Section 12. Sections 12 and 13 are attempts to prove the restriction theorem for $\mathcal{J}'$. Also in section 13, we explain how $\mathcal{J}'(X, \Delta)$

appears in the study of the Hodge theory of singular varieties. In Section 14, we introduce a characteristic p analog of the maximal non-lc-ideal $\mathcal{J}'(X, \Delta)$, called a non-F-pure ideal, and investigate its basic properties. In Section 15, we explore the relationship between non-F-pure ideals and maximal non-lc ideals, which is followed by Section 16 where we prove a restriction theorem for non-F-pure ideals.

We will work over the complex number field $\mathbb{C}$ throughout Part I. But we note that by using the Lefschetz principle, we can extend everything to the case where the base field is an algebraically closed field of characteristic zero. Also, we will use the following notation freely.

Notation. (i) For an $\mathbb{R}$ -Weil divisor $D = \sum_{j=1}^r d_j D_j$ such that D_j is a prime divisor for every j and $D_i \neq D_j$ for $i \neq j$, we define the *round-up* $\lceil D \rceil = \sum_{j=1}^r \lceil d_j \rceil D_j$ (resp. the *round-down* $\lfloor D \rfloor = \sum_{j=1}^r \lfloor d_j \rfloor D_j$), where for every real number x , $\lceil x \rceil$ (resp. $\lfloor x \rfloor$) is the integer defined by $x \leq \lceil x \rceil < x + 1$ (resp. $x - 1 < \lfloor x \rfloor \leq x$). The *fractional part* $\{D\}$ of D denotes $D - \lfloor D \rfloor$. We define

$$D^{=k} = \sum_{d_j=k} d_j D_j = k \sum_{d_j=k} D_j, \quad D^{\leq k} = \sum_{d_j \leq k} d_j D_j,$$

$$D^{< k} = \sum_{d_j < k} d_j D_j, \quad D^{\geq k} = \sum_{d_j \geq k} d_j D_j \quad \text{and} \quad D^{> k} = \sum_{d_j > k} d_j D_j$$

for every $k \in \mathbb{R}$. We put

$${}^k D = \text{Supp } D^{=k}.$$

We note that ${}^0 D = \text{Supp } D^{=0} = 0$ and ${}^1 D = \text{Supp } D^{=1} = D^{=1}$. We call D a *boundary* $\mathbb{R}$ -divisor if $0 \leq d_j \leq 1$ for every j . We note that $\sim_{\mathbb{Q}}$ (resp. $\sim_{\mathbb{R}}$) denotes the $\mathbb{Q}$ -linear (resp. $\mathbb{R}$ -linear) equivalence of $\mathbb{Q}$ -divisors (resp. $\mathbb{R}$ -divisors).

(ii) For a proper birational morphism $f : X \rightarrow Y$, the *exceptional locus* $\text{Exc}(f) \subset X$ is the locus where f is not an isomorphism.

Acknowledgments. The first author was partially supported by the Grant-in-Aid for Young Scientists (A) #20684001 from JSPS and by the Inamori Foundation. The second author was partially supported by an NSF postdoctoral fellowship and by the NSF grant DMS-1064485. The third author was partially supported by the Grant-in-Aid for Young Scientists (B) #20740019 from JSPS and by the Program for Improvement of Research Environment for Young Researchers from SCF commissioned by MEXT of Japan. They would like the referee for many useful suggestions. A part of this work was done during visits of all the authors to MSRI. They are grateful to MSRI for its hospitality and support.

Part 1. Variants of non-lc ideals $\mathcal{J}_{NLC}$

This part is devoted to the study of variants of non-lc ideal sheaves $\mathcal{J}_{NLC}$ on complex algebraic varieties.

3. LC CENTERS, NON-KLT CENTERS, AND NON-LC CENTERS

In this section, we quickly recall the notion of lc and klt pairs and define *lc centers*, *non-klt centers*, and *non-lc centers*.

3.1 (Discrepancies, lc and klt pairs, etc.). Let X be a normal variety and B an effective $\mathbb{R}$ -divisor on X such that $K_X + B$ is $\mathbb{R}$ -Cartier. Let $f : Y \rightarrow X$ be a resolution such that $\text{Exc}(f) \cup f_*^{-1}B$ has a simple normal crossing support, where $f_*^{-1}B$ is the strict transform of B on Y . We write

$$K_Y = f^*(K_X + B) + \sum_i a_i E_i$$

and $a(E_i, X, B) = a_i$. We say that (X, B) is

$$\begin{cases} \text{log canonical (lc, for short)} & \text{if } a_i \geq -1 \text{ for every } i, \text{ and,} \\ \text{kawamata log terminal (klt, for short)} & \text{if } a_i > -1 \text{ for every } i. \end{cases}$$

Note that the *discrepancy* $a(E, X, B) \in \mathbb{R}$ can be defined for every prime divisor E over X . By definition, there exists the largest Zariski open set U (resp. U') of X such that (X, B) is lc (resp. klt) on U (resp. U'). We put $\text{Nlc}(X, B) = X \setminus U$ (resp. $\text{Nklt}(X, B) = X \setminus U'$) and call it the *non-lc locus* (resp. *non-klt locus*) of the pair (X, B) . We sometimes simply denote $\text{Nlc}(X, B)$ by X_{NLC} . We will discuss various scheme structures on $\text{Nlc}(X, B)$ (resp. $\text{Nklt}(X, B)$) in Section 9 (resp. in Section 10).

Let E be a prime divisor over X . The closure of the image of E on X is denoted by $c_X(E)$ and called the *center* of E on X .

3.2 (Lc centers, non-klt centers, and non-lc centers). Let X be a normal variety and B an effective $\mathbb{R}$ -divisor on X such that $K_X + B$ is $\mathbb{R}$ -Cartier. Let E be a prime divisor over X . In this paper, we use the following terminology. The center $c_X(E)$ is

$$\begin{cases} \text{an lc center} & \text{if } a(E, X, B) = -1 \text{ and } c_X(E) \not\subset \text{Nlc}(X, B), \\ \text{a non-klt center} & \text{if } a(E, X, B) \leq -1, \text{ and} \\ \text{a non-lc center} & \text{if } a(E, X, B) < -1. \end{cases}$$

The above terminology is slightly different from the usual one. We note that it is very important to distinguish lc centers, non-klt centers, and non-lc centers in our theory. In the traditional theory of multiplier ideal sheaves, we can not distinguish among lc centers, non-klt centers, and non-lc centers. In our new framework, the notion of lc centers plays very important roles. It is because our arguments heavily depend on the new cohomological package reviewed in Section 4. It is much more powerful than the Kawamata–Viehweg–Nadel vanishing theorem. We note that an lc center is a non-klt center.

The next lemma is almost obvious by the definition of lc centers.

Lemma 3.3. *The number of lc centers of (X, B) is finite even if (X, B) is not log canonical.*

We note the following elementary example.

Example 3.4. Let $X = \mathbb{C}^2 = \text{Spec } \mathbb{C}[x, y]$ and $C = (y^2 = x^3)$. We consider the pair (X, C) . Then we can easily check that there is a prime divisor E over X such that $a(E, X, C) = -1$ and $c_X(E)$ is the origin $(0, 0)$ of $\mathbb{C}^2$ and that (X, C) is not lc at $(0, 0)$. Therefore, the center $c_X(E)$ is a non-klt center but not an lc center of (X, C) .

4. NEW COHOMOLOGICAL PACKAGE

We quickly review Ambro's formulation of torsion-free and vanishing theorems in a simplified form. For more advanced topics and the proof, see [F3, Chapter 2]. The paper [F1] may help the reader to understand the proof of Theorem 4.2. We think that it is not so easy to grasp the importance of Theorem 4.2. We recommend the reader to learn how to use Theorem 4.2 in [F2], [F3], [F4], and this paper.

4.1 (Global embedded simple normal crossing pairs). Let Y be a simple normal crossing divisor on a smooth variety M and D an $\mathbb{R}$ -divisor on M such that $\text{Supp}(D+Y)$ is simple normal crossing and that D and Y have no common irreducible components. We put $B = D|_Y$ and consider the pair (Y, B) . Let $\nu : Y^\nu \rightarrow Y$ be the normalization. We put $K_{Y^\nu} + \Theta = \nu^*(K_Y + B)$. A *stratum* of (Y, B) is an irreducible component of Y or the image of some lc center of $(Y^\nu, \Theta=1)$.

When Y is smooth and B is an $\mathbb{R}$ -divisor on Y such that $\text{Supp } B$ is simple normal crossing, we put $M = Y \times \mathbb{A}^1$ and $D = B \times \mathbb{A}^1$. Then $(Y, B) \simeq (Y \times \{0\}, B \times \{0\})$ satisfies the above conditions.

Theorem 4.2. *Let (Y, B) be as above. Assume that B is a boundary $\mathbb{R}$ -divisor. Let $f : Y \rightarrow X$ be a proper morphism and L a Cartier divisor on Y .*

(1) *Assume that $L - (K_Y + B)$ is f -semi-ample. Let q be an arbitrary non-negative integer. Then every non-zero local section of $R^q f_* \mathcal{O}_Y(L)$ contains in its support the f -image of some stratum of (Y, B) .*

(2) *Let $\pi : X \rightarrow V$ be a proper morphism and assume that $L - (K_Y + B) \sim_{\mathbb{R}} f^* H$ for some π -ample $\mathbb{R}$ -Cartier $\mathbb{R}$ -divisor H on X . Then, $R^q f_* \mathcal{O}_Y(L)$ is π_* -acyclic, that is, $R^p \pi_* R^q f_* \mathcal{O}_Y(L) = 0$ for every $p > 0$ and $q \geq 0$.*

Remark 4.3. It is obvious that the statement of Theorem 4.2 (1) is equivalent to the following one.

(1') *Assume that $L - (K_Y + B)$ is f -semi-ample. Let q be an arbitrary non-negative integer. Then every associated prime of $R^q f_* \mathcal{O}_Y(L)$ is the generic point of the f -image of some stratum of (Y, B) .*

For the proof of Theorem 4.2, see [F3, Theorem 2.39].

Remark 4.4. In Theorem 4.2 (2), it is sufficient to assume that H is π -nef and π -log big. See [F3, Theorem 2.47]. We omit the technical details on nef and log big divisors in order to keep this paper readable.

5. NON-LC IDEAL SHEAVES

Let us recall the definition of non-lc ideal sheaves (cf. [F2, Section 2] and [F4, Section 7]).

Definition 5.1 (Non-lc ideal sheaf). Let X be a normal variety and B an $\mathbb{R}$ -divisor on X such that $K_X + B$ is $\mathbb{R}$ -Cartier. Let $f : Y \rightarrow X$ be a resolution with $K_Y + B_Y = f^*(K_X + B)$ such that $\text{Supp } B_Y$ is simple normal crossing. Then we put

$$\begin{aligned} \mathcal{J}_{NLC}(X, B) &= f_* \mathcal{O}_Y(\lceil -(B_Y^{<1})^\top - \lfloor B_Y^{>1} \rfloor) \\ &= f_* \mathcal{O}_Y(-\lfloor B_Y \rfloor + B_Y^=1) \end{aligned}$$

and call it the *(minimal) non-lc ideal sheaf associated to (X, B)* . If B is effective, then $\mathcal{J}_{NLC}(X, B) \subset \mathcal{O}_X$.

The ideal sheaf $\mathcal{J}_{NLC}(X, B)$ is independent of the choice of resolution, and thus well-defined, by the following easy lemma.

Lemma 5.2. *Let $g : Z \rightarrow Y$ be a proper birational morphism between smooth varieties and B_Y an $\mathbb{R}$ -divisor on Y such that $\text{Supp } B_Y$ is simple normal crossing. Assume that $K_Z + B_Z = g^*(K_Y + B_Y)$ and that $\text{Supp } B_Z$ is simple normal crossing. Then we have*

$$g_*\mathcal{O}_Z(\lceil -(B_Z^{\leq 1})^\top - \lfloor B_Z^{\geq 1} \rfloor) \simeq \mathcal{O}_Y(\lceil -(B_Y^{\leq 1})^\top - \lfloor B_Y^{\geq 1} \rfloor).$$

Proof. By $K_Z + B_Z = g^*(K_Y + B_Y)$, we obtain

$$\begin{aligned} K_Z &= g^*(K_Y + B_Y^{-1} + \{B_Y\}) \\ &\quad + g^*(\lfloor B_Y^{\leq 1} \rfloor + \lfloor B_Y^{\geq 1} \rfloor) - (\lfloor B_Z^{\leq 1} \rfloor + \lfloor B_Z^{\geq 1} \rfloor) - B_Z^{-1} - \{B_Z\}. \end{aligned}$$

If $a(\nu, Y, B_Y^{-1} + \{B_Y\}) = -1$ for a prime divisor ν over Y , then we can check that $a(\nu, Y, B_Y) = -1$ by using [KM, Lemma 2.45]. Since $g^*(\lfloor B_Y^{\leq 1} \rfloor + \lfloor B_Y^{\geq 1} \rfloor) - (\lfloor B_Z^{\leq 1} \rfloor + \lfloor B_Z^{\geq 1} \rfloor)$ is Cartier, we can easily see that

$$g^*(\lfloor B_Y^{\leq 1} \rfloor + \lfloor B_Y^{\geq 1} \rfloor) = \lfloor B_Z^{\leq 1} \rfloor + \lfloor B_Z^{\geq 1} \rfloor + E,$$

where E is an effective f -exceptional Cartier divisor. Thus, we obtain

$$g_*\mathcal{O}_Z(\lceil -(B_Z^{\leq 1})^\top - \lfloor B_Z^{\geq 1} \rfloor) \simeq \mathcal{O}_Y(\lceil -(B_Y^{\leq 1})^\top - \lfloor B_Y^{\geq 1} \rfloor).$$

This completes the proof. $\square$

The next lemma is obvious by definition: Definition 5.1.

Lemma 5.3. *Let X be a normal variety and B an effective $\mathbb{R}$ -divisor on X such that $K_X + B$ is $\mathbb{R}$ -Cartier. Then (X, B) is lc if and only if $\mathcal{J}_{NLC}(X, B) = \mathcal{O}_X$.*

In the following sections, we consider variants of non-lc ideal sheaves.

6. OBSERVATIONS TOWARDS NON-LC IDEAL SHEAVES

First, we informally define $\mathcal{J}'$ as a limit of multiplier ideal sheaves. We will call $\mathcal{J}'(X, B)$ the *maximal non-lc ideal sheaf* of the pair (X, B) . For the details, see Section 7.

6.1. Let D be an effective $\mathbb{R}$ -divisor on a smooth variety X . Let $f : Y \rightarrow X$ be a resolution such that $\text{Exc}(f) \cup \text{Supp } f_*^{-1}D$ is simple normal crossing. Then the *multiplier ideal sheaf* $\mathcal{J}(X, D) \subset \mathcal{O}_X$ associated to D was defined to be

$$\mathcal{J}(X, D) = f_*\mathcal{O}_Y(K_{Y/X} - \lfloor f^*D \rfloor),$$

where $K_{Y/X} = K_Y - f^*K_X$. In this situation, we put

$$\mathcal{J}'(X, D) = \mathcal{J}(X, (1 - \varepsilon)D)$$

for $0 < \varepsilon \ll 1$. We note that the right hand side is independent of ε for $0 < \varepsilon \ll 1$. Therefore, we can write

$$\mathcal{J}'(X, D) = \bigcap_{0 < \varepsilon} \mathcal{J}(X, (1 - \varepsilon)D)$$

since

$$\mathcal{J}(X, (1 - \varepsilon)D) \subset \mathcal{J}(X, (1 - \varepsilon')D)$$

for $0 < \varepsilon < \varepsilon'$. We write $K_Y + \Delta_Y = f^*(K_X + D)$. Then

$$\mathcal{J}(X, D) = f_*\mathcal{O}_Y(-\perp\Delta_Y\lrcorner),$$

and

$$\mathcal{J}(X, (1 - \varepsilon)D) = f_*\mathcal{O}_Y(-\perp\Delta_Y\lrcorner + \sum_{k=-\infty}^{\infty} {}^k\Delta_Y)$$

for $0 < \varepsilon \ll 1$. Since ${}^k\Delta_Y$ is f -exceptional for $k < 0$, we can write

$$\mathcal{J}'(X, D) = f_*\mathcal{O}_Y(-\perp\Delta_Y\lrcorner + \sum_{k=1}^{\infty} {}^k\Delta_Y),$$

This expression is very useful for generalizations.

By definition, we can easily check that

$$\mathcal{J}(X, (1 + \varepsilon)D) = \mathcal{J}(X, D)$$

for $0 < \varepsilon \ll 1$ and that $\mathcal{J}(X, (1 - \varepsilon)D) = \mathcal{J}(X, D)$ for $0 < \varepsilon \ll 1$ if and only if $t = 1$ is not a jumping number of $\mathcal{J}(X, tD)$. In this paper, we are mainly interested in the case when D is a reduced divisor. In this case, $t = 1$ is a jumping number of $\mathcal{J}(X, tD)$ and then $\mathcal{J}'(X, D) \supsetneq \mathcal{J}(X, D)$.

Next, we observe various properties which should be satisfied by *non-lc ideal sheaves*.

6.2. Let X be a smooth projective variety and B an effective integral Cartier divisor on X such that $\text{Supp } B$ is simple normal crossing. We can write $B = \sum_{k=1}^{\infty} {}^k B_k$, where $B_k := {}^k B = \text{Supp } B^{\geq k}$. We would like to define an ideal sheaf $I(X, B) \subset \mathcal{O}_X$ such that $\text{Supp } \mathcal{O}_X/I(X, B) = \text{Nlc}(X, B)$. Let us put

$$I(X, B) = \mathcal{O}_X(-\sum_{k=2}^{\infty} m_k B_k)$$

for some $m_k \geq 1$ for every $k \geq 2$. Then $I(X, B)$ defines the non-lc locus of the pair (X, B) . Let L be a Cartier divisor on X such that $A := L - (K_X + B)$ is ample. For various geometric applications, we think that it is natural to require

$$H^i(X, \mathcal{O}_X(L) \otimes I(X, B)) = 0$$

for all $i > 0$. Since

$$\begin{aligned} \mathcal{O}_X(L) \otimes I(X, B) &= \mathcal{O}_X(K_X + B + A - \sum_{k=2}^{\infty} m_k B_k) \\ &= \mathcal{O}_X(K_X + B_1 + \sum_{k=2}^{\infty} (k - m_k) B_k + A), \end{aligned}$$

In view of the Norimatsu vanishing theorem (cf. [L, Lemma 4.3.5]), if we hope for vanishing, we should make m_k equal k or $k - 1$ for every $k \geq 2$. If $m_k = k$ for every $k \geq 2$, then

$$I(X, B) = \mathcal{J}_{\text{NLC}}(X, B).$$

If $m_k = k - 1$ for every $k \geq 2$, then

$$I(X, B) = \mathcal{J}'(X, B).$$

Let $f : Y \rightarrow X$ be a blow-up along a stratum of $\text{Supp } B$, where a *stratum* of $\text{Supp } B$ means an lc center of $(X, \text{Supp } B)$. We put $K_Y + B_Y = f^*(K_X + B)$. Then it is natural to require

$$I(Y, B_Y) = \mathcal{O}_Y(-\sum_{k=2}^{\infty} n_k B_Y)$$

such that $n_k = k$ or $k - 1$ for every $k \geq 2$ and

$$f_* I(Y, B_Y) = I(X, B).$$

We think that the most *natural* choices for non-lc ideal sheaves are

$$I(X, B) = \mathcal{J}_{NLC}(X, B) = \mathcal{O}_X(-\sum_{k=2}^{\infty} k B_k)$$

or

$$I(X, B) = \mathcal{J}'(X, B) = \mathcal{O}_X(-\sum_{k=2}^{\infty} (k - 1) B_k).$$

The ideal sheaf $\mathcal{J}_{NLC}(X, B)$ should be called *minimal* non-lc ideal sheaf of (X, B) and $\mathcal{J}'(X, B)$ should be called *maximal* non-lc ideal sheaf of (X, B) .

The smaller $B_1 + \sum_{k=2}^{\infty} (k - m_k) B_k$ is, the more easily we can apply our torsion-free theorem (cf. Theorem 4.2 (1)) to $I(X, B)$. It is one of the main reasons why the first author adopted $\mathcal{J}_{NLC}(X, B)$ to define $\text{Nlc}(X, B)$.

Finally, we put

$$I(X, B) = \mathcal{O}_X(-\sum_{k=2}^{1-l} k B_k - \sum_{k=2-l}^{\infty} (k - 1) B_k) =: \mathcal{J}'_l(X, B)$$

for $l = 0, -1, \dots, -\infty$. Then

$$\mathcal{J}_{NLC}(X, B) = \mathcal{J}'_{-\infty}(X, B) \subset \mathcal{J}'_l(X, B) \subset \mathcal{J}'_0(X, B) = \mathcal{J}'(X, B)$$

and $\mathcal{J}'_l(X, B)$ satisfies all the above desired properties for every l . We will discuss $\mathcal{J}'_l(X, B)$ for every negative integer l in Section 8. We do not know whether $\mathcal{J}'_l(X, B)$ with $l \neq 0, -\infty$ is useful or not for geometric applications.

7. MAXIMAL NON-LC IDEAL SHEAVES

Let us define *maximal non-lc ideal sheaves*.

Definition 7.1. Let X be a normal variety and Δ an $\mathbb{R}$ -divisor on X such that $K_X + \Delta$ is $\mathbb{R}$ -Cartier. Let $f : Y \rightarrow X$ be a resolution with $K_Y + \Delta_Y = f^*(K_X + \Delta)$ such that $\text{Supp } \Delta_Y$ is simple normal crossing. Then we put

$$\mathcal{J}'(X, \Delta) = f_* \mathcal{O}_Y(\lceil K_Y - f^*(K_X + \Delta) + \varepsilon F \rceil)$$

for $0 < \varepsilon \ll 1$, where $F = \text{Supp } \Delta_Y^{\geq 1}$.

It is easy to see that the right hand side does not depend on ε if $0 < \varepsilon \ll 1$. We note that

$$\begin{aligned} \lceil K_Y - f^*(K_X + \Delta) + \varepsilon F \rceil &= \lceil -\Delta_Y + \varepsilon F \rceil \\ &= -\lfloor \Delta_Y \rfloor + \sum_{k=1}^{\infty} k \Delta_Y \end{aligned}$$

for $0 < \varepsilon \ll 1$. Therefore, we can write

$$\begin{aligned} \mathcal{J}'(X, \Delta) &= f_* \mathcal{O}_Y(\lceil -\Delta_Y + \varepsilon F \rceil) \\ &= f_* \mathcal{O}_Y(-\lfloor \Delta_Y \rfloor + \sum_{k=1}^{\infty} k \Delta_Y). \end{aligned}$$

By Lemma 7.3 below, $\mathcal{J}'(X, \Delta)$ does not depend on the resolution $f : Y \rightarrow X$.

We note that

$$\mathcal{J}(X, \Delta) = f_* \mathcal{O}_Y(-\lfloor \Delta_Y \rfloor)$$

is the *multiplier ideal sheaf* associated to the pair (X, Δ) and that

$$\mathcal{J}_{NLC}(X, \Delta) = f_* \mathcal{O}_Y(-\lfloor \Delta_Y \rfloor + \Delta_Y^{\leq -1})$$

is the (*minimal*) *non-lc ideal sheaf* associated to the pair (X, Δ) (cf. Definition 5.1). It is obvious that

$$\mathcal{J}(X, \Delta) \subset \mathcal{J}_{NLC}(X, \Delta) \subset \mathcal{J}'(X, \Delta)$$

by the above definitions, and it is also easy to check that the definition of $\mathcal{J}'(X, \Delta)$ agrees with that given in 6.1 when X is smooth and Δ is effective.

From now on, we assume that Δ is effective. Then $\mathcal{J}'(X, \Delta)$ is an ideal sheaf on X . We are mainly interested in the case when Δ is effective due to following fact.

Lemma 7.2. *Assume that Δ is effective. Then (X, Δ) is log canonical if and only if $\mathcal{J}'(X, \Delta) = \mathcal{O}_X$.*

Lemma 7.3. *Let X be a smooth variety and Δ an $\mathbb{R}$ -divisor on X such that $\text{Supp } \Delta$ is simple normal crossing. Let $f : Y \rightarrow X$ be a proper birational morphism such that $\text{Exc}(f) \cup \text{Supp } f_*^{-1} \Delta$ is simple normal crossing. We put $K_Y + \Delta_Y = f^*(K_X + \Delta)$. Then*

$$f_* \mathcal{O}_Y(\lceil -\Delta_Y + \varepsilon' F' \rceil) \simeq \mathcal{O}_X(\lceil -\Delta + \varepsilon F \rceil)$$

for $0 < \varepsilon, \varepsilon' \ll 1$, where $F = \text{Supp } \Delta^{\geq 1}$ and $F' = \text{Supp } \Delta_Y^{\geq 1}$.

Proof. Since $K_Y + \Delta_Y = f^*(K_X + \Delta)$, we can write

$$K_Y = f^*(K_X + \{\Delta - \varepsilon F\} + \varepsilon F) + f^* \lfloor \Delta \rfloor - \varepsilon F \rfloor - \Delta_Y.$$

We note that $\{\Delta - \varepsilon F\} + \varepsilon F$ is a boundary $\mathbb{R}$ -divisor whose support is simple normal crossing for $0 < \varepsilon \ll 1$ and that $(X, \{\Delta - \varepsilon F\})$ is klt. Thus, $a(\nu, X, \{\Delta - \varepsilon F\} + \varepsilon F) \geq -1$ for every ν (assuming again $0 < \varepsilon \ll 1$). We can easily check that $a(\nu, X, F) = -1$ if $a(\nu, X, \{\Delta - \varepsilon F\} + \varepsilon F) = -1$ and that $a(\nu, X, F) = -1$ induces $a(\nu, X, \Delta) \leq -1$ (cf. [KM, Lemma 2.45]). Therefore, the round-up of $f^* \lfloor \Delta \rfloor - \varepsilon F \rfloor - \Delta_Y + \varepsilon' F'$ is effective. So, we can write

$$f^* \lfloor \Delta \rfloor - \varepsilon F \rfloor - \lfloor \Delta_Y \rfloor - \varepsilon' F' \rfloor = E,$$

where E is an effective Cartier divisor on Y . We can easily check that E is f -exceptional for $0 < \varepsilon, \varepsilon' \ll 1$. Thus, we obtain

$$f_*\mathcal{O}_Y(\ulcorner -\Delta_Y + \varepsilon' F'^\urcorner) \simeq \mathcal{O}_X(\ulcorner -\Delta + \varepsilon F^\urcorner)$$

since $\ulcorner -\Delta_Y + \varepsilon' F'^\urcorner = f^*(\ulcorner -\Delta + \varepsilon F^\urcorner) + E$. $\square$

We can also define $\mathcal{J}'$ for ideal sheaves.

Definition 7.4. Let X be a normal variety and Δ an $\mathbb{R}$ -divisor on X such that $K_X + \Delta$ is $\mathbb{R}$ -Cartier. Let $\mathfrak{a} \subset \mathcal{O}_X$ be a non-zero ideal sheaf on X and c a real number. Let $f : Y \rightarrow X$ be a resolution such that $K_Y + \Delta_Y = f^*(K_X + \Delta)$ and $f^{-1}\mathfrak{a} = \mathcal{O}_Y(-E)$, where $\text{Supp } \Delta_Y \cup \text{Supp } E$ has a simple normal crossing support. We put

$$\mathcal{J}'((X, \Delta); \mathfrak{a}^c) = f_*\mathcal{O}_Y(-\lfloor \Delta_Y + cE \rfloor + \sum_{k=1}^{\infty} k(\Delta_Y + cE)).$$

We sometime write

$$\mathcal{J}'((X, \Delta); c \cdot \mathfrak{a}) = \mathcal{J}'((X, \Delta); \mathfrak{a}^c).$$

Of course, $\mathcal{J}'((X, \Delta); \mathfrak{a}^c)$ dose not depend on $f : Y \rightarrow X$ by Lemma 7.3. We recall that

$$\mathcal{J}((X, \Delta); \mathfrak{a}^c) = f_*\mathcal{O}_Y(-\lfloor \Delta_Y + cE \rfloor).$$

Lemma 7.5. *Let X be a normal variety and Δ an effective $\mathbb{R}$ -divisor on X such that $K_X + \Delta$ is $\mathbb{R}$ -Cartier. Then we have*

$$\mathcal{J}'(X, \Delta) = \mathcal{J}((X, \Delta); \mathcal{J}(X, \Delta)^{-\varepsilon})$$

for $0 < \varepsilon \ll 1$. In particular,

$$\mathcal{J}'(X, \Delta) = \bigcap_{0 < \varepsilon} \mathcal{J}((X, \Delta); \mathcal{J}(X, \Delta)^{-\varepsilon}).$$

Although we will not use Lemma 7.5 in the proof of the restriction theorem (cf. Theorem 12.7), this lemma may help us understand $\mathcal{J}'$.

Proof. Let $f : Y \rightarrow X$ be a resolution with $f^{-1}\mathcal{J}(X, \Delta) = \mathcal{O}_Y(-E)$ such that $\text{Exc}(f)$, $\text{Supp } f_*^{-1}\Delta$, $\text{Supp } E$, and $\text{Exc}(f) \cup \text{Supp } f_*^{-1}\Delta \cup \text{Supp } E$ are simple normal crossing divisors. We put $K_Y + \Delta_Y = f^*(K_X + \Delta)$. Then

$$\mathcal{J}((X, \Delta); \mathcal{J}(X, \Delta)^{-\varepsilon}) = f_*\mathcal{O}_Y(-\lfloor \Delta_Y - \varepsilon E \rfloor)$$

by definition. Since $\text{Supp } \Delta_Y^{\geq 1} \subset \text{Supp } E$ and $\Delta_Y^{< 0}$ is f -exceptional, we can easily check that

$$f_*\mathcal{O}_Y(-\lfloor \Delta_Y - \varepsilon E \rfloor) = f_*\mathcal{O}_Y(\ulcorner -\Delta_Y + \varepsilon E^\urcorner) = \mathcal{J}'(X, \Delta).$$

This completes the proof. $\square$

By this lemma, $\mathcal{J}'(X, \Delta)$ itself is a multiplier ideal sheaf. The vanishing theorem holds for $\mathcal{J}'$.

Theorem 7.6 (Vanishing theorem). *Let X be a normal variety and Δ an $\mathbb{R}$ -divisor on X such that $K_X + \Delta$ is $\mathbb{R}$ -Cartier. Let $\pi : X \rightarrow S$ be a projective morphism onto an algebraic variety S and L a Cartier divisor on X . Assume that $L - (K_X + \Delta)$ is π -ample. Then we have*

$$R^i \pi_*(\mathcal{J}'(X, \Delta) \otimes \mathcal{O}_X(L)) = 0$$

for all $i > 0$.

Proof. Let $f : Y \rightarrow X$ be a resolution of X with $K_Y + \Delta_Y = f^*(K_X + \Delta)$ such that $\text{Supp } \Delta_Y$ is simple normal crossing. Then

$$A - N + G + f^*L - (K_Y + \Delta_Y^{\leq 1} + \{\Delta_Y\} + G) = f^*(L - (K_X + \Delta)),$$

where $A = \lceil -(\Delta_Y^{\leq 1}) \rceil$, $N = \lfloor \Delta_Y^{\geq 1} \rfloor$, and $G = \sum_{k=2}^{\infty} k \Delta_Y$. Therefore, $R^i \pi_*(f_* \mathcal{O}_Y(A - N + G + f^*L)) = 0$ for all $i > 0$ by Theorem 4.2 (2). Thus, we obtain the desired vanishing theorem since

$$f_* \mathcal{O}_Y(A - N + G + f^*L) \simeq \mathcal{J}'(X, \Delta) \otimes \mathcal{O}_X(L).$$

We finish the proof. $\square$

Remark 7.7. When Δ is effective in Theorem 7.6, the assumption that $L - (K_X + \Delta)$ is π -ample can be replaced by the following weaker assumption: π is only proper, $L - (K_X + \Delta)$ is π -nef and π -big, $(L - (K_X + \Delta))|_{\text{Nlc}(X, \Delta)}$ is π -ample, and $(L - (K_X + \Delta))|_C$ is π -big for every lc center C of (X, Δ) . For details, see [F3, Theorem 2.47] and Remark 4.4.

We close this section with the following simple example. Here, we use the notation in [L, 9.3.C Monomial Ideals].

Theorem 7.8. *Let $\mathfrak{a}$ be a monomial ideal on $X = \mathbb{C}^n$. Then $\mathcal{J}'(c \cdot \mathfrak{a}) = \mathcal{J}'((X, 0); c \cdot \mathfrak{a})$ is the monomial ideal generated by all monomials x^v whose exponent vectors satisfy the condition that*

$$v + \mathbf{1} \in P(c \cdot \mathfrak{a}),$$

where $P(c \cdot \mathfrak{a})$ is the Newton polyhedron of $c \cdot \mathfrak{a}$.

Proof. It is obvious by Howald's theorem (cf. [L, Theorem 9.3.27]) since $\mathcal{J}'(X, c \cdot \mathfrak{a}) = \mathcal{J}(X, (1 - \varepsilon)c \cdot \mathfrak{a})$ for $0 < \varepsilon \ll 1$. $\square$

8. INTERMEDIATE NON-LC IDEAL SHEAVES

This section is a continuation of Section 6.

8.1. Let X be a normal variety and Δ an $\mathbb{R}$ -divisor on X such that $K_X + \Delta$ is $\mathbb{R}$ -Cartier. Let $f : Y \rightarrow X$ be a resolution with $K_Y + \Delta_Y = f^*(K_X + \Delta)$ such that $\text{Supp } \Delta_Y$ is simple normal crossing. Then we set

$$\mathcal{J}'_l(X, \Delta) = f_* \mathcal{O}_Y(-\lfloor \Delta_Y \rfloor + \Delta_Y^{\leq 1} + \sum_{k=2-l}^{\infty} k \Delta_Y)$$

for $l = 0, -1, \dots, -\infty$. We have the natural inclusions

$$\begin{aligned} \mathcal{J}_{NLC}(X, \Delta) &= \mathcal{J}'_{-\infty}(X, \Delta) \subset \dots \subset \mathcal{J}'_l(X, \Delta) \\ &\subset \mathcal{J}'_{l+1}(X, \Delta) \subset \dots \subset \mathcal{J}'_0(X, \Delta) = \mathcal{J}'(X, \Delta). \end{aligned}$$

Of course, it is obvious that there are only finitely many distinct ideals in the set $\{\mathcal{J}'_l(X, \Delta)\}_{l=0, -1, \dots, -\infty}$.

We note that

$$\lceil -\Delta_Y + \Delta_Y^{\leq 1} + \varepsilon F^\top \rceil = -\lfloor \Delta_Y \rfloor + \Delta_Y^{\leq 1} + \sum_{k=2-l}^{\infty} k \Delta_Y$$

when $F = \text{Supp } \Delta^{\geq 2-l}$ and $0 < \varepsilon \ll 1$. Thus, $\mathcal{J}'_l(X, \Delta)$ is well-defined by the following lemma.

Lemma 8.2. *Let X be a smooth variety and Δ an $\mathbb{R}$ -divisor on X such that $\text{Supp } \Delta$ is simple normal crossing. Let $f : Y \rightarrow X$ be a proper birational morphism such that $\text{Exc}(f) \cup \text{Supp } f_*^{-1} \Delta$ is simple normal crossing. We put $K_Y + \Delta_Y = f^*(K_X + \Delta)$. Then*

$$f_* \mathcal{O}_Y(\lceil -\Delta_Y + \Delta_Y^{\leq 1} + \varepsilon' F'^\top \rceil) \simeq \mathcal{O}_X(\lceil -\Delta + \Delta^{\leq 1} + \varepsilon F^\top \rceil)$$

for $0 < \varepsilon, \varepsilon' \ll 1$, where $F = \text{Supp } \Delta^{\geq m}$ and $F' = \text{Supp } \Delta_Y^{\geq m}$ for every positive integer $m \geq 2$.

Proof. Since $K_Y + \Delta_Y = f^*(K_X + \Delta)$, we can write

$$K_Y = f^*(K_X + \{\Delta - \Delta^{\leq 1} - \varepsilon F\} + \Delta^{\leq 1} + \varepsilon F) + f^* \lfloor \Delta - \Delta^{\leq 1} - \varepsilon F \rfloor - \Delta_Y.$$

We note that $\{\Delta - \Delta^{\leq 1} - \varepsilon F\} + \Delta^{\leq 1} + \varepsilon F$ is a boundary $\mathbb{R}$ -divisor whose support is simple normal crossing. Thus,

$$a(\nu, X, \{\Delta - \Delta^{\leq 1} - \varepsilon F\} + \Delta^{\leq 1} + \varepsilon F) \geq -1$$

for every ν . We can easily check that $a(\nu, X, \Delta^{\leq 1} + F) = -1$ if $a(\nu, X, \{\Delta - \Delta^{\leq 1} - \varepsilon F\} + \Delta^{\leq 1} + \varepsilon F) = -1$ and that $a(\nu, X, \Delta^{\leq 1} + F) = -1$ induces $a(\nu, X, \Delta) = -1$ or $a(\nu, X, \Delta) \leq -m$ (cf. [KM, Lemma 2.45]). Therefore, the round-up of $f^* \lfloor \Delta - \Delta^{\leq 1} - \varepsilon F \rfloor - \Delta_Y + \Delta_Y^{\leq 1} + \varepsilon' F'$ is effective. So, we can write

$$f^* \lfloor \Delta - \Delta^{\leq 1} - \varepsilon F \rfloor - \lfloor \Delta_Y \rfloor - \Delta_Y^{\leq 1} - \varepsilon' F' \rfloor = E,$$

where E is an effective Cartier divisor on Y . We can easily check that E is f -exceptional for $0 < \varepsilon, \varepsilon' \ll 1$. Thus, we obtain

$$f_* \mathcal{O}_Y(\lceil -\Delta_Y + \Delta_Y^{\leq 1} + \varepsilon' F'^\top \rceil) \simeq \mathcal{O}_X(\lceil -\Delta + \Delta^{\leq 1} + \varepsilon F^\top \rceil)$$

since $\lceil -\Delta_Y + \Delta_Y^{\leq 1} + \varepsilon' F'^\top \rceil = f^*(\lceil -\Delta + \Delta^{\leq 1} + \varepsilon F^\top \rceil) + E$. $\square$

The next property is obvious by the definition of $\mathcal{J}'_l$.

Lemma 8.3. *Let X be a normal variety and Δ an effective $\mathbb{R}$ -divisor on X such that $K_X + \Delta$ is $\mathbb{R}$ -Cartier. Then, for every l , (X, Δ) is log canonical if and only if $\mathcal{J}'_l(X, \Delta) = \mathcal{O}_X$.*

We note the following Bertini type theorem.

Lemma 8.4. *Let X be a normal variety and Δ an effective $\mathbb{R}$ -divisor on X such that $K_X + \Delta$ is $\mathbb{R}$ -Cartier. Let Λ be a linear system on X and $D \in \Lambda$ a general member of Λ . Then*

$$\mathcal{J}'_l(X, \Delta) = \mathcal{J}'_l(X, \Delta + tD)$$

outside the base locus $\text{Bs}\Lambda$ of Λ for all $0 \leq t \leq 1$ and l .

Proof. By replacing X with $X \setminus \text{Bs}\Lambda$, we can assume that $\text{Bs}\Lambda = \emptyset$. Let $f : Y \rightarrow X$ be a resolution as in 8.1. Since D is a general member of Λ , $f^*D = f_*^{-1}D$ is a smooth divisor on Y such that $\text{Supp } f^*D \cup \text{Supp } \Delta_Y$ is simple normal crossing. Therefore, we can check that

$$\mathcal{J}'_l(X, \Delta + tD) = \mathcal{J}'_l(X, \Delta)$$

for all $0 \leq t \leq 1$ and l . $\square$

The vanishing theorem also holds for $\mathcal{J}'_l$.

Theorem 8.5 (Vanishing theorem). *Let X be a normal variety and Δ an $\mathbb{R}$ -divisor on X such that $K_X + \Delta$ is $\mathbb{R}$ -Cartier. Let $\pi : X \rightarrow S$ be a projective morphism onto an algebraic variety S and L a Cartier divisor on X . Assume that $L - (K_X + \Delta)$ is π -ample. Then we have*

$$R^i \pi_*(\mathcal{J}'_l(X, \Delta) \otimes \mathcal{O}_X(L)) = 0$$

for all $i > 0$ and every l .

Proof. We put $G = \sum_{k=2-l}^{\infty} k\Delta_Y$. Then the proof of Theorem 7.6 works without any changes. $\square$

Note that Remark 7.7 works for every l . We also note that the restriction theorem does not necessarily hold for $l \neq 0, -\infty$.

Example 8.6. Let $X = \mathbb{C}^2 = \text{Spec } \mathbb{C}[x, y]$, $S = (x = 0)$, $C = (y^2 = x^3)$ and $B = 2C$. We put $K_S + B_S = (K_X + S + B)|_S$, and we will compare the intermediate non-lc ideals of $(X, S + B)$ and of (S, B_S) .

We consider the following sequence of blow-ups:

$$X \xleftarrow{f_1} X_1 \xleftarrow{f_2} X_2 \xleftarrow{f_3} X_3.$$

We denote by E_i the exceptional curve of f_i (and we use the same letter for its strict transform). Let $f_1 : X_1 \rightarrow X$ be the blow-up at the origin, $f_2 : X_2 \rightarrow X_1$ be the blow-up at the intersection point of E_1 and C and $f_3 : X_3 \rightarrow X_2$ be the blow-up at the intersection point of E_1, E_2 and C . Then $\pi := f_3 \circ f_2 \circ f_1 : X_3 \rightarrow X$ is a log resolution of $(X, S + B)$, and we have $K_{X_3/X} = E_1 + 2E_2 + 4E_3$, $\pi^*B = 4E_1 + 6E_2 + 12E_3 + B$ and $\pi^*S = E_1 + E_2 + 2E_3 + S$. By the projection formula, we obtain

$$\begin{aligned} \mathcal{J}'_{-1}(X, S + B) &= \pi_* \mathcal{O}_{X_3}(-3E_1 - 4E_2 - 9E_3 - B) \\ &= \pi_*(\mathcal{O}_{X_3}(E_1 + 2E_2 + 3E_3) \otimes \pi^* \mathcal{O}_X(-B)) \\ &= \mathcal{O}_X(-B). \end{aligned}$$

On the other hand, since $B_S = (y^4 = 0)$ in S , one can easily see that

$$\mathcal{J}'_{-1}(S, B_S) = \mathfrak{m}^3,$$

where $\mathfrak{m}$ is the maximal ideal corresponding to $0 \in S$. Of course, we have

$$\mathcal{J}'_{-1}(X, S + B)|_S = \mathcal{O}_X(-B)|_S = \mathfrak{m}^4.$$

Thus, we obtain

$$\mathcal{J}'_{-1}(X, S + B)|_S \subsetneq \mathcal{J}'_{-1}(S, B_S).$$

9. SUPPLEMENTARY REMARKS

This section is a supplement to [F4]. We consider various scheme structures on the non-lc locus of the pair (X, B) .

Definition 9.1. Let X be a normal variety and B an effective $\mathbb{R}$ -divisor on X such that $K_X + B$ is $\mathbb{R}$ -Cartier. We put

$$\mathcal{O}_{\mathrm{Nlc}(X, B)_l} = \mathcal{O}_X / \mathcal{J}'_l(X, B)$$

for every $l = -\infty, \dots, 0$. We simply write

$$\mathcal{O}_{\mathrm{Nlc}(X, B)} = \mathcal{O}_{\mathrm{Nlc}(X, B)_{-\infty}} = \mathcal{O}_X / \mathcal{J}_{\mathrm{NLC}}(X, B)$$

and

$$\mathcal{O}_{\mathrm{Nlc}(X, B)'} = \mathcal{O}_{\mathrm{Nlc}(X, B)_0} = \mathcal{O}_X / \mathcal{J}'_0(X, B) = \mathcal{O}_X / \mathcal{J}'(X, B).$$

We note that there exists natural surjection

$$\mathcal{O}_{\mathrm{Nlc}(X, B)_l} \rightarrow \mathcal{O}_{\mathrm{Nlc}(X, B)_k}$$

for every $l < k$.

The following theorem is a slight generalization of [F4, Theorem 8.1]. The proof of [F4, Theorem 8.1] works without any modifications.

Theorem 9.2. *Let l be an arbitrary non-positive integer or $-\infty$. Let X be a normal variety and B an effective $\mathbb{R}$ -divisor on X such that $K_X + B$ is $\mathbb{R}$ -Cartier. Let D be a Cartier divisor on X . Assume that $D - (K_X + B)$ is π -ample, where $\pi : X \rightarrow S$ is a projective morphism onto a variety S . Let $\{C_i\}$ be any set of lc centers of the pair (X, B) . We put $W = \bigcup C_i$ with the reduced scheme structure. Assume that W is disjoint from $\mathrm{Nlc}(X, B)$.*

Then we have

$$R^i \pi_*(\mathcal{J} \otimes \mathcal{O}_X(D)) = 0$$

for every $i > 0$, where $\mathcal{J} = \mathcal{I}_W \cdot \mathcal{J}'_l(X, B) \subset \mathcal{O}_X$ and $\mathcal{I}_W$ is the defining ideal sheaf of W on X . Therefore, the restriction map

$$\pi_* \mathcal{O}_X(D) \rightarrow \pi_* \mathcal{O}_W(D) \oplus \pi_* \mathcal{O}_{\mathrm{Nlc}(X, B)_l}(D)$$

is surjective and

$$R^i \pi_* \mathcal{O}_W(D) = 0$$

for every $i > 0$. In particular, the restriction maps

$$\pi_* \mathcal{O}_X(D) \rightarrow \pi_* \mathcal{O}_W(D)$$

and

$$\pi_* \mathcal{O}_X(D) \rightarrow \pi_* \mathcal{O}_{\mathrm{Nlc}(X, B)_l}(D)$$

are surjective.

We close this section with the next supplementary result. The proof is obvious.

Proposition 9.3. *In the non-vanishing theorem (cf. [F4, Theorem 12.1]) and the base point free theorem (cf. [F4, Theorem 13.1]), we assumed that $\mathcal{O}_{\mathrm{Nlc}(X,B)}(mL)$ is $\pi|_{\mathrm{Nlc}(X,B)}$ -generated for every $m \gg 0$. However, it is sufficient to assume that $\mathcal{O}_{\mathrm{Nlc}(X,B)_l}(mL)$ is $\pi|_{\mathrm{Nlc}(X,B)_l}$ -generated for every $m \gg 0$, where l is any non-positive integer. We note that, for every l ,*

$$\pi_* \mathcal{O}_X(mL) \rightarrow \pi_* \mathcal{O}_{\mathrm{Nlc}(X,B)_l}(mL)$$

is surjective for $m \geq a$ by the vanishing theorem (cf. Theorem 9.2).

Therefore, in [F4], we can adopt $\mathcal{J}'_l(X, B)$ for any l instead of $\mathcal{J}_{\mathrm{NLC}}(X, B)$. However, from the point of view of the minimal model program, we believe that $\mathcal{J}_{\mathrm{NLC}}(X, B)$ is the most natural defining ideal sheaf of the non-lc locus $\mathrm{Nlc}(X, B)$ of (X, B) . Also see Remark 10.7 below.

10. NON-KLT IDEAL SHEAVES

In this section, we consider *non-klt ideal sheaves*.

10.1. Let X be a normal variety and Δ an $\mathbb{R}$ -divisor on X such that $K_X + \Delta$ is $\mathbb{R}$ -Cartier. Let $f : Y \rightarrow X$ be a resolution with $K_Y + \Delta_Y = f^*(K_X + \Delta)$ such that $\mathrm{Supp} \Delta_Y$ is simple normal crossing. Then we put

$$\mathcal{J}_l(X, \Delta) = f_* \mathcal{O}_Y(-\lfloor \Delta_Y \rfloor + \sum_{k=2-l}^{\infty} k \Delta_Y)$$

for $l = 0, -1, \dots, -\infty$. We have the natural inclusions

$$\begin{aligned} \mathcal{J}(X, \Delta) = \mathcal{J}_{-\infty}(X, \Delta) &\subset \dots \subset \mathcal{J}_l(X, \Delta) \\ &\subset \mathcal{J}_{l+1}(X, \Delta) \subset \dots \subset \mathcal{J}_0(X, \Delta). \end{aligned}$$

We note that $\mathcal{J}(X, \Delta) = \mathcal{J}_{-\infty}(X, \Delta)$ is the usual multiplier ideal sheaf associated to the pair (X, Δ) . Of course, it is obvious that there are only finite numbers of different ideals in $\{\mathcal{J}_l(X, \Delta)\}_{l=0, -1, \dots, -\infty}$. It is also obvious that

$$\mathcal{J}_l(X, \Delta) \subset \mathcal{J}'_l(X, \Delta)$$

for every l . It is easy to check that $\mathcal{J}_l(X, \Delta)$ is well-defined, that is, it does not depend on the resolution $f : Y \rightarrow X$ (cf. Lemma 8.2). We note that $\mathcal{J}_l(X, \Delta) \subset \mathcal{O}_X$ when Δ is effective.

Lemma 10.2. *Assume that Δ is effective. Then we have*

$$\mathcal{J}_l(X, \Delta)|_U = \mathcal{J}'_l(X, \Delta)|_U$$

for every l , where $U = X \setminus W$ and W is the union of all the lc centers of (X, Δ) .

Proof. By shrinking X , we can assume that $U = X$. In this case, $f(\Delta_Y^{\leq 1}) \subset \mathrm{Nlc}(X, \Delta)$. Therefore, we see that $\mathcal{J}_l(X, \Delta) = \mathcal{J}'_l(X, \Delta)$ for every l . $\square$

Lemma 10.3. *Let X be a normal variety and Δ an effective $\mathbb{R}$ -divisor on X such that $K_X + \Delta$ is $\mathbb{R}$ -Cartier. Then, for every l , (X, Δ) is kawamata log terminal if and only if $\mathcal{J}_l(X, \Delta) = \mathcal{O}_X$.*

We obtain the following vanishing theorem without any difficulties as an application of Theorem 4.2 (2).

Theorem 10.4 (Vanishing theorem). *Let X be a normal variety and Δ an $\mathbb{R}$ -divisor on X such that $K_X + \Delta$ is $\mathbb{R}$ -Cartier. Let $\pi : X \rightarrow S$ be a projective morphism onto an algebraic variety S and L a Cartier divisor on X . Assume that $L - (K_X + \Delta)$ is π -ample. Then we have*

$$R^i \pi_*(\mathcal{I}_l(X, \Delta) \otimes \mathcal{O}_X(L)) = 0$$

for all $i > 0$ and every l .

Proof. See the proof of Theorem 7.6. $\square$

Remark 10.5. When Δ is effective in Theorem 10.4, the assumption that $L - (K_X + \Delta)$ is π -ample can be replaced by the following weaker assumption: π is only proper, $L - (K_X + \Delta)$ is π -nef and π -big, and $(L - (K_X + \Delta))|_{\text{Nlc}(X, \Delta)}$ is π -ample (cf. [F3, Theorem 2.47] and Remark 4.4). It is well known that it is sufficient to assume $L - (K_X + \Delta)$ is π -nef and π -big for $l = -\infty$. It is nothing but the Kawamata–Viehweg–Nadel vanishing theorem.

10.6. We close this section with the following very important remark.

Remark 10.7. Let D be an effective $\mathbb{Q}$ -divisor on a smooth variety X . Definition 9.3.9 in [L] says that the pair (X, D) is log canonical if $\mathcal{J}(X, (1 - \varepsilon)D) = \mathcal{O}_X$ for all $0 < \varepsilon < 1$.

Again note that $\mathcal{J}'(X, D)|_U$ does not always coincide with $\mathcal{J}(X, D)|_U$, where $U = X \setminus W$ and W is the union of all the lc centers of (X, D) . This may be a desirable property in certain circumstances, and so the scheme structure on $\text{Nlc}(X, D)$ induced by $\mathcal{J}'(X, D) = \mathcal{J}(X, (1 - \varepsilon)D)$ for $0 < \varepsilon \ll 1$ may be less suitable in some applications than the scheme structure induced by $\mathcal{J}_{\text{NLC}}(X, D)$.

11. DIFFERENTS

Let us recall the definition and the basic properties of Shokurov's *differents* following [Sh, §3] and [A, 9.2.1]. See also [F4, Section 14].

11.1 (Differents). Let X be a normal variety and $S + B$ an $\mathbb{R}$ -divisor on X such that $K_X + S + B$ is $\mathbb{R}$ -Cartier. Assume that S is reduced and that S and B have no common irreducible components. Let $f : Y \rightarrow X$ be a resolution such that

$$K_Y + S_Y + B_Y = f^*(K_X + S + B)$$

and that $\text{Supp}(S_Y + B_Y)$ is simple normal crossing and S_Y is smooth, where S_Y is the strict transform of S on Y . Let $\nu : S^\nu \rightarrow S$ be the normalization. Then $f : S_Y \rightarrow S$ can be decomposed as

$$f : S_Y \xrightarrow{\pi} S^\nu \xrightarrow{\nu} S.$$

We define $B_{S_Y} = B_Y|_{S_Y}$. Then we obtain

$$(K_Y + S_Y + B_Y)|_{S_Y} = K_{S_Y} + B_{S_Y}$$

by adjunction. We put $B_{S^\nu} = \pi_* B_{S_Y}$. Then we obtain that

$$K_{S^\nu} + B_{S^\nu} = \nu^*(K_X + S + B).$$

The $\mathbb{R}$ -divisor B_{S^ν} on S^ν is called the *different* of $(X, S + B)$ on S^ν . We can easily check that B_{S^ν} is independent of the resolution $f : Y \rightarrow X$. So, B_{S^ν} is a well-defined $\mathbb{R}$ -divisor on S^ν . We can check the following properties.

- (i) $K_{S^\nu} + B_{S^\nu}$ is $\mathbb{R}$ -Cartier and $K_{S^\nu} + B_{S^\nu} = \nu^*(K_X + S + B)$.
- (ii) If B is a $\mathbb{Q}$ -divisor, then so is B_{S^ν} .
- (iii) B_{S^ν} is effective if B is effective in a neighborhood of S .
- (iv) (S^ν, B_{S^ν}) is log canonical if $(X, S + B)$ is log canonical in a neighborhood of S .
- (v) Let D be an $\mathbb{R}$ -Cartier divisor on X such that S and D have no common irreducible components. Then we have

$$(B + D)_{S^\nu} = B_{S^\nu} + \nu^*D.$$

We sometimes write $D|_{S^\nu} = \nu^*D$ for simplicity.

The properties except (iii) follow directly from the definition. We give a proof of (iii) for the reader's convenience.

Proof of (iii). By shrinking X , we can assume that X is quasi-projective and B is effective. By taking hyperplane cuts, we can also assume that X is a surface. Run the log minimal model program over X with respect to $K_Y + S_Y$. Let C be a curve on Y such that $(K_Y + S_Y) \cdot C < 0$ and $f(C)$ is a point. Then $K_Y \cdot C < 0$ because S_Y is the strict transform of S . Therefore, each step of the log minimal model program over X with respect to $K_Y + S_Y$ is the contraction of a (-1) -curve E with $(K_Y + S_Y) \cdot E < 0$. So, by replacing (Y, S_Y) with the output of the above log minimal model program, we can assume that Y is smooth, (Y, S_Y) is plt, and $K_Y + S_Y$ is f -nef. We note that S_Y is a smooth curve since (Y, S_Y) is plt. By the negativity lemma and the assumption that B is effective, B_Y is effective. We note the following equality

$$-B_Y = K_Y + S_Y - f^*(K_X + S + B).$$

By adjunction, we obtain

$$(K_Y + S_Y + B_Y)|_{S_Y} = K_{S_Y} + B_Y|_{S_Y}.$$

It is obvious that $B_Y|_{S_Y}$ is effective. This implies that $B_{S^\nu} = B_Y|_{S_Y}$ is effective. $\square$

When X is singular, B_{S^ν} is not necessarily zero even if $B = 0$.

11.2 (Inversion of adjunction). Let us recall Kawakita's inversion of adjunction on log canonicity (see [Kk]).

Theorem 11.3. *Let X be a normal variety, S a reduced divisor on X , and B an effective $\mathbb{R}$ -divisor on X such that $K_X + S + B$ is $\mathbb{R}$ -Cartier. Assume that S has no common irreducible component with the support of B . Let $\nu : S^\nu \rightarrow S$ be the normalization and B_{S^ν} the different on S^ν such that $K_{S^\nu} + B_{S^\nu} = \nu^*(K_X + S + B)$. Then $(X, S + B)$ is log canonical in a neighborhood of S if and only if (S^ν, B_{S^ν}) is log canonical.*

By adjunction, it is obvious that (S^ν, B_{S^ν}) is log canonical if $(X, S + B)$ is log canonical in a neighborhood of S . It is the property (iv) above. So, the above theorem is usually called the *inversion of adjunction on log canonicity*. We used

Theorem 11.3 in the proof of the restriction theorem for $\mathcal{J}_{NLC}$: Theorem 12.1 (see [F2]).

12. RESTRICTION THEOREMS

In this section, we consider the restriction theorem for $\mathcal{J}'$. First, let us recall the restriction theorem for $\mathcal{J}_{NLC}$. It is the main theorem of [F2].

Theorem 12.1 (Restriction theorem). *Let X be a normal variety and $S + B$ an effective $\mathbb{R}$ -divisor on X such that S is reduced and normal and that S and B have no common irreducible components. Assume that $K_X + S + B$ is $\mathbb{R}$ -Cartier. Let B_S be the different on S such that $K_S + B_S = (K_X + S + B)|_S$. Then we obtain*

$$\mathcal{J}_{NLC}(S, B_S) = \mathcal{J}_{NLC}(X, S + B)|_S.$$

In particular, (S, B_S) is log canonical if and only if $(X, S + B)$ is log canonical in a neighborhood of S .

There is a natural question on $\mathcal{J}'$.

Question 12.2. Let X be a normal variety and $S + B$ an effective $\mathbb{R}$ -divisor on X such that S is reduced and normal and that S and B have no common irreducible components. Assume that $K_X + S + B$ is $\mathbb{R}$ -Cartier. Let B_S be the different on S such that $K_S + B_S = (K_X + S + B)|_S$. Is the following equality

$$\mathcal{J}'(S, B_S) = \mathcal{J}'(X, S + B)|_S.$$

true?

In this section, we will give partial answers to Question 12.2.

12.3. We prove the restriction theorem for $\mathcal{J}'$ under the assumption that X is smooth and $B = 0$. The following theorem is contained in the main theorem from the next section, but the proof in this special case is sufficiently simple that we reproduce it here.

Theorem 12.4. *Let X be a smooth variety and S a reduced normal divisor on X . Then we have*

$$0 \rightarrow \mathcal{O}_X(-S) \rightarrow \mathcal{J}'(X, S) \rightarrow \mathcal{J}'(S, 0) \rightarrow 0.$$

In particular, we obtain

$$\mathcal{J}'(X, S)|_S = \mathcal{J}'(S, 0)$$

We note that $K_S = (K_X + S)|_S$ by adjunction.

Proof. Let $f : Y \rightarrow X$ be a resolution of S such that f is an isomorphism outside the singular locus of S . We put $E = \text{Exc}(f)$. We can assume that f is a composition of blow-ups. Each step can be assumed to be the blow-up described in Proposition 12.5 below. We note that the codimension of the singular locus of S in X is ≥ 3 since S is normal. Therefore, we have $R^1 f_* \mathcal{O}_Y(K_Y + E) = 0$ by Proposition 12.5 and the Leray spectral sequence. We consider the following short exact sequence

$$0 \rightarrow \mathcal{O}_Y(K_Y + E) \rightarrow \mathcal{O}_Y(K_Y + S_Y + E) \rightarrow \mathcal{O}_{S_Y}(K_{S_Y} + E|_{S_Y}) \rightarrow 0,$$

where $S_Y = f_*^{-1}S$. By taking $\otimes \mathcal{O}_Y(-f^*(K_X + S))$ and applying f_* , we obtain

$$0 \rightarrow \mathcal{O}_X(-S) \rightarrow \mathcal{J}'(X, S) \rightarrow \mathcal{J}'(S, 0) \rightarrow 0.$$

In particular, we have

$$\mathcal{J}'(X, S)|_S = \mathcal{J}'(S, 0).$$

We finish the proof. $\square$

Proposition 12.5 (Vanishing lemma). *Let V be a smooth variety and D a simple normal crossing divisor on V . Let $f : W \rightarrow V$ be the blow-up along C such that C is smooth, irreducible, and has simple normal crossings with D . We put $F = f_*^{-1}D + E$, where E is the exceptional divisor of f . We further assume that, if $C \not\subset D$, then the codimension of C is ≥ 3 . We then obtain*

$$f_*\mathcal{O}_W(K_W + F) \simeq \mathcal{O}_V(K_V + D)$$

and

$$R^1f_*\mathcal{O}_W(K_W + F) = 0.$$

Note that W is smooth and F is a simple normal crossing divisor on W .

We provide two proofs, the first relies on more standard methods while the second relies on the theory of Du Bois singularities.

Proof #1. First, we can easily check that

$$K_W + F = f^*(K_V + D) + G,$$

where G is an effective f -exceptional Cartier divisor. Therefore, we obtain

$$f_*\mathcal{O}_W(K_W + F) \simeq \mathcal{O}_V(K_V + D).$$

Next, we consider the following short exact sequence

$$0 \rightarrow \mathcal{O}_W(K_W + f_*^{-1}D) \rightarrow \mathcal{O}_W(K_W + F) \rightarrow \mathcal{O}_E(K_E + f_*^{-1}D|_E) \rightarrow 0.$$

It is sufficient to prove

$$R^1f_*\mathcal{O}_W(K_W + f_*^{-1}D) = R^1f_*\mathcal{O}_E(K_E + f_*^{-1}D|_E) = 0.$$

Assume that $C \not\subset D$. Then $f_*^{-1}D = f^*D$. In this case, $R^1f_*\mathcal{O}_W(K_W + f^*D) = 0$ by the Grauert–Riemenschneider vanishing theorem and $R^1f_*\mathcal{O}_E(K_E + f^*D|_E) = 0$ since $f : E \rightarrow C$ is a $\mathbb{P}^m$ -bundle with $m \geq 2$. Here, we used the assumption that the codimension of C in V is ≥ 3 . So, we can assume that $C \subset D$. In this case, $R^1f_*\mathcal{O}_W(K_W + f_*^{-1}D) = 0$ by the vanishing theorem of Reid–Fukuda type (cf. [Fk, Lemma]). On the other hand, $R^1f_*\mathcal{O}_E(K_E + f_*^{-1}D|_E) = 0$ by the relative Kodaira vanishing theorem. We note that $\pi = f|_E : E \rightarrow C$ is a $\mathbb{P}^m$ -bundle for some $m \geq 1$ and $\pi(f_*^{-1}D) = C$. Thus, $f_*^{-1}D|_E$ is π -ample.

We have now proved that $R^1f_*\mathcal{O}_W(K_W + F) = 0$. $\square$

Proof #2. Note that the map f is a log resolution of the scheme $D \cup C$. The scheme $D \cup C$ is in simple normal crossings so it has Du Bois singularities. It follows that $Rf_*\mathcal{O}_F \simeq_{\text{qis}} \mathcal{O}_{D \cup C}$ by [Sc1]. Thus Grothendieck duality implies that

$$(1) \quad Rf_*\omega_F^\bullet \simeq_{\text{qis}} \omega_{D \cup C}^\bullet.$$

We can map the isomorphism from Equation (1) into the isomorphism $Rf_*\omega_W^\bullet \simeq_{\text{qis}} \omega_V^\bullet$, and then, from the resulting exact triangle, obtain $Rf_*\mathcal{O}_W(K_W + F)[\dim W] \simeq_{\text{qis}} Rf_*\omega_W^\bullet(F) \simeq_{\text{qis}} R\mathcal{H}om_W^\bullet(\mathcal{I}_{D \cup C}, \omega_V^\bullet)$.

We assume that $C \not\subseteq D$ (as the other case is even easier). The dualizing complex of $D \cup C$ has zero cohomology for i between the degrees $-\dim D$ and $-\dim C$. To see this, simply take cohomology and form the long exact sequence from the triangle

$$\omega_{D \cap C}^\bullet \longrightarrow \omega_D^\bullet \oplus \omega_C^\bullet \longrightarrow \omega_{D \cup C}^\bullet \xrightarrow{+1}$$

noting that $\mathcal{H}^i(\omega_Z^\bullet) = 0$ for all $i < -\dim Z$. Thus

$$0 \cong R^{-\dim D+1} f_* \omega_F^\bullet \cong R^{-\dim D} f_* \mathcal{O}_W(K_W + F)[\dim W] = R^1 f_* \mathcal{O}_W(K_W + F)$$

and we have proven the vanishing. For the isomorphism, simply notice that

$$\begin{aligned} f_* \mathcal{O}_W(K_W + F) &\cong \mathcal{H}^{-\dim V} (R \mathcal{H} \mathrm{om}_V^\bullet(\mathcal{I}_{D \cup C}, \omega_V^\bullet)) \\ &\cong \mathcal{H} \mathrm{om}_W(\mathcal{I}_{D \cup C}, \omega_V) \\ &\cong \mathcal{O}_V(K_V + D) \end{aligned}$$

since the last two sheaves are reflexive and agree outside of C . $\square$

12.6. We prove the restriction theorem for $\mathcal{J}'$ on the assumption that X has only mild singularities. Theorem 12.7 is an easy corollary of Theorem 12.1 and is not covered by the main result of the next section.

Theorem 12.7. *Let X be a normal variety and $S + B$ an effective $\mathbb{R}$ -divisor on X such that S is reduced and normal and that S and B have no common irreducible components. Assume that $B = B_1 + B_2$ such that both B_1 and B_2 are effective $\mathbb{R}$ -divisors around S , $(X, S + B_1)$ is log canonical in a neighborhood of S . Then we obtain*

$$\mathcal{J}'(X, S + B)|_S = \mathcal{J}'(S, B_S),$$

where B_S is the different on S such that $(K_X + S + B)|_S = K_S + B_S$.

Lemma 12.8. *With the same notation and assumptions as in Theorem 12.7, we have*

$$\mathcal{J}'(X, S + B) = \mathcal{J}_{NLC}(X, S + B_1 + (1 - \varepsilon)B_2)$$

in a neighborhood of S for $0 < \varepsilon \ll 1$, and

$$\mathcal{J}'(S, B_S) = \mathcal{J}'(S, B_{1S} + B_{2|S}) = \mathcal{J}_{NLC}(S, B_{1S} + (1 - \varepsilon)B_{2|S})$$

for $0 < \varepsilon \ll 1$.

Proof. By shrinking X around S , we can assume that $(X, S + B_1)$ is log canonical and B_2 is effective. By the definitions of $\mathcal{J}'$ and $\mathcal{J}_{NLC}$, it is almost obvious that

$$\mathcal{J}'(X, S + B) = \mathcal{J}_{NLC}(X, S + B_1 + (1 - \varepsilon)B_2)$$

for $0 < \varepsilon \ll 1$. By the assumption, (S, B_{1S}) is log canonical, where B_{1S} is the different such that $(K_X + S + B_1)|_S = K_S + B_{1S}$. Thus, $\mathcal{J}'(S, B_S) = \mathcal{J}_{NLC}(S, B_{1S} + (1 - \varepsilon)B_{2|S})$ holds for $0 < \varepsilon \ll 1$. $\square$

Proof of Theorem 12.7. We have the following equalities.

$$\begin{aligned}\mathcal{J}'(X, S + B)|_S &= \mathcal{J}_{NLC}(X, S + B_1 + (1 - \varepsilon)B_2)|_S \\ &= \mathcal{J}_{NLC}(S, B_{1S} + (1 - \varepsilon)B_2|_S) \\ &= \mathcal{J}'(S, B_{1S} + B_2|_S) \\ &= \mathcal{J}'(S, B_S)\end{aligned}$$

by Lemma 12.8 and Theorem 12.1. $\square$

We close this section with the following nontrivial example.

Example 12.9. Let $X = \mathbb{C}^2 = \text{Spec } \mathbb{C}[x, y]$, $S = (x = 0)$, and $C = (y^2 = x^3)$. We put $K_S + C_S = (K_X + S + C)|_S$. We use the same notation as in Example 8.6. Then we have

$$\begin{aligned}\mathcal{J}_{NLC}(X, S + C) &= \pi_* \mathcal{O}_{X_3}(-2E_1 - 2E_2 - 4E_3) \\ &= f_{1*} \mathcal{O}_{X_1}(-2E_1) \\ &= \mathfrak{n}^2, \\ \mathcal{J}'(X, S + C) &= \pi_* \mathcal{O}_{X_3}(-E_1 - E_2 - 3E_3) \\ &= \pi_* \mathcal{O}_{X_3}(-E_1 - 2E_2 - 3E_3) \\ &= f_{1*}(f_{2*} \mathcal{O}_{X_2}(-E_2) \otimes \mathcal{O}_{X_1}(-E_1)) \\ &= (x^2, y),\end{aligned}$$

where $\mathfrak{n}$ is the maximal ideal corresponding to $(0, 0) \in X$. On the other hand, by easy calculations, we obtain

$$\mathcal{J}_{NLC}(S, C_S) = \mathfrak{m}^2, \quad \mathcal{J}'(S, C_S) = \mathfrak{m},$$

where $\mathfrak{m}$ is the maximal ideal corresponding to $0 \in S$. Hence we can check the both restriction theorems (cf. Theorem 12.7)

$$\begin{aligned}\mathcal{J}_{NLC}(S, C_S) &= \mathcal{J}_{NLC}(X, S + C)|_S, \\ \mathcal{J}'(S, C_S) &= \mathcal{J}'(X, S + C)|_S\end{aligned}$$

in this case.

13. THE RESTRICTION THEOREM FOR COMPLETE INTERSECTIONS

In this section we prove a restriction theorem, Theorem 13.13, for maximal non-lc ideals $\mathcal{J}'(X, D)$ in a complete intersection. It is important to note that we do not use Kawakita's proof of inversion of adjunction for log canonicity [Kk]. We also only use fairly mundane vanishing theorems – Kawamata–Viehweg vanishing in the form of local vanishing for multiplier ideals, see [Km], [V], and [L].

Our method is related to techniques used to study Du Bois singularities, and so some of the auxiliary notation we use draws from this perspective, for an introduction to Du Bois singularities, see [KS]. We briefly recall why one might expect to use techniques from Du Bois singularities to study non-lc ideal sheaves.

Suppose that X is a reduced scheme of finite type over a field k of characteristic zero. One can then associate an object $\underline{\Omega}_X^0$ in the bounded derived category with

coherent cohomology (this object has its origin in Deligne's mixed Hodge theory for singular varieties). This object $\underline{\Omega}_X^0$ is used to determine whether or not X has Du Bois singularities (recalling that Du Bois singularities are very closely related to log canonical singularities, see [KK]). Furthermore, in the case that X is normal and K_X is Cartier, it follows from [KSS] that the most interesting cohomology ($-\dim X$) of the Grothendieck dual of $\underline{\Omega}_X^0$ is equal to $\mathcal{J}'(X, 0)$. This suggests that two things:

- $\mathcal{J}'(X, 0)$ is natural object from the point of view of Du Bois singularities (or more generally, from the point of view of the Hodge theory of singular varieties), and
- some of the ideas from Du Bois singularities might be useful in proving restriction theorems for $\mathcal{J}'(X, \Delta)$.

We will take advantage of the second idea in this section.

Now we begin our main definitions.

Suppose that Y is a smooth affine variety and $X \subset Y$ is a reduced closed subscheme with ideal sheaf $\mathcal{I}_X$. Let $\mathfrak{a}$ be an ideal on Y and $t > 0$ a real number. Let $\pi : \tilde{Y} \rightarrow Y$ be a log resolution of $(Y, X, \mathfrak{a}^t)$ and set $\mathfrak{a}\mathcal{O}_{\tilde{Y}} = \mathcal{O}_{\tilde{Y}}(-G)$ and $\mathcal{I}_X\mathcal{O}_{\tilde{Y}} = \mathcal{O}_{\tilde{Y}}(-\bar{X})$.

Consider the following short exact sequence:

$$0 \rightarrow \mathcal{O}_{\tilde{Y}}(\lfloor tG - \epsilon\bar{X} \rfloor) \rightarrow \mathcal{O}_{\tilde{Y}}(\lfloor tG \rfloor) \rightarrow M_{X, \mathfrak{a}^t} \rightarrow 0,$$

where $\epsilon > 0$ is arbitrarily small. Furthermore, one can replace $\bar{X}$ with the reduced pre-image of X , and assuming ϵ was chosen to be sufficiently small, the sequence does not change.

Definition 13.1. We define $\underline{\Omega}_{X, \mathfrak{a}^t}^0$ to be $R\pi_* M_{X, \mathfrak{a}^t}$.

Remark 13.2. The object $\underline{\Omega}_{X, \mathfrak{a}^t}^0$ is purely an auxiliary object from the point of view of this paper. However, in the case that $\mathfrak{a} = \mathcal{O}_Y$, it agrees with $\underline{\Omega}_X^0$, the zeroth graded piece of the Deligne–Du Bois complex, see [D], [Es] and [Sc1].

Lemma 13.3. *The object $\underline{\Omega}_{X, \mathfrak{a}^t}^0$ is independent of the choice of π (assuming ϵ is chosen sufficiently small).*

Proof. Suppose that $\rho : Y' \rightarrow \tilde{Y}$ is a further log resolution. Set $\mathfrak{a}\mathcal{O}_{Y'} = \mathcal{O}_{Y'}(-G')$ and $\mathcal{I}_X\mathcal{O}_{Y'} = \mathcal{O}_{Y'}(-X')$. It is sufficient to show that $R\rho_*\mathcal{O}_{Y'}(\lfloor tG' - \epsilon X' \rfloor) \simeq_{\text{qis}} \mathcal{O}_{\tilde{Y}}(\lfloor tG - \epsilon\bar{X} \rfloor)$. Therefore, by Grothendieck duality, it is sufficient to show that

$$R\rho_*\mathcal{O}_{Y'}(\lceil K_{Y'} - tG' + \epsilon X' \rceil) \rightarrow \mathcal{O}_{\tilde{Y}}(\lceil K_{\tilde{Y}} - tG + \epsilon\bar{X} \rceil)$$

is an isomorphism. Twisting by $\mathcal{O}_{\tilde{Y}}(-K_{\tilde{Y}} - \bar{X})$, it is sufficient to show that

$$R\rho_*\mathcal{O}_{Y'}(\lceil K_{Y'/\tilde{Y}} - tG' - (1 - \epsilon)X' \rceil) \rightarrow \mathcal{O}_{\tilde{Y}}(\lceil -tG - (1 - \epsilon)\bar{X} \rceil)$$

is an isomorphism. But this is just the independence of the choice of resolution for multiplier ideals. $\square$

Instead of computing a full log resolution, it will be convenient only to compute a log resolution of $(Y, \mathfrak{a}^t)$ which is an embedded resolution of X . In our next lemma, we show that our auxiliary object $\underline{\Omega}_{X, \mathfrak{a}^t}^0$ can be computed via such a resolution. But first a definition:

Definition 13.4. Given a closed subvariety X in a smooth variety Y , a *factorizing embedded resolution* of X in Y is a proper birational morphism $\pi : \tilde{Y} \rightarrow Y$ such that

- (1) $\tilde{Y}$ is smooth and π is an isomorphism at every generic point of $X \subseteq Y$,
- (2) the exceptional locus $\text{exc}(\pi)$ is a simple normal crossings divisor,
- (3) the strict transform $\tilde{X}$ of X in $\tilde{Y}$ is smooth and has simple normal crossings with $\text{exc}(\pi)$,
- (4) $\mathcal{I}_X \mathcal{O}_{\tilde{Y}} = \mathcal{I}_{\tilde{X}} \mathcal{O}_{\tilde{Y}}(-E)$ where $\mathcal{I}_X$ (resp. $\mathcal{I}_{\tilde{X}}$) is the ideal sheaf of X (resp. $\tilde{X}$) and E is a π -exceptional divisor on $\tilde{Y}$.

Such resolutions always exist, see [BEV] or [W].

Lemma 13.5. *Suppose that X, Y and $\mathfrak{a}$ are as above and suppose that no component of X is contained in $V(\mathfrak{a})$. Further suppose that $\pi : \tilde{Y} \rightarrow Y$ is a log resolution of $(Y, \mathfrak{a}^t)$ that is simultaneously a factorizing embedded resolution of X in Y (and so that the pullbacks of the various objects we were working with are in simple normal crossings, see [BEV] or [W]). Set $\tilde{X}$ to be the strict transform of X in $\tilde{Y}$ so that we can write $\mathcal{I}_X \mathcal{O}_{\tilde{Y}} = \mathcal{I}_{\tilde{X}} \mathcal{O}_{\tilde{Y}}(-E)$. Then if we let M be the cokernel of*

$$\mathcal{O}_{\tilde{Y}}(\lfloor tG - \epsilon E \rfloor) \otimes \mathcal{I}_{\tilde{X}} \rightarrow \mathcal{O}_{\tilde{Y}}(\lfloor tG \rfloor)$$

we have that $R\pi_ M \simeq_{\text{qis}} \underline{\Omega}_{X, \mathfrak{a}^t}^0$. (Note the statement does not change if we replace E by E_{red} .)*

Proof. We note that by blowing up $\tilde{X}$ (which is smooth), we can obtain an actual log resolution $\eta : Y' \rightarrow Y$ of $(Y, X, \mathfrak{a}^t)$ as pictured in the diagram below:

$$\begin{array}{ccc} Y' & \xrightarrow{\rho} & \tilde{Y} \\ & \searrow \eta & \swarrow \pi \\ & Y & \end{array}$$

Set $\mathcal{I}_{\tilde{X}} \mathcal{O}_{Y'} = \mathcal{O}_{Y'}(-X')$, set $\mathcal{O}_{Y'}(-E') = \rho^* \mathcal{O}_{\tilde{Y}}(-E)$ and set $\mathfrak{a} \mathcal{O}_{Y'} = \mathcal{O}_{Y'}(-G') = \rho^* \mathcal{O}_{\tilde{Y}}(-G)$. Note that ρ induces a bijection between the components (and coefficients) of E with those of E' (and also of G with those of G' since no component of X is contained in $V(\mathfrak{a})$). It is sufficient to show that

$$R\rho_* \mathcal{O}_{Y'}(\lfloor tG' - \epsilon(E' + X') \rfloor) \simeq_{\text{qis}} \mathcal{O}_{\tilde{Y}}(\lfloor tG - \epsilon E \rfloor) \otimes \mathcal{I}_{\tilde{X}}.$$

Now twist by $\mathcal{O}_{\tilde{Y}}(-\lfloor tG - \epsilon E \rfloor)$, it is thus sufficient to show that

$$R\rho_* \mathcal{O}_{Y'}(\lfloor tG' - \epsilon(E' + X') \rfloor - \rho^* \lfloor tG - \epsilon E \rfloor) \simeq_{\text{qis}} \mathcal{I}_{\tilde{X}}.$$

Note that, over each component of $\tilde{X}$, there is exactly one new divisor created by ρ , they are all disjoint, and $\lfloor tG - \epsilon E \rfloor$ does not contain $\mathcal{I}_{\tilde{X}}$ in its support, thus the left side is just $R\rho_* \mathcal{O}_{Y'}(\lfloor -\epsilon X' \rfloor)$ which is isomorphic to $\mathcal{I}_{\tilde{X}}$ since $\tilde{X}$ is smooth. $\square$

In what follows, we use the symbol $\mathbf{D}$ to denote the Grothendieck dual of a complex in D_{coh}^b , for example $\mathbf{D}(\underline{\Omega}_X^0) \cong R\mathcal{H}\text{om}_Y^{\bullet}(\underline{\Omega}_X^0, \omega_Y^{\bullet})$. See [Ha].

We now make a transition in concept. Instead of simply looking at $\underline{\Omega}_{X, \mathfrak{a}^t}^0$, we consider $\underline{\Omega}_{X, \mathfrak{a}^{t-\epsilon'}}^0$. In particular, viewing ϵ' as a very small positive number. It

is straightforward to verify that $\mathcal{O}_{\tilde{Y}}(\lfloor (t - \epsilon')G - \epsilon\overline{X} \rfloor)$ is constant for sufficiently small positive ϵ' and ϵ , and therefore so is $\underline{\Omega}_{X, \mathfrak{a}^{t-\epsilon'}}^0$. Furthermore, because we are subtracting (and not adding) ϵ' from t , we may in fact choose $\epsilon' = \epsilon$ (as long as they are both sufficiently small). Therefore, we conflate the two ϵ 's and write $\underline{\Omega}_{X, \mathfrak{a}^{t-\epsilon}}^0$ to denote this object for $\epsilon = \epsilon'$ sufficiently small.

Proposition 13.6. *With the notation above, $\mathcal{H}^i \mathbf{D} \left(\underline{\Omega}_{X, \mathfrak{a}^{t-\epsilon}}^0 \right) = 0$ for $i < -\dim X$.*

Proof. We first note that we may assume that no irreducible component of X is contained in $V(\mathfrak{a})$. To see this, suppose $X = X_1 \cup X_2$ where X_1 is the union of those irreducible components of X that are not contained in $V(\mathfrak{a})$ and X_2 is the union of the remaining components. Then notice that

$$\lfloor (t - \epsilon)G - \epsilon\overline{X} \rfloor = \lfloor (t - \epsilon)G - \epsilon(\overline{X_1 \cup X_2}) \rfloor = \lfloor (t - \epsilon)G - \epsilon\overline{X_1} \rfloor$$

since we choose ϵ arbitrarily small. It follows that $\underline{\Omega}_{X, \mathfrak{a}^{t-\epsilon}}^0 = \underline{\Omega}_{X_1, \mathfrak{a}^{t-\epsilon}}^0$. In particular, note that if $X \subset V(\mathfrak{a})$, then $\underline{\Omega}_{X, \mathfrak{a}^{t-\epsilon}}^0 \simeq_{\text{qis}} 0$. Therefore if $\mathcal{H}^i \mathbf{D} \left(\underline{\Omega}_{X_1, \mathfrak{a}^{t-\epsilon}}^0 \right) = 0$ for $i < -\dim X_1$, then for all $i < -\dim X \leq -\dim X_1$, we have $\mathcal{H}^i \mathbf{D} \left(\underline{\Omega}_{X, \mathfrak{a}^{t-\epsilon}}^0 \right) = 0$.

We proceed by induction on the dimension of X . If $\dim X = 0$, then X is disjoint from the support $\mathfrak{a}$ and the result follows from the theory of Du Bois singularities since $\mathcal{H}^i \mathbf{D}(\underline{\Omega}_X^0) = 0$ for $i < -\dim X$, see [KSS, Lemma 3.6].

For the induction step, define $\Gamma \subset Y$ to be the reduced scheme

$$\Gamma := (\text{Sing } X) \cup V(\mathfrak{a}),$$

where $\text{Sing } X$ is the singular locus of X . In particular Γ contains $V(\mathfrak{a})$. Decompose $\Gamma = \Gamma_{\mathfrak{a}} \cup \Sigma_{\times}$ where $\Gamma_{\mathfrak{a}}$ is the union of components of Γ that are contained in $V(\mathfrak{a})$ and $\Sigma_{\times}$ is the union of the components of Γ that are not contained in $V(\mathfrak{a})$.

Let $\pi : \tilde{Y} \rightarrow Y$ be an embedded log resolution of X and log resolution of $(Y, \mathfrak{a}^{t-\epsilon})$ as in Lemma 13.5. Set $E_{\mathfrak{a}}$ to be the reduced pre-image of $\Gamma_{\mathfrak{a}}$ in $\tilde{Y}$, $E_{\times}$ to be the reduced pre-image of $\Sigma_{\times}$ in $\tilde{Y}$, $\mathfrak{a}\mathcal{O}_{\tilde{Y}} = \mathcal{O}_{\tilde{Y}}(-G)$ and set $\tilde{X}$ to be the strict transform of X . We may assume that π is an isomorphism outside of $V(\mathfrak{a})$ and $\Sigma_{\times}$. Now write $\mathcal{I}_X \mathcal{O}_{\tilde{Y}} = \mathcal{I}_{\tilde{X}} \mathcal{O}_{\tilde{Y}}(-E)$ and note that

$$(2) \quad \lfloor (t - \epsilon)G - \epsilon(E_{\times} + E_{\mathfrak{a}}) \rfloor = \lfloor (t - \epsilon)G - \epsilon E \rfloor = \lfloor (t - \epsilon)G - \epsilon E_{\times} \rfloor$$

since we pick $\epsilon > 0$ to be arbitrarily small.

Notice we have the following short exact sequence.

$$0 \rightarrow \mathcal{O}_{\tilde{Y}}(\lfloor (t - \epsilon)G - \epsilon E \rfloor) \otimes \mathcal{I}_{\tilde{X}} \rightarrow \mathcal{O}_{\tilde{Y}}(\lfloor (t - \epsilon)G - \epsilon E_{\times} \rfloor) \rightarrow \mathcal{O}_{\tilde{X}}(\lfloor (t - \epsilon)G - \epsilon E_{\times} \rfloor|_{\tilde{X}}) \rightarrow 0.$$

By pushing forward, it follows that there exists the following commutative diagram of exact triangles:

$$\begin{array}{ccccccc}
R\pi_*\mathcal{O}_{\tilde{X}}(\lfloor(t-\epsilon)G - \epsilon E_{\times}\rfloor|_{\tilde{X}})[-1] & \longrightarrow & 0 & \longrightarrow & R\pi_*\mathcal{O}_{\tilde{X}}(\lfloor(t-\epsilon)G - \epsilon E_{\times}\rfloor|_{\tilde{X}}) \xrightarrow{+1} \\
\downarrow & & \downarrow & & \downarrow \\
R\pi_*\mathcal{O}_{\tilde{Y}}(\lfloor(t-\epsilon)G - \epsilon E\rfloor) \otimes \mathcal{I}_{\tilde{X}} & \longrightarrow & R\pi_*\mathcal{O}_{\tilde{Y}}(\lfloor(t-\epsilon)G\rfloor) & \longrightarrow & \underline{\Omega}_{X,\mathfrak{a}^{t-\epsilon}}^0 \xrightarrow{+1} \\
\downarrow & & \downarrow \sim & & \downarrow \\
R\pi_*\mathcal{O}_{\tilde{Y}}(\lfloor(t-\epsilon)G - \epsilon E_{\times}\rfloor) & \longrightarrow & R\pi_*\mathcal{O}_{\tilde{Y}}(\lfloor(t-\epsilon)G\rfloor) & \longrightarrow & \underline{\Omega}_{\Sigma_{\times},\mathfrak{a}^{t-\epsilon}}^0 \xrightarrow{+1} \\
\downarrow +1 & & \downarrow +1 & & \downarrow +1
\end{array}$$

While in general, restricting divisors does not commute with round-downs, in our case we do have $\lfloor(t-\epsilon)G - \epsilon E_{\times}\rfloor|_{\tilde{X}} = \lfloor(t-\epsilon)G|_{\tilde{X}} - \epsilon E_{\times}|_{\tilde{X}}\rfloor$ because the divisors and the object we are restricting them to are in simple normal crossings. We dualize the right vertical column and obtain

$$\mathbf{D}\left(\underline{\Omega}_{\Sigma_{\times},\mathfrak{a}^{t-\epsilon}}^0\right) \longrightarrow \mathbf{D}\left(\underline{\Omega}_{X,\mathfrak{a}^{t-\epsilon}}^0\right) \longrightarrow R\pi_*\omega_{\tilde{X}}^{\bullet}(\lceil -(t-\epsilon)G|_{\tilde{X}} + \epsilon E_{\times}|_{\tilde{X}} \rceil) \xrightarrow{+1}.$$

By taking cohomology, and using the inductive hypothesis on $\mathbf{D}\left(\underline{\Omega}_{\Sigma_{\times},\mathfrak{a}^{t-\epsilon}}^0\right)$, we obtain our desired result. $\square$

The following corollary is a key application of what we have proven so far. It allows us to relate the auxiliary objects $\underline{\Omega}_{X,\mathfrak{a}^{t-\epsilon}}^0$ with the maximal non-lc ideals $\mathcal{J}'(X; \mathfrak{a}^t)$.

Corollary 13.7. *Assume that X is normal and equidimensional, $\mathfrak{a}$ is an ideal sheaf on X and that no component of X is contained inside $V(\mathfrak{a})$. Let $\pi' : \tilde{X} \rightarrow X$ be a log resolution, let F be the exceptional divisor of π and set $\mathfrak{a}\mathcal{O}_{\tilde{X}} = \mathcal{O}_{\tilde{X}}(-H)$ then*

$$(3) \quad \mathcal{H}^{-\dim X} \mathbf{D}\left(\underline{\Omega}_{X,\mathfrak{a}^{t-\epsilon}}^0\right) = \pi'_*\mathcal{O}_{\tilde{X}}(\lceil K_{\tilde{X}} - (t-\epsilon)H + \epsilon F \rceil)$$

for all sufficiently small ϵ and any embedding of $X \subseteq Y$ into a smooth variety. In particular, $\mathcal{H}^{-\dim X} \mathbf{D}\left(\underline{\Omega}_{X,\mathfrak{a}^{t-\epsilon}}^0\right)$ is independent of the choice of embedding of X into Y .

Proof. Since the right side is independent of the choice of resolution by Lemma 7.3, we assume that $\pi' := \pi|_{\tilde{X}}$ is induced as in Lemma 13.5 and furthermore that π is an isomorphism outside of $V(\mathfrak{a})$ and $\text{Sing } X$. The result follows immediately from the final exact triangle used in the proof of Proposition 13.6 when one notes that $\dim \Sigma_{\times} \leq \dim X - 2$ (since X is normal). Note that while there may be components of F and H which coincide, choosing small enough epsilon allows us to ignore such complication as in Equation (2). $\square$

In the case that X is a complete intersection, we will show that $\mathcal{H}^i \mathbf{D}\left(\underline{\Omega}_{X,\mathfrak{a}^{t-\epsilon}}^0\right) = 0$ for $i > -\dim X$. First however, we need the following lemma which will be a key point in an inductive argument, the proof is similar to that of Proposition 13.6.

Lemma 13.8. *Suppose that Y , $\mathfrak{a}$ and X are as above and suppose that no component of X is contained inside $V(\mathfrak{a})$. Suppose further that X is a codimension 1 subset of a reduced equidimensional scheme $Z \subseteq Y$ also such that no component of Z is*

contained inside $V(\mathfrak{a})$. Let $\pi : \tilde{Y} \rightarrow Y$ be a log resolution of $(Y, X \cup (\text{Sing } Z), \mathfrak{a}^t)$ that is simultaneously an embedded resolution of Z as in Lemma 13.5. Furthermore, we assume that π is an isomorphism outside of $(\text{Sing } Z) \cup X \cup V(\mathfrak{a})$.

Set $\tilde{Z}$ to be the strict transform of Z , set $\mathfrak{a}\mathcal{O}_{\tilde{Y}} = \mathcal{O}_{\tilde{Y}}(-G)$, set $\mathcal{I}_X\mathcal{O}_{\tilde{Y}} = \mathcal{O}_{\tilde{Y}}(-\bar{X})$ and set E_Z to be the divisor on $\tilde{Y}$ such that $\mathcal{I}_Z\mathcal{O}_{\tilde{Y}} = \mathcal{I}_{\tilde{Z}}\mathcal{O}_{\tilde{Y}}(-E_Z)$. Finally, let Σ denote the union of components of $\text{Sing } Z$ which are not contained in $V(\mathfrak{a})$. Then there is an exact triangle

$$R\pi_*\mathcal{O}_{\tilde{Z}}(\lfloor(t-\epsilon)G\rfloor_{\tilde{Z}} - \epsilon(E_Z + \bar{X})|_{\tilde{Z}}) \longrightarrow \underline{\Omega}_{Z, \mathfrak{a}^{t-\epsilon}}^0 \longrightarrow \underline{\Omega}_{X \cup \Sigma, \mathfrak{a}^{t-\epsilon}}^0 \xrightarrow{+1}.$$

Proof. We begin with the following short exact sequence:

$$\begin{aligned} 0 \rightarrow \mathcal{O}_{\tilde{Y}}(\lfloor(t-\epsilon)G - \epsilon(E_Z + \bar{X})\rfloor) \otimes \mathcal{I}_{\tilde{Z}} &\rightarrow \mathcal{O}_{\tilde{Y}}(\lfloor(t-\epsilon)G - \epsilon(E_Z + \bar{X})\rfloor) \\ &\rightarrow \mathcal{O}_{\tilde{Z}}(\lfloor(t-\epsilon)G - \epsilon(E_Z + \bar{X})\rfloor|_{\tilde{Z}}) \rightarrow 0. \end{aligned}$$

We set $E_\times$ to be the union of the components of $\text{Supp}(E_Z \cup \bar{X})$ whose images in Y are contained in $X \cup \Sigma$ so that $\pi(E_\times) = X \cup \Sigma$. Furthermore, notice that $\text{Supp}(E_Z \cup \bar{X}) \setminus E_\times \subseteq \text{Supp } G$ so that

$$\lfloor(t-\epsilon)G - \epsilon((E_Z)_{\text{red}} \cup \bar{X}_{\text{red}})\rfloor = \lfloor(t-\epsilon)G - \epsilon(E_Z + \bar{X})\rfloor = \lfloor(t-\epsilon)G - \epsilon(E_\times)\rfloor$$

again because ϵ is sufficiently small.

We now form a diagram with exact triangles as columns and rows as in the proof of Proposition 13.6:

$$\begin{array}{ccccc} R\pi_*\mathcal{O}_{\tilde{Z}}(\lfloor(t-\epsilon)G - \epsilon(E_Z + \bar{X})\rfloor|_{\tilde{Z}})[-1] & \longrightarrow & 0 & \longrightarrow & R\pi_*\mathcal{O}_{\tilde{Z}}(\lfloor(t-\epsilon)G - \epsilon(E_Z + \bar{X})\rfloor|_{\tilde{Z}}) \\ \downarrow & & \downarrow & & \downarrow \\ R\pi_*\mathcal{O}_{\tilde{Y}}(\lfloor(t-\epsilon)G - \epsilon(E_Z + \bar{X})\rfloor) \otimes \mathcal{I}_{\tilde{Z}} & \twoheadrightarrow & R\pi_*\mathcal{O}_{\tilde{Y}}(\lfloor(t-\epsilon)G\rfloor) & \longrightarrow & \underline{\Omega}_{Z, \mathfrak{a}^{t-\epsilon}}^0 \\ \downarrow & & \downarrow \sim & & \downarrow \\ R\pi_*\mathcal{O}_{\tilde{Y}}(\lfloor(t-\epsilon)G - \epsilon E_\times\rfloor) & \longrightarrow & R\pi_*\mathcal{O}_{\tilde{Y}}(\lfloor(t-\epsilon)G\rfloor) & \longrightarrow & \underline{\Omega}_{X \cup \Sigma, \mathfrak{a}^{t-\epsilon}}^0 \end{array}$$

Our desired exact triangle is the right vertical column. $\square$

Theorem 13.9. *With the notation as in Proposition 13.6, suppose that X is a complete intersection variety in Y , and no component of X is contained inside $V(\mathfrak{a})$, then $\mathcal{H}^i \mathbf{D}(\underline{\Omega}_{X, \mathfrak{a}^{t-\epsilon}}^0) = 0$ for $i > -\dim X$.*

Proof. We proceed by induction on the codimension of X in Y . We begin with the case where X is a hypersurface. We have the following exact triangle:

$$R\pi_*\mathcal{O}_{\tilde{Y}}(\lfloor(t-\epsilon)G - \epsilon\bar{X}\rfloor) \longrightarrow R\pi_*\mathcal{O}_{\tilde{Y}}(\lfloor(t-\epsilon)G\rfloor) \longrightarrow \underline{\Omega}_{X, \mathfrak{a}^{t-\epsilon}}^0 \xrightarrow{+1},$$

where $\bar{X}$ is the pullback of X (if one takes $\bar{X}$ to be the reduced pre-image of X , you get the same result since ϵ is arbitrarily small). Dualizing gives us:

$$\mathbf{D}(\underline{\Omega}_{X, \mathfrak{a}^{t-\epsilon}}^0) \rightarrow R\pi_*\omega_{\tilde{Y}}^\bullet(\lceil -(t-\epsilon)G \rceil) \rightarrow R\pi_*\omega_{\tilde{Y}}^\bullet(\lceil -(t-\epsilon)G + \epsilon\bar{X} \rceil) \xrightarrow{+1}.$$

Taking cohomology gives us the claimed vanishing since the cohomology of the right-most two terms vanish for $i > -\dim Y = -\dim X - 1$. To see this explicitly, note that the middle term vanishes due to Kawamata–Viehweg vanishing in the form of

local vanishing for multiplier ideals. The right most term also vanishes for the same reason once one notices that

$$R\pi_*\omega_{\tilde{Y}}^\bullet(\lceil -(t-\epsilon)G + \epsilon\overline{X} \rceil) \otimes \mathcal{O}_Y(-X) \simeq_{\text{qis}} R\pi_*\omega_{\tilde{Y}}^\bullet(\lceil -(t-\epsilon)G - (1-\epsilon)\pi^*X \rceil).$$

Now we assume that X is a complete intersection in Y . Choose hypersurfaces $H_1, \dots, H_l$ to be general hypersurfaces containing X such that $X = H_1 \cap \dots \cap H_l$. Let Z be an intersection of the first $l-1 = (\dim Y - \dim X) - 1$ such hypersurfaces. In this way, X is a Cartier divisor in Z and $Z \setminus X$ is smooth. Notice also that Z is certainly S_2 and it is smooth at all points where X is smooth (which includes all the generic points of X). In particular, Z is smooth in codimension 1 and thus it is normal. Let $\pi : \tilde{Y} \rightarrow Y$ be a log resolution of $(Y, X, \mathfrak{a}^t)$ that is simultaneously a factorizing embedded resolution of Z , as in Lemma 13.8. Dualizing the triangle from Lemma 13.8, we obtain a triangle:

$$(4) \quad \mathbf{D} \left(\underline{\Omega}_{X, \mathfrak{a}^{t-\epsilon}}^0 \right) \rightarrow \mathbf{D} \left(\underline{\Omega}_{Z, \mathfrak{a}^{t-\epsilon}}^0 \right) \rightarrow R\pi_*\omega_{\tilde{Z}}^\bullet(\lceil -(t-\epsilon)G|_{\tilde{Z}} + \epsilon(E_Z + \overline{X})|_{\tilde{Z}} \rceil) \xrightarrow{+1}.$$

Since $Z \setminus X$ is smooth, observe that

$$\begin{aligned} & \mathcal{H}^i R\pi_*\omega_{\tilde{Z}}^\bullet(\lceil -(t-\epsilon)G + \epsilon(E_Z + \overline{X})|_{\tilde{Z}} \rceil) \otimes \mathcal{O}_Z(-X) \\ &= \mathcal{H}^i R\pi_*\omega_{\tilde{Z}}^\bullet(\lceil -(t-\epsilon)G|_{\tilde{Z}} + \epsilon(\overline{X})|_{\tilde{Z}} \rceil) \otimes \mathcal{O}_Z(-X) \end{aligned}$$

which vanishes for $i > -\dim Z$ using the projection formula and local vanishing for multiplier ideals. Furthermore, $\mathcal{H}^i \mathbf{D} \left(\underline{\Omega}_{Z, \mathfrak{a}^{t-\epsilon}}^0 \right) = 0$ for $i > -(\dim X + 1) = -\dim Z$ by the induction hypothesis. Thus taking cohomology of Equation (4) for $i > -\dim Z$ gives us the desired result. $\square$

Corollary 13.10. *If Z is a normal complete intersection, X is a Weil divisor in Z and $V(\mathfrak{a})$ doesn't contain any component of X or Z , then there is a short exact sequence*

$$\begin{aligned} 0 & \rightarrow \mathcal{H}^{-\dim Z} \mathbf{D} \left(\underline{\Omega}_{Z, \mathfrak{a}^{t-\epsilon}}^0 \right) \\ & \rightarrow \pi_* \mathcal{O}_{\tilde{Z}}(\lceil K_{\tilde{Z}} - (t-\epsilon)G|_{\tilde{Z}} + \epsilon(E_Z + \overline{X})|_{\tilde{Z}} \rceil) \\ & \rightarrow \mathcal{H}^{-\dim X} \mathbf{D} \left(\underline{\Omega}_{X \cup \Sigma, \mathfrak{a}^{t-\epsilon}}^0 \right) \\ & \rightarrow 0 \end{aligned}$$

where Σ and the remaining notation comes from Lemma 13.8.

Proof. Simply dualize the sequence from Lemma 13.8. Then take cohomology and apply the vanishing results Proposition 13.6 and Theorem 13.9. $\square$

Lemma 13.11. *With the notation from Corollary 13.10, further assume that X is normal and Cartier. Then $\mathcal{H}^{-\dim X} \mathbf{D} \left(\underline{\Omega}_{X \cup \Sigma, \mathfrak{a}^{t-\epsilon}}^0 \right) \cong \mathcal{H}^{-\dim X} \mathbf{D} \left(\underline{\Omega}_{X, \mathfrak{a}^{t-\epsilon}}^0 \right)$. In particular, we have a short exact sequence*

$$\begin{aligned} 0 & \rightarrow \mathcal{H}^{-\dim Z} \mathbf{D} \left(\underline{\Omega}_{Z, \mathfrak{a}^{t-\epsilon}}^0 \right) \\ & \rightarrow \pi_* \mathcal{O}_{\tilde{Z}}(\lceil K_{\tilde{Z}} - (t-\epsilon)G|_{\tilde{Z}} + \epsilon(E_Z + \overline{X})|_{\tilde{Z}} \rceil) \\ & \rightarrow \mathcal{H}^{-\dim X} \mathbf{D} \left(\underline{\Omega}_{X, \mathfrak{a}^{t-\epsilon}}^0 \right) \\ & \rightarrow 0 \end{aligned}$$

Proof. We will need the following:

Claim. *There is a triangle*

$$\underline{\Omega}_{X \cup \Sigma, \mathfrak{a}^{t-\epsilon}}^0 \longrightarrow \underline{\Omega}_{X, \mathfrak{a}^{t-\epsilon}}^0 \oplus \underline{\Omega}_{\Sigma, \mathfrak{a}^{t-\epsilon}}^0 \longrightarrow \underline{\Omega}_{\Sigma \cap X, \mathfrak{a}^{t-\epsilon}}^0 \xrightarrow{+1}.$$

Proof of Claim 13. We fix $\pi : \tilde{Y} \rightarrow Y$ to be an embedded resolution of Z which is also a log resolution of X , Σ , $X \cap \Sigma$ and $\mathfrak{a}$ and write $\mathfrak{a} \cdot \mathcal{O}_{\tilde{Y}} = \mathcal{O}_{\tilde{Y}}(-G)$. Let E_1 (respectively E_2) denote the reduced pre-image of X in $\tilde{Y}$ (respectively, of Σ in $\tilde{Y}$) and note that $E_1 \cap E_2$ is the reduced pre-image of $X \cap \Sigma$ (which we assumed was also a divisor). Furthermore, let

- E'_1 denote the union of the components of E_1 which are not components of G , and let
- E'_2 denote the union of the components of E_2 which are not components of G .

Consider $E'_1 \cap E'_2$. I claim that this is a divisor and thus is equal to the union of the components of $E_1 \cap E_2$ which are not contained in G . Suppose that L is an irreducible component of $E'_1 \cap E'_2$ and suppose L is not a divisor. On the other hand, $L \subseteq \pi^{-1}(X \cap \Sigma)$ by construction, and so it must be contained in some divisor H lying over $X \cap \Sigma$. Since L is an irreducible component of $E'_1 \cap E'_2$, we must have that H is not a component of $E'_1 \cap E'_2$, and so in particular, H is contained in G . Therefore L is contained in G .

Since L is not a divisor, it must be codimension 2 (since it is a component of the intersection of two divisors in a smooth space). Suppose L is in the intersection of a component $F_1 \subseteq E'_1$ and a component F_2 of E'_2 (neither F_1 or F_2 are in G by construction). But then F_1 , F_2 and H would not be in simple normal crossings at the generic point of L (because the generic point of L is a two dimensional regular local ring). Therefore L is not contained in G , a contradiction.

It follows that we have the following short exact sequence

$$\begin{array}{ccccccc} 0 \longrightarrow \mathcal{O}_{\tilde{Y}}(-(E'_1 \cup E'_2)) & \longrightarrow & \mathcal{O}_{\tilde{Y}}(-E'_1) \oplus \mathcal{O}_{\tilde{Y}}(-E'_2) & \xrightarrow{-} & \mathcal{O}_{\tilde{Y}}(-(E'_1 \cap E'_2)) & \longrightarrow & 0 \\ & & \parallel & & \parallel & & \\ 0 \rightarrow \mathcal{O}_{\tilde{Y}}(\lfloor -\epsilon(E'_1 \cup E'_2) \rfloor) & \rightarrow & \mathcal{O}_{\tilde{Y}}(\lfloor -\epsilon E'_1 \rfloor) \oplus \mathcal{O}_{\tilde{Y}}(\lfloor -\epsilon E'_2 \rfloor) & \xrightarrow{-} & \mathcal{O}_{\tilde{Y}}(\lfloor -\epsilon(E'_1 \cap E'_2) \rfloor) & \rightarrow & 0 \end{array}$$

where the third horizontal map sends (a, b) to $a - b$. We tensor this short exact sequence with $\mathcal{O}_{\tilde{Y}}(\lfloor (t - \epsilon)G \rfloor)$ and then map the result into

$$0 \rightarrow \mathcal{O}_{\tilde{Y}}(\lfloor (t - \epsilon)G \rfloor) \rightarrow \mathcal{O}_{\tilde{Y}}(\lfloor (t - \epsilon)G \rfloor) \oplus \mathcal{O}_{\tilde{Y}}(\lfloor (t - \epsilon)G \rfloor) \xrightarrow{-} \mathcal{O}_{\tilde{Y}}(\lfloor (t - \epsilon)G \rfloor) \rightarrow 0.$$

By taking the cokernel, we obtain a short exact sequence from the nine-lemma,

$$0 \rightarrow M_{X \cup \Sigma, \mathfrak{a}^{t-\epsilon}} \rightarrow M_{X, \mathfrak{a}^{t-\epsilon}} \oplus M_{\Sigma, \mathfrak{a}^{t-\epsilon}} \xrightarrow{-} M_{X \cap \Sigma, \mathfrak{a}^{t-\epsilon}} \rightarrow 0.$$

Pushing forward completes the proof of the claim. $\square$

Dualizing the triangle from the claim, we obtain

$$\mathbf{D} \left(\underline{\Omega}_{\Sigma \cap X, \mathfrak{a}^{t-\epsilon}}^0 \right) \rightarrow \mathbf{D} \left(\underline{\Omega}_{X, \mathfrak{a}^{t-\epsilon}}^0 \right) \oplus \mathbf{D} \left(\underline{\Omega}_{\Sigma, \mathfrak{a}^{t-\epsilon}}^0 \right) \rightarrow \mathbf{D} \left(\underline{\Omega}_{X \cup \Sigma, \mathfrak{a}^{t-\epsilon}}^0 \right) \xrightarrow{+1}.$$

We then take cohomology. Notice that $\Sigma \cap X \subseteq (\text{Sing } Z) \cap X \subseteq \text{Sing } X$ since X is Cartier. The result follows by Proposition 13.6 since $\dim(\Sigma \cap X) \leq \dim X - 2$ (since X is normal) and $\dim \Sigma \leq \dim Z - 2 = \dim X - 1$ (since Z is normal). $\square$

When we combine Lemma 13.11 with Corollary 13.7, we obtain

Corollary 13.12. *Suppose that X is a normal irreducible Cartier divisor in Z which itself is a normal irreducible complete intersection and suppose that $\mathfrak{a}$ is a non-zero ideal sheaf on Z such that $V(\mathfrak{a})$ does not contain X . Let $\pi : \tilde{Z} \rightarrow Z$ be a log resolution of Z , X and $\mathfrak{a}$, and let $\tilde{X}$ denote the strict transform of X . Set E to be the exceptional set of π , F to be the exceptional set of $(\pi|_{\tilde{X}})$, and write $\mathfrak{a} \cdot \mathcal{O}_{\tilde{Z}} = \mathcal{O}_{\tilde{Z}}(-G)$. Then we have a short exact sequence*

$$\begin{aligned} 0 &\rightarrow \pi_* \mathcal{O}_{\tilde{Z}}([K_{\tilde{Z}} - (t - \epsilon)G + \epsilon E]) \\ &\rightarrow \pi_* \mathcal{O}_{\tilde{Z}}([K_{\tilde{Z}} - (t - \epsilon)G + \epsilon(E + \pi^* X)]) \\ &\rightarrow \pi_* \mathcal{O}_{\tilde{X}}([K_{\tilde{X}} - (t - \epsilon)G|_{\tilde{X}} + \epsilon F]) \\ &\rightarrow 0 \end{aligned}$$

Proof. This follows from an application of Corollary 13.7 to the short exact sequence of Lemma 13.11. $\square$

Theorem 13.13. *If Z is a normal complete intersection, $X \subset Z$ is a normal Cartier divisor and $\mathfrak{a}$ is an ideal sheaf on Z such that $V(\mathfrak{a})$ does not contain any component of Z or X , then there is a short exact sequence:*

$$0 \rightarrow \mathcal{J}'(Z; \mathfrak{a}^t) \otimes \mathcal{O}_Z(-X) \rightarrow \mathcal{J}'((Z, X); \mathfrak{a}^t) \rightarrow \mathcal{J}'(X; \mathfrak{a}^t) \rightarrow 0.$$

Proof. Tensor the exact sequence from Corollary 13.12 with $\mathcal{O}_Z(-K_Z - X)$ noticing that

$$\begin{aligned} &\pi_* \mathcal{O}_{\tilde{X}}([K_{\tilde{X}} - (t - \epsilon)G|_{\tilde{X}} + \epsilon F]) \otimes \mathcal{O}_Z(-K_Z - X) \\ &\cong \pi_* \mathcal{O}_{\tilde{X}}([K_{\tilde{X}} - (t - \epsilon)G|_{\tilde{X}} + \epsilon F]) \otimes \mathcal{O}_X \otimes \mathcal{O}_Z(-K_Z - X) \\ &\cong \pi_* \mathcal{O}_{\tilde{X}}([K_{\tilde{X}} - (t - \epsilon)G|_{\tilde{X}} + \epsilon F]) \otimes \mathcal{O}_X(-K_X) \\ &\subseteq \mathcal{O}_X. \end{aligned}$$

$\square$

Part 2. A characteristic p analog of maximal non-lc ideals

This part is devoted to the study of a positive characteristic analog of the maximal non-lc ideal sheaves. As mentioned earlier, this is independent of the previous part, except for the definition of the maximal non-lc ideal sheaves. We should also mention that this is a first attempt. The authors expect that further refinements of the definition may be necessary, in particular see Remark 14.7.

14. NON-F-PURE IDEALS

In this section, we introduce a characteristic p analog of maximal non-lc ideals, called non-F-pure ideals, and study their basic properties.

From this point forward, all rings are Noetherian commutative rings with identity. For a reduced ring R , we denote by R° the set of elements of R that are not in any minimal prime.

Let R be a reduced ring of characteristic $p > 0$. For an ideal I of R and a power q of p , we denote by $I^{[q]}$ the ideal generated by the q^{th} powers of elements of I . Given an R -module M and an integer $e \geq 1$, we will use $F_*^e M$ to denote the R -module which agrees with M as an additive group, but where the multiplication is defined by $r \cdot m = r^{p^e} m$. For example, $I \cdot F_*^e M \cong F_*^e(I^{[p^e]} M)$. We say that R is *F-finite* if $F_*^1 R$ is a finitely generated R -module. For example, a field k is *F-finite* if and only if $[k : k^p] < \infty$, and any algebra essentially of finite type over an *F-finite* field is *F-finite*.

Definition 14.1. A *triple* $(R, \Delta, \mathfrak{a}^t)$ is the combined information of

- (i) an *F-finite* reduced ring R ,
- (ii) an ideal $\mathfrak{a} \subseteq R$ such that $\mathfrak{a} \cap R^\circ \neq \emptyset$,
- (iii) a real number $t > 0$.

Furthermore, if R is a normal domain, then we also consider

- (iv) an effective $\mathbb{R}$ -divisor Δ on $X = \text{Spec } R$.

If $\mathfrak{a} = R$ (resp., $\Delta = 0$) then we call the triple $(R, \Delta, \mathfrak{a}^t)$ a *pair* and denote it by (R, Δ) (resp., $(R, \mathfrak{a}^t)$).

First we recall the definitions of generalized test ideals, *F-purity* and *F-regularity* for triples.

Definition 14.2. Let $(R, \Delta, \mathfrak{a}^t)$ be a triple.

- (i) (cf. [BSTZ, Definition-Proposition 3.3], [HT, Lemma 2.1], [HY, Definition-Theorem 6.5], [T1, Definition 2.6]) The *big generalized test ideal* $\tau_b((R, \Delta); \mathfrak{a}^t)$ is defined to be

$$\tau_b((R, \Delta); \mathfrak{a}^t) = \sum_{e \geq n} \sum_{\varphi} \varphi(F_*^e(\mathfrak{a}^{\lceil tp^e \rceil} d)),$$

where n is an arbitrary positive integer and φ ranges over all elements of $\text{Hom}_R(F_*^e R(\lceil p^e \Delta \rceil), R) \subseteq \text{Hom}_R(F_*^e R, R)$, and where $d \in R^\circ$ is a big test element for R . We do not give the definition of big test elements here (see [Ho] for the definition of big test elements), but, for example, if the localized ring R_d is regular, then some power d^n is a big test element for R by [Ho, p.63, Theorem].

- (ii) ([Sc5, Definition 3.2] cf. [HW, Definition 2.1], [Sc2, Definition 3.1], [T2, Definition 3.1]) Let $\overline{\mathfrak{a}^{\lceil t \bullet \rceil}} := \{\mathfrak{a}_m\}_{m \in \mathbb{N}}$ be a graded family of ideals defined by $\mathfrak{a}_m = \overline{\mathfrak{a}^{\lceil t(m-1) \rceil}}$, where $\overline{\mathfrak{b}}$ is the integral closure of an ideal $\mathfrak{b} \subseteq R$. We say that $(R, \Delta, \overline{\mathfrak{a}^{\lceil t \bullet \rceil}})$ is *sharply F-pure* if there exist an integer $e \geq 1$ and a map

$$\varphi \in \text{Hom}_R(F_*^e R(\lceil (p^e - 1) \Delta \rceil), R) \cdot F_*^e \mathfrak{a}_{p^e - 1}$$

such that $\varphi(1) = 1$.

- (iii) ([Sc5, Definition 3.2] cf. [HW, Definition 2.1], [T2, Definition 3.1]) We say that $(R, \Delta, \mathfrak{a}^t)$ is *strongly F-regular* if for every $d \in R^\circ$, there exist an integer $e \geq 1$ and a map

$$\varphi \in \text{Hom}_R(F_*^e R(\lceil (p^e - 1) \Delta \rceil), R) \cdot F_*^e(\mathfrak{a}^{\lceil t(p^e - 1) \rceil} d)$$

such that $\varphi(1) = 1$.

Remark 14.3. (1) Considering the case where $d = 1$, one can easily see that if $(R, \Delta, \mathfrak{a}^t)$ is strongly F-regular, then $(R, \Delta, \overline{\mathfrak{a}^{[t \bullet]}})$ is sharply F-pure. Also, it follows from [Sc5, Corollary 5.7] that $(R, \Delta, \mathfrak{a}^t)$ is strongly F-regular if and only if $\tau_b((R, \Delta); \mathfrak{a}^t) = R$.

(2) In the case where $\Delta = 0$ and $\mathfrak{a} = 0$, $(R, \Delta, \overline{\mathfrak{a}^{[t \bullet]}})$ is sharply F-pure (resp. $(R, \Delta, \mathfrak{a}^t)$ is strongly F-regular) if and only if the ring R is F-pure (resp. strongly F-regular), and $\tau_b((R, \Delta); \mathfrak{a}^t) = \tau_b(R)$ is the big test ideal. Refer to [HR], [HH] and [Ho] for basic properties of F-pure rings, strongly F-regular rings and the big test ideal, respectively.

Thanks to [HY] and [T1], the generalized test ideal can be viewed as a characteristic p analog of the multiplier ideal. In particular, a strongly F-regular triple $(X = \text{Spec } R, \Delta, \mathfrak{a}^t)$ corresponds to a klt triple under the condition that $K_X + \Delta$ is $\mathbb{Q}$ -Cartier. Also, a sharply F-pure triple is expected to correspond to a lc triple under the same condition. Employing this philosophy, we introduce a characteristic p analog of maximal non-lc ideals.

Definition 14.4 (cf. [B, Corollary 2.14]). Let $(R, \Delta, \mathfrak{a}^t)$ be a triple. We denote the integral closure of an ideal $\mathfrak{b}$ of R by $\overline{\mathfrak{b}}$. We then define the family of ideals $\{\sigma_n((R, \Delta); \mathfrak{a}^t)\}_{n \in \mathbb{N}}$ inductively as follows¹:

$$\begin{aligned}\sigma_1((R, \Delta); \mathfrak{a}^t) &= \sum_{e \geq 1} \sum_{\varphi} \varphi(F_*^e \overline{\mathfrak{a}^{[t(p^e-1)]}}), \\ \sigma_2((R, \Delta); \mathfrak{a}^t) &= \sum_{e \geq 1} \sum_{\varphi} \varphi(F_*^e(\sigma_1((R, \Delta); \mathfrak{a}^t) \overline{\mathfrak{a}^{[t(p^e-1)]}})), \\ \sigma_n((R, \Delta); \mathfrak{a}^t) &= \sum_{e \geq 1} \sum_{\varphi} \varphi(F_*^e(\sigma_{n-1}((R, \Delta); \mathfrak{a}^t) \overline{\mathfrak{a}^{[t(p^e-1)]}})),\end{aligned}$$

where φ runs through all elements of $\text{Hom}_R(F_*^e R(\lceil (p^e-1)\Delta \rceil), R) \subseteq \text{Hom}_R(F_*^e R, R)$. Just for the convenience, we decree that $\sigma_0((R, \Delta); \mathfrak{a}^t) = R$. It follows from [B, Proposition 2.13] that the descending chain

$$R \supseteq \sigma_1((R, \Delta); \mathfrak{a}^t) \supseteq \sigma_2((R, \Delta); \mathfrak{a}^t) \supseteq \sigma_3((R, \Delta); \mathfrak{a}^t) \supseteq \cdots$$

stabilizes at some $n \in \mathbb{N}$. Then the *non-F-pure ideal* $\sigma((R, \Delta); \mathfrak{a}^t)$ is defined to be

$$\sigma((R, \Delta); \mathfrak{a}^t) = \sigma_n((R, \Delta); \mathfrak{a}^t) = \sigma_{n+1}((R, \Delta); \mathfrak{a}^t) = \cdots.$$

When $\mathfrak{a} = R$ (resp. $\Delta = 0$), we simply denote this ideal by $\sigma(R, \Delta)$ (resp. $\sigma(R, \mathfrak{a}^t)$).

Remark 14.5. (1) If $\Delta = 0$ and $\mathfrak{a} = R$, then $\sigma((R, \Delta); \mathfrak{a}^t) = \sigma(R)$ is just the image of the map $\text{Hom}_R(F_*^e R, R) \rightarrow R$ which sends ϕ to $\phi(1)$, at least for $e \gg 0$. In the case that R is local and Gorenstein with perfect residue field, this image is stable for $e \gg 0$ by the Matlis dual of a celebrated result of Hartshorne and Speiser, see [HS, Proposition 1.11]. Thus, Blickle's stabilization result [B, Proposition 2.13] should be viewed as a large generalization of Hartshorne and Speiser's result.

(2) The real exponent t on the ideal $\mathfrak{a}$ in the definition of big generalized test ideals (Definition 14.2 (i)) is just a formal notation, but it is compatible with “real” powers

¹This definition of $\sigma_n((R, \Delta); \mathfrak{a}^t)$ is slightly different from the one given in a previous version. We changed the definition because the proof of Theorem 15.1 didn't work for the previous one.

of $\mathfrak{a}$. That is, setting $\mathfrak{b} = \mathfrak{a}^n$ for $n \in \mathbb{N}$, one has $\tau_b((R, \Delta); \mathfrak{b}^t) = \tau_b((R, \Delta); \mathfrak{a}^{nt})$. However, in the case of non- F -pure ideals, it is not compatible in general. For example, if $R = k[x]$ is the one-dimensional polynomial ring over an F -finite field k of characteristic $p > 0$, then $\sigma(R, (x)) = R$ but $\sigma(R, (x^p)^{1/p}) = x$.

Remark 14.6. Classically, the test ideal is defined as the annihilator ideal of some submodule of the injective hull. We can define the non- F -pure ideal in a similar way.

Let $(R, \mathfrak{m})$ be an F -finite reduced local ring of characteristic $p > 0$, and denote by $E = E_R(R/\mathfrak{m})$ the injective hull of the residue field $R/\mathfrak{m}$. For each integer $e \geq 1$, we denote $\mathbb{F}^{e, \Delta}(E) := F_*^e R(\lceil (p^e - 1)\Delta \rceil) \otimes_R E$ and regard it as an R -module by the action of $F_*^e R \cong R$ from the left. Then the e^{th} iteration of the Frobenius map induces a map $F^e : E \rightarrow \mathbb{F}^{e, \Delta}(E)$. The image of $z \in E$ via this map is denoted by $z^{p^e} := 1 \otimes z = F^e(z) \in \mathbb{F}^{e, \Delta}(E)$.

$N_1((R, \Delta); \mathfrak{a}^t)$ is defined to be the submodule of E consisting of all elements $z \in E$ such that $\overline{\mathfrak{a}^{\lceil t(p^e - 1) \rceil}} z^{p^e} = 0$ in $\mathbb{F}^{e, \Delta}(E)$ for all $e \in \mathbb{N}$. $N_2((R, \Delta); \mathfrak{a}^t)$ is defined to be the submodule of E consisting of all elements $z \in E$ such that $\text{Ann}_R(N_1((R, \Delta); \mathfrak{a}^t)) \overline{\mathfrak{a}^{\lceil t(p^e - 1) \rceil}} z^{p^e} = 0$ in $\mathbb{F}^{e, \Delta}(E)$ for all $e \in \mathbb{N}$. Inductively we define $N_n((R, \Delta); \mathfrak{a}^t)$ to be the submodule of E consisting of all elements $z \in E$ such that $\text{Ann}_R(N_{n-1}((R, \Delta); \mathfrak{a}^t)) \overline{\mathfrak{a}^{\lceil t(p^e - 1) \rceil}} z^{p^e} = 0$ in $\mathbb{F}^{e, \Delta}(E)$ for all $e \in \mathbb{N}$. Then the ascending chain

$$N_1((R, \Delta); \mathfrak{a}^t) \subseteq N_2((R, \Delta); \mathfrak{a}^t) \subseteq N_3((R, \Delta); \mathfrak{a}^t) \subseteq \cdots \subseteq E$$

stabilizes at some $n \in \mathbb{N}$ and $\sigma((R, \Delta); \mathfrak{a}^t) = \text{Ann}_R(N_n((R, \Delta); \mathfrak{a}^t))$.

Remark 14.7. There are numerous other non- F -pure ideals that one can define. The ideal $\sigma((R, \Delta); \mathfrak{a}^t)$ is the largest ideal that we know of that both commutes with localization and seems naturally determined.

We briefly enumerate some of the other potential non- F -pure ideals and describe some of their advantages and disadvantages. All of these ideals define the non- F -pure locus of $(R, \Delta, \mathfrak{a}^t)$ (if they stabilize in the right way).

- (1) $\sigma_1((R, \Delta); \mathfrak{a}^t)$. This ideal still defines the non- F -pure locus and its formation commutes with localization. However, we do not believe that Theorem 15.2 holds for this ideal. The same comments also hold for the other σ_i .
- (2) For any fixed n , consider the ideal $\sigma'_n((R, \Delta); \mathfrak{a}^t) = \sum_{e \geq n} \sum_{\varphi} \varphi(F_*^e \overline{\mathfrak{a}^{\lceil t(p^e - 1) \rceil}})$. This ideal is non-canonically determined, because of the choice of n (for various reasons, it is desirable to choose n sufficiently large). It may be that $\sigma'((R, \Delta), \mathfrak{a}^t) := \cap \sigma'_n((R, \Delta); \mathfrak{a}^t)$ is a good alternative, but we do not know if this intersection stabilizes for sufficiently large n (if it does, then $\sigma'((R, \Delta), \mathfrak{a}^t)$ also commutes with localization). If it does, then the ideal $\sigma'((R, \Delta); \mathfrak{a}^t)$ defines the non- F -pure locus and is *a priori* larger than $\sigma((R, \Delta); \mathfrak{a}^t)$, see Lemma 14.8 below. If R is local with injective hull E of the residue field, $\Delta = 0$ and $\mathfrak{a} = R$, then $\sigma'((R, \Delta); \mathfrak{a}^t) = \sigma'(R)$ also coincides with

$$\text{Ann}_R 0_E^F = \text{Ann}_R \{z \in E \mid z^{p^e} = 0 \text{ for } e \gg 0\},$$

the annihilator ideal of the Frobenius closure 0_E^F of the zero submodule in the injective hull E .

- (3) Suppose now that R is local and that $(p^{e_0} - 1)(K_R + \Delta)$ is Cartier (for some e_0). Then consider the ideal $\sum_{e=ne_0, n>0} \sum_{\varphi} \varphi(F_*^e \mathfrak{a}^{\lceil t(p^e-1) \rceil})$. This ideal suffers from the same issue that (2) does, but it is even smaller. If it can be shown to stabilize for sufficiently large and divisible e_0 , then it would be useful. In particular, one could prove versions of the restriction theorem (Theorem 16.7) for triples $(R, \Delta, \mathfrak{a}^t)$.
- (4) Associated to a triple $(R, \Delta, \mathfrak{a}^t)$, one can define a *Cartier-algebra* on R , see [B] for the definition and details. The ideal $\sigma((R, \Delta); \mathfrak{a}^t)$ is a natural object associated to this algebra. If one replaces this Cartier-algebra by a Veronese sub-algebra, one obtains a different non- F -pure ideal. If there is some stabilization of these non- F -pure ideals for sufficiently fine Veronese sub-algebras, then this could be very useful.

Suppose that R is $\mathbb{Q}$ -Gorenstein with index not divisible by $p > 0$. Further suppose that $\Delta = 0$ and $\mathfrak{a} = R$. Then $\sigma(R)$ coincides with $\sigma'_n(R)$ from (2) for $n \gg 0$. This ideal also coincides with the ideal from (3) for sufficiently large and divisible e_0 and coincides with ideal from (4) for a sufficiently fine Veronese subalgebra.

To see that $\sigma'_n(R) = \sigma(R)$ for $n \gg 0$, first assume without loss of generality that R is local. Then, notice that the evaluation-at-1 map $\text{Hom}_R(F_*^{e+1} R, R) \rightarrow R$ factors through $\text{Hom}_R(F_*^e R, R)$. Thus $\sigma'_n(R)$ is simply the image of $\text{Hom}_R(F_*^n R, R) \rightarrow R$. If R is $\mathbb{Q}$ -Gorenstein and $(p^{e_0} - 1)K_R$ is Cartier, then it follows from [Sc4, Corollary 3.10] that the image of $\text{Hom}_R(F_*^{me_0} R, R) \rightarrow R$ is contained in $\sigma_m(R)$. This implies that $\sigma'_n(R) \subseteq \sigma(R)$ for $n \gg 0$. The reverse containment is done (in much greater generality) in Lemma 14.8 below.

The equality with the ideals from (3) and (4) follow similarly.

Before discussing the basic properties of non- F -pure ideals, we start with the following technical lemma.

Lemma 14.8. *Let $(R, \Delta, \mathfrak{a}^t)$ be a triple. Then for each $n \in \mathbb{N}$ and $i \in \mathbb{Z}_{\geq 0}$, one has*

$$\sigma_{n+i}((R, \Delta); \mathfrak{a}^t) \subseteq \sum_{e \geq i+1} \sum_{\varphi} \varphi(F_*^e(\sigma_{n-1}((R, \Delta); \mathfrak{a}^t) \overline{\mathfrak{a}^{\lceil t(p^e-1) \rceil}}))$$

where φ ranges over all elements of $\text{Hom}_R(F_*^e R(\lceil (p^e - 1)\Delta \rceil), R)$. In particular, for all $n \in \mathbb{N}$, the ideal $\sigma_n((R, \Delta); \mathfrak{a}^t)$ is contained in

$$\sum_{e \geq n} \sum_{\varphi} \varphi(F_*^e \overline{\mathfrak{a}^{\lceil t(p^e-1) \rceil}}) = \sigma'_n((R, \Delta); \mathfrak{a}^t).$$

Proof. From the point of view of Blickle's theory of Cartier algebras, this statement is essentially obvious. We write down a proof in detail however.

For all integers $i \geq 0$, set

$$\sigma_{i,n}((R, \Delta); \mathfrak{a}^t) := \sum_{e \geq i+1} \sum_{\varphi} \varphi(F_*^e(\sigma_{n-1}((R, \Delta); \mathfrak{a}^t) \overline{\mathfrak{a}^{\lceil t(p^e-1) \rceil}})).$$

We will prove the assertion by induction on i . Obviously we may assume that $i \geq 1$. Let $e \geq 1$ be an integer, and fix any $\varphi \in \text{Hom}_R(F_*^e R(\lceil (p^e - 1)\Delta \rceil), R)$, $a \in F_*^e \overline{\mathfrak{a}^{\lceil t(p^e-1) \rceil}}$ and $b \in F_*^e \sigma_{n+i-1}((R, \Delta); \mathfrak{a}^t)$. It then suffices to show that $\varphi(ab) \in \sigma_{i,n}((R, \Delta); \mathfrak{a}^t)$.

It follows from the induction hypothesis that $b \in F_*^e \sigma_{i-1,n}((R, \Delta); \mathfrak{a}^t)$, that is,

$$b \in \sum_{l \geq i} \sum_{\psi} \psi(F_*^{e+l}(\sigma_{n-1}((R, \Delta); \mathfrak{a}^t) \overline{\mathfrak{a}^{[t(p^l-1)]}})),$$

where ψ ranges over all elements of $\text{Hom}_{F_*^e R}(F_*^{e+l}R(\lceil (p^l-1)\Delta \rceil), F_*^e R)$. Since $p^l \lceil (p^e-1)\Delta \rceil + \lceil (p^l-1)\Delta \rceil \geq \lceil (p^{e+l}-1)\Delta \rceil$, the map $\varphi \circ \psi$ can be regarded as an element of $\text{Hom}_R(F_*^{e+l}R(\lceil (p^{e+l}-1)\Delta \rceil), R)$. Also, since $p^l \lceil t(p^e-1) \rceil + \lceil t(p^l-1) \rceil \geq \lceil t(p^{e+l}-1) \rceil$, we have $(F_*^e \overline{\mathfrak{a}^{[t(p^e-1)]}})(F_*^{e+l} \overline{\mathfrak{a}^{[t(p^l-1)]}}) \subseteq F_*^{e+l} \overline{\mathfrak{a}^{[t(p^{e+l}-1)]}}$. Thus,

$$\begin{aligned} \varphi(ab) &\in \sum_{l \geq i} \sum_{\psi} (\varphi \circ \psi)(F_*^e \overline{\mathfrak{a}^{[t(p^e-1)]}} F_*^{e+l}(\sigma_{n-1}((R, \Delta); \mathfrak{a}^t) \overline{\mathfrak{a}^{[t(p^l-1)]}})) \\ &\subseteq \sum_{l \geq i} \sum_{\psi} (\varphi \circ \psi)(F_*^{e+l}(\sigma_{n-1}((R, \Delta); \mathfrak{a}^t) \overline{\mathfrak{a}^{[t(p^{e+l}-1)]}})) \\ &\subseteq \sigma_{i,n}((R, \Delta); \mathfrak{a}^t). \end{aligned}$$

□

Question 14.9. Let the notation be the same as in Lemma 14.8. If n is sufficiently large, then does $\sigma((R, \Delta); \mathfrak{a}^t)$ coincide with $\sigma'_n((R, \Delta); \mathfrak{a}^t)$?

We list basic properties of non-F-pure ideals.

Proposition 14.10. *Let $(R, \Delta, \mathfrak{a}^t)$ be a triple.*

- (1) *For any effective $\mathbb{R}$ -divisor $\Delta' \leq \Delta$ on $\text{Spec } R$, for any ideal $\mathfrak{a} \subseteq \mathfrak{b} \subseteq R$ and for any real number $s \leq t$, one has*

$$\sigma((R, \Delta); \mathfrak{a}^t) \subseteq \sigma((R, \Delta'); \mathfrak{b}^s).$$

If $\mathfrak{a} \subseteq \mathfrak{b} \subseteq \overline{\mathfrak{a}}$, then

$$\sigma((R, \Delta); \mathfrak{a}^t) = \sigma((R, \Delta); \mathfrak{b}^t).$$

- (2) $\sigma((R, \Delta); \mathfrak{a}^t) \mathfrak{a} \subseteq \sigma((R, \Delta); \mathfrak{a}^{t+1})$.
 (3) *Let W be a multiplicatively closed subset of R , and let Δ_W and $\mathfrak{a}_W$ be the images of Δ and $\mathfrak{a}$ in R_W , respectively. Then*

$$\sigma((R_W, \Delta_W); \mathfrak{a}_W^t) = \sigma((R, \Delta); \mathfrak{a}^t) R_W.$$

- (4) *If R is locally an integral domain and if the non-strongly-F-regular locus of $\text{Spec } R$ is contained in $V(\mathfrak{a})$, then for any $\epsilon > 0$,*

$$\sigma((R, \Delta); \mathfrak{a}^t) \subseteq \tau_b((R, (1-\epsilon)\Delta); \mathfrak{a}^{t-\epsilon}).$$

- (5) $(R, \Delta, \overline{\mathfrak{a}^{[t \bullet]}})$ *is sharply F-pure if and only if $\sigma((R, \Delta); \mathfrak{a}^t) = R$.*

Proof. (1) It is obvious, also see [B, Proposition 3.2].

- (2) By the definition of $\sigma_1((R, \Delta); \mathfrak{a}^t)$, we have

$$\begin{aligned} \sigma_1((R, \Delta); \mathfrak{a}^t) \mathfrak{a} &= \sum_{e \geq 1} \sum_{\varphi} \varphi(F_*^e(\overline{\mathfrak{a}^{[t(p^e-1)]}}) \mathfrak{a}^{[p^e]}) \\ &\subseteq \sum_{e \geq 1} \sum_{\varphi} \varphi(F_*^e \overline{\mathfrak{a}^{[t(t+1)(p^e-1)]}}) \\ &= \sigma_1((R, \Delta); \mathfrak{a}^{t+1}). \end{aligned}$$

Applying this inclusion to the definition of $\sigma_2((R, \Delta); \mathfrak{a}^t)$, we have

$$\begin{aligned} \sigma_2((R, \Delta); \mathfrak{a}^t) \mathfrak{a} &= \sum_{e \geq 1} \sum_{\varphi} \varphi(F_*^e(\sigma_1((R, \Delta); \mathfrak{a}^t) \overline{\mathfrak{a}^{[t(p^e-1)]}} \mathfrak{a}^{[p^e]})) \\ &\subseteq \sum_{e \geq 1} \sum_{\varphi} \varphi(F_*^e(\sigma_1((R, \Delta); \mathfrak{a}^{t+1}) \overline{\mathfrak{a}^{[(t+1)(p^e-1)]}})) \\ &= \sigma_2((R, \Delta); \mathfrak{a}^{t+1}). \end{aligned}$$

Inductively we have $\sigma_n((R, \Delta); \mathfrak{a}^t) \mathfrak{a} \subseteq \sigma_n((R, \Delta); \mathfrak{a}^{t+1})$ for all integers $n \geq 1$, so that $\sigma((R, \Delta); \mathfrak{a}^t) \mathfrak{a} \subseteq \sigma((R, \Delta); \mathfrak{a}^{t+1})$.

(3) It is immediate from [B, Lemma 2.18].

(4) We may assume that R is a local domain and set $X = \operatorname{Spec} R$. It follows from [Ho, p.63, Theorem] that there exists an integer $m \geq 1$ such that every nonzero element of $\mathfrak{a}^m$ is a big test element for R . Then $\tau_b((R, (1-\epsilon)\Delta); \mathfrak{a}^{t-\epsilon})$ is equal to $\sum_{e \geq n} \sum_{\psi} \psi(F_*^e \mathfrak{a}^{[(t-\epsilon)p^e]+m})$, where n is an arbitrary positive integer and ψ ranges over all the elements of $\operatorname{Hom}_R(F_*^e R(\lceil p^e(1-\epsilon)\Delta \rceil), R)$. By [Hu], there exists a positive integer k such that $\overline{\mathfrak{a}^{n+k}} \subseteq \mathfrak{a}^n$ for all $n \geq 0$. We take a sufficiently large n such that for all $e \geq n$, $\lceil t(p^e-1) \rceil \geq \lceil (t-\epsilon)p^e \rceil + m + k$, $\lceil (p^e-1)\Delta \rceil \geq \lceil p^e(1-\epsilon)\Delta \rceil$ and that $\sigma((R, \Delta); \mathfrak{a}^t)$ is contained in $\sum_{e \geq n} \sum_{\varphi} \varphi(F_*^e \overline{\mathfrak{a}^{[t(p^e-1)]}})$, where φ ranges over all the elements of $\operatorname{Hom}_R(F_*^e R(\lceil (p^e-1)\Delta \rceil), R)$. Then

$$\begin{aligned} \sigma((R, \Delta); \mathfrak{a}^t) &\subseteq \sum_{e \geq n} \sum_{\varphi} \varphi(F_*^e \overline{\mathfrak{a}^{[t(p^e-1)]}}) \\ &\subseteq \sum_{e \geq n} \sum_{\psi} \psi(F_*^e \overline{\mathfrak{a}^{[(t-\epsilon)p^e]+m+k}}) \\ &\subseteq \sum_{e \geq n} \sum_{\psi} \psi(F_*^e \mathfrak{a}^{[(t-\epsilon)p^e]+m}) \\ &= \tau_b((R, (1-\epsilon)\Delta); \mathfrak{a}^{t-\epsilon}). \end{aligned}$$

(5) By the definition of sharp F-purity, $(R, \Delta, \overline{\mathfrak{a}^{[t\bullet]}})$ is sharply F-pure if and only if $\sigma_1((R, \Delta); \mathfrak{a}^t) = R$. However, $\sigma_1((R, \Delta); \mathfrak{a}^t) = R$ if and only if $\sigma_n((R, \Delta); \mathfrak{a}^t) = R$ for all integers $n \geq 1$. $\square$

Remark 14.11 (Compare with 6.1). Even if R is regular, the equality

$$\sigma((R, \Delta); \mathfrak{a}^t) = \tau_b((R, (1-\epsilon)\Delta); \mathfrak{a}^{t-\epsilon})$$

does not hold for any $\epsilon > 0$ in general. We give two easy examples.

- (1) Let $R = k[x]$ be the one-dimensional polynomial ring over an F -finite field k of characteristic $p > 0$ and let $\mathfrak{a} = (x^p)$. Then $\sigma(R, \mathfrak{a}^{\frac{1}{p}}) = x$, while $\tau_b(R, \mathfrak{a}^{\frac{1}{p}-\epsilon}) = R$ for any $\epsilon > 0$.
- (2) Let $R = k[x, y]$ be the two-dimensional polynomial ring over an F -finite field k of characteristic $p > 0$ and let $\Delta = \operatorname{div}(x^3 - y^2)$. If $p \equiv 2 \pmod{3}$, then $\sigma(R, \frac{5p-1}{6p}\Delta) = (x, y)$, while $\tau_b(R, (\frac{5p-1}{6p} - \epsilon)\Delta) = R$ for any $\epsilon > 0$.

The first example is also a counterexample to Theorem 15.1 below when the denominator of t is divisible by p .

15. NON-F-PURE IDEALS VS. NON-LC IDEAL SHEAVES

In this section, we explore the relationship between non-F-pure ideals and non-lc ideal sheaves.

In Theorem 7.8, we gave a combinatorial description of the non-lc ideal sheaf $\mathcal{J}'(X, \mathfrak{a}^t)$ associated to a monomial ideal $\mathfrak{a}$ on $X = \mathbb{C}^n$. We show that the non-F-pure ideal $\sigma(R, \mathfrak{a}^t)$ has a similar description when $\mathfrak{a}$ is a monomial ideal of the polynomial ring $R = k[x_1, \dots, x_n]$ over an F -finite field k .

Theorem 15.1 (Compare with Theorem 7.8). *Let $\mathfrak{a}$ be a monomial ideal of the polynomial ring $R := k[x_1, \dots, x_n]$ over an F -finite field k of characteristic $p > 0$. Let $t > 0$ be a rational number whose denominator is not divisible by p . Then the non-F-pure ideal $\sigma(R, \mathfrak{a}^t)$ is the monomial ideal generated by all monomials x^v whose exponent vectors satisfy the condition that*

$$v + \mathbf{1} \in P(t \cdot \mathfrak{a}),$$

where $P(t \cdot \mathfrak{a})$ is the Newton polyhedron of $t \cdot \mathfrak{a}$.

Proof. We denote by $I(R, \mathfrak{a}^t)$ the monomial ideal generated by all monomials x^v whose exponent vectors satisfy the condition that $v + \mathbf{1} \in P(t \cdot \mathfrak{a})$. It follows from [HY, Theorem 4.8] that for sufficiently small $1 \gg \epsilon > 0$, the generalized test ideal $\tau_b(R, \mathfrak{a}^{t-\epsilon})$ coincides with $I(R, \mathfrak{a}^t)$. By Proposition 14.10 (4), $\sigma(R, \mathfrak{a}^t)$ is contained in $I(R, \mathfrak{a}^t)$.

Hence, we will prove the converse inclusion. For each integer $e \geq 1$, let $\phi^e : F_*^e R \rightarrow R$ be the R -linear map such that

$$\phi^e(x_1^{l_1} x_2^{l_2} \dots x_n^{l_n}) = \begin{cases} 1 & \text{if } l_1 = l_2 = \dots = l_n = p^e - 1 \\ 0 & \text{whenever } l_i \leq p^e - 1 \text{ for all } i \text{ and } l_i < p^e - 1 \text{ for some } i. \end{cases}$$

Then the ideal $\phi^e(F_*^e \overline{\mathfrak{a}^{[t(p^e-1)]}})$ is generated by monomials, because everything involved is $\mathbb{Z}^n$ -graded. The monomial x^v is in the ideal $\phi^e(F_*^e \overline{\mathfrak{a}^{[t(p^e-1)]}})$ if and only if

$$(5) \quad p^e v \in P(\lceil t(p^e - 1) \rceil \cdot \mathfrak{a}) - (p^e - 1)\mathbf{1}.$$

Since the denominator of t is not divisible by p , there are infinitely many $e \in \mathbb{N}$ such that $t(p^e - 1)$ is an integer. For such e , dividing out by $p^e - 1$, we can rephrase (5) into the condition that $\frac{p^e}{p^e-1}v + \mathbf{1} \in P(t \cdot \mathfrak{a})$. By taking a sufficiently large e , this is equivalent to saying that $v + \mathbf{1} \in P(t \cdot \mathfrak{a})$. Thus, by the definition of $\sigma_1(R, \mathfrak{a}^t)$, the ideal $I(R, \mathfrak{a}^t)$ is contained in $\sigma_1(R, \mathfrak{a}^t)$.

Similarly, for each $e \geq 1$, the monomial x^v is in $\phi^e(F_*^e(\sigma_1(R, \mathfrak{a}^t) \overline{\mathfrak{a}^{[t(p^e-1)]}}))$ if it is in $\phi^e(F_*^e(I(R, \mathfrak{a}^t) \overline{\mathfrak{a}^{[t(p^e-1)]}}))$ which happens if and only if

$$(6) \quad p^e v \in (P(t \cdot \mathfrak{a}) - \mathbf{1}) \cap \mathbb{Z}_{\geq 0}^n + P(\lceil t(p^e - 1) \rceil \cdot \mathfrak{a}) - (p^e - 1)\mathbf{1}.$$

We will show that for all sufficiently large e such that also $t(p^e - 1)$ is an integer, Equation (6) is equivalent to the condition that $v + \mathbf{1} \in P(t \cdot \mathfrak{a})$, that is, $x^v \in I(R, \mathfrak{a}^t)$. First suppose that (6) holds for such e . In particular,

$$p^e v \in (P(t \cdot \mathfrak{a}) - \mathbf{1}) + P(t(p^e - 1) \cdot \mathfrak{a}) - (p^e - 1)\mathbf{1} = p^e P(t \cdot \mathfrak{a}) - p^e \mathbf{1}.$$

Dividing out by p^e , we see that $v + \mathbf{1} \in P(t \cdot \mathbf{a})$. Conversely, suppose that $v + \mathbf{1} \in P(t \cdot \mathbf{a})$. Since v is in $\mathbb{Z}_{\geq 0}^n$, this can be rephrased to say that $v \in (P(t \cdot \mathbf{a}) - \mathbf{1}) \cap \mathbb{Z}_{\geq 0}^n$. Multiplying both sides by $p^e - 1$, we have that

$$(p^e - 1)v \in (p^e - 1)(P(t \cdot \mathbf{a}) - \mathbf{1}) = P(\lceil t(p^e - 1) \rceil \cdot \mathbf{a}) - (p^e - 1)\mathbf{1}$$

for all e such that $t(p^e - 1)$ is an integer. Finally, we see that for such e ,

$$\begin{aligned} p^e v &= v + (p^e - 1)v \\ &= (P(t \cdot \mathbf{a}) - \mathbf{1}) \cap \mathbb{Z}_{\geq 0}^n + P(\lceil t(p^e - 1) \rceil \cdot \mathbf{a}) - (p^e - 1)\mathbf{1}. \end{aligned}$$

So, what it comes down to is that

$$I(R, \mathbf{a}^t) \subseteq \phi^e(F_*^e(\sigma_1(R, \mathbf{a}^t) \overline{\mathbf{a}^{\lceil t(p^e - 1) \rceil}})) \subseteq \sigma_2(R, \mathbf{a}^t).$$

Inductively we conclude that $I(R, \mathbf{a}^t)$ is contained in $\sigma(R, \mathbf{a}^t)$, which completes the proof of Theorem 15.1. $\square$

Theorem 15.2. *Let R be an F -finite normal ring of characteristic $p > 0$ and Δ be an effective $\mathbb{R}$ -divisor on $X := \operatorname{Spec} R$ such that $K_X + \Delta$ is $\mathbb{R}$ -Cartier. Let $\mathbf{a} \subseteq R$ be a nonzero ideal and $t > 0$ be a real number. If $f : \tilde{X} \rightarrow X$ is a proper birational morphism from a normal scheme $\tilde{X}$ such that $\mathbf{a}\mathcal{O}_{\tilde{X}} = \mathcal{O}_{\tilde{X}}(-Z)$ is invertible and $K_{\tilde{X}} + \Delta_{\tilde{X}} = f^*(K_X + \Delta) + tZ$, then one has an inclusion*

$$\sigma((R, \Delta); \mathbf{a}^t) \subseteq H^0(\tilde{X}, \mathcal{O}_{\tilde{X}}(\lceil K_{\tilde{X}} - f^*(K_X + \Delta) - tZ + \varepsilon F \rceil))$$

for sufficiently small $0 \leq \varepsilon \ll 1$, where $F = \operatorname{Supp} \Delta_{\tilde{X}}^{\geq 1}$.

Proof. The proof is very similar to those of [HW, Theorem 3.3] and [T1, Theorem 2.13].

We may assume that R is local. Let $c \in \sigma((R, \Delta); \mathbf{a}^t)$. By Lemma 14.8, we may assume that there exist a sufficiently large $q = p^e$, a nonzero element $a \in \overline{\mathbf{a}^{\lceil t(q-1) \rceil}}$ and an R -linear map $\varphi : F_*^e R(\lceil (q-1)\Delta \rceil) \rightarrow R$ sending a to c . The map φ induces an R -linear map $\varphi' : F_*^e R(\lceil (q-1)\Delta \rceil + \operatorname{div}_X(a)) \rightarrow R(\operatorname{div}_X(c))$ sending 1 to 1. Since

$$\begin{aligned} &\mathcal{H}\operatorname{om}_R(F_*^e R(\lceil (q-1)\Delta \rceil + \operatorname{div}_X(a)), R(\operatorname{div}_X(c))) \\ &\cong F_*^e \mathcal{O}_X((1-q)K_X + q\operatorname{div}_X(c) - \lceil (q-1)\Delta \rceil - \operatorname{div}_X(a)) \end{aligned}$$

by Grothendieck duality, we can regard φ' as a rational section of $\mathcal{O}_{\tilde{X}}((1-q)K_{\tilde{X}} + qf_*^{-1}\operatorname{div}_X(c))$. Let D be the divisor on $\tilde{X}$ corresponding to φ' . Then D is linearly equivalent to $(1-q)K_{\tilde{X}} + qf_*^{-1}\operatorname{div}_X(c)$ and $f_*D \geq \lceil (q-1)\Delta \rceil + \operatorname{div}_X(a)$.

Set $Y := \tilde{X} \setminus \operatorname{Supp} D^{<0}$. Since $\operatorname{Supp} D^{<0}$ is supported on the exceptional locus, φ' lies in the global section of

$$F_*^e \mathcal{O}_Y((1-q)K_Y + qf_*^{-1}\operatorname{div}_X(c)|_Y) \cong F_*^e \mathcal{H}\operatorname{om}_{\mathcal{O}_Y}(F_*^e \mathcal{O}_Y, \mathcal{O}_Y(f_*^{-1}\operatorname{div}_X(c)|_Y)).$$

We will prove that the coefficient of $D - qf_*^{-1}\operatorname{div}_X(c)$ in each irreducible component is less than or equal to $q - 1$. Assume to the contrary that there exists an irreducible component D_0 of D whose coefficient is greater than or equal to $q(\operatorname{ord}_{D_0}(f_*^{-1}\operatorname{div}_X(c)) + 1)$, where ord_{D_0} denotes the order along D_0 . Note that D_0 intersects Y . Set $B := f_*^{-1}\operatorname{div}_X(c) - \operatorname{ord}_{D_0}(f_*^{-1}\operatorname{div}_X(c))D_0$. Then φ' lies in the global section of

$$F_*^e \mathcal{O}_Y((1-q)K_Y + qB|_Y - qD_0|_Y) \cong F_*^e \mathcal{H}\operatorname{om}_{\mathcal{O}_Y}(F_*^e \mathcal{O}_Y(qD_0|_Y), \mathcal{O}_Y(B|_Y)).$$

Combining φ' with the natural inclusion map $\mathcal{O}_Y(D_0|_Y) \hookrightarrow F_*^e \mathcal{O}_Y(qD_0|_Y)$, we obtain the $\mathcal{O}_Y$ -linear map $\psi : \mathcal{O}_Y(D_0|_Y) \rightarrow \mathcal{O}_Y(B|_Y)$. Since φ' sends 1 to 1, ψ also sends 1 to 1. However, since D_0 is not contained in $\text{Supp } B$, this is a contradiction. Thus, every coefficient of $D - qf_*^{-1} \text{div}_X(c)$ is less than or equal to $q - 1$.

Now we put $G := \frac{1}{q-1}D - f_*^{-1}\Delta$. Then G is $\mathbb{Q}$ -linearly equivalent to $-(K_{\tilde{X}} + f_*^{-1}\Delta) + \frac{q}{q-1}f_*^{-1} \text{div}_X(c)$, so that f_*G is $\mathbb{Q}$ -linearly equivalent to $-(K_X + \Delta) + \frac{q}{q-1} \text{div}_X(c)$. There are finitely many prime divisors E_j on $\tilde{X}$ such that

$$K_{\tilde{X}} \underset{\mathbb{R}\text{-lin.}}{\sim} f^*(K_X + \Delta) + tZ + \sum_j a_j E_j$$

where a_j are real numbers chosen as $f_*^{-1}\Delta + tZ + \sum_j a_j E_j$ is f -exceptional. Hence, one has

$$G - f^*f_*G + f_*^{-1}\Delta + tZ + \frac{q}{q-1}(\text{div}_{\tilde{X}}(c) - f_*^{-1} \text{div}_X(c)) + \sum_j a_j E_j = 0,$$

because it is an f -exceptional divisor $\mathbb{R}$ -linearly equivalent to zero. On the other hand,

$$f_*G \geq \frac{1}{q-1}[(q-1)\Delta] + \frac{1}{q-1} \text{div}_X(a) - \Delta \geq \frac{1}{q-1} \text{div}_X(a).$$

Putting it all together, we obtain

$$\begin{aligned} -a_j &= \text{ord}_{E_j} \left(G - f^*f_*G + f_*^{-1}\Delta + tZ + \frac{q}{q-1}(\text{div}_{\tilde{X}}(c) - f_*^{-1} \text{div}_X(c)) \right) \\ &\leq \frac{1}{q-1} \text{ord}_{E_j} (D - \text{div}_{\tilde{X}}(a) + t(q-1)Z - qf_*^{-1} \text{div}_X(c)) + \frac{q}{q-1} \nu_{E_j}(c) \\ &\leq \frac{1}{q-1} \text{ord}_{E_j} (D - qf_*^{-1} \text{div}_X(c)) + \frac{q}{q-1} \nu_{E_j}(c) \\ &\leq 1 + \frac{q}{q-1} \nu_{E_j}(c), \end{aligned}$$

where ord_{E_j} denotes the order along E_j and ν_{E_j} is the valuation corresponding to E_j . Since q is sufficiently large, we conclude that $a_j + \nu_{E_j}(c) \geq -1$ for all j , which implies that c lies in $H^0(\tilde{X}, \mathcal{O}_{\tilde{X}}(\lceil K_{\tilde{X}} - f^*(K_X + \Delta) - tZ + \varepsilon F \rceil))$ for sufficiently small $0 \leq \varepsilon \ll 1$. $\square$

Conjecture 15.3. Let R be a normal ring essentially of finite type over a field of characteristic zero, and let Δ be an effective $\mathbb{Q}$ -divisor on $X := \text{Spec } R$ such that $K_X + \Delta$ is $\mathbb{Q}$ -Cartier. Let $\mathfrak{a} \subseteq R$ be a nonzero ideal and $t > 0$ be a real number. Denoting by $(R_p, \Delta_p, \mathfrak{a}_p)$ the reduction modulo p of the triple $(R, \Delta, \mathfrak{a})$ and by $\mathcal{J}'((X, \Delta); \mathfrak{a}^t)_p$ that of the maximal non-lc ideal $\mathcal{J}'((X, \Delta); \mathfrak{a}^t)$, one has

$$\mathcal{J}'((X, \Delta); \mathfrak{a}^t)_p = \sigma((R_p, \Delta_p); \mathfrak{a}_p^t)$$

for infinitely many primes p .

Remark 15.4. It follows from Theorem 15.2 that $\sigma((R_p, \Delta_p); \mathfrak{a}_p^t)$ is contained in $\mathcal{J}'((X, \Delta); \mathfrak{a}^t)_p$ for all sufficiently large primes p . However, the converse inclusion does not hold for all sufficiently large primes p in general. For example, let $E \subseteq \mathbb{P}_{\mathbb{Q}}^2$ be an elliptic curve over the rational numbers and $X = \text{Spec } R$ be the affine cone

over E . Since X has only log canonical singularities, $\mathcal{J}'(X, 0) = R$. On the other hand, $\sigma(R_p, 0) = R_p$ if and only if p is not supersingular prime for E . It is known by Elkies [El] that there are infinitely many supersingular primes for E . Hence, it cannot happen that $\mathcal{J}'(X, 0)_p = \sigma(R_p, 0)$ for all sufficiently large primes p . The reader is referred to [MTW, Example 4.6] for a more detailed explanation.

16. THE RESTRICTION THEOREM FOR NON-F-PURE IDEALS

In this section, we formulate the restriction theorem for non-F-pure ideals when $\mathfrak{a}$ is the unit ideal.

For simplicity, we may assume that R is an F -finite normal local ring of characteristic $p > 0$, and set $X = \operatorname{Spec} R$. Then there exists a bijection of sets:

$$\left\{ \begin{array}{l} \text{Effective } \mathbb{Q}\text{-divisors } \Delta \text{ on } X \text{ such} \\ \text{that } (p^e - 1)(K_X + \Delta) \text{ is Cartier} \end{array} \right\} \leftrightarrow \{ \text{Nonzero elements of } \operatorname{Hom}_R(F_*^e R, R) \} / \sim$$

where the equivalence relation on the right hand side identifies two maps $\phi_1, \phi_2 \in \operatorname{Hom}_R(F_*^e R, R)$ if there exists some unit $u \in R$ such that $\phi_1(x) = \phi_2(ux)$. The reader is referred to [Sc4] for the details of this correspondence.

Given a map $\phi \in \operatorname{Hom}_R(F_*^e R, R)$ and an integer $l \geq 1$, the l^{th} iteration ϕ^l of ϕ is defined as follows:

$$\phi^l = \phi \circ (F_*^e \phi) \circ \cdots \circ (F_*^{(l-1)e} \phi) \in \operatorname{Hom}_R(F_*^{le} R, R).$$

We remark that if ϕ corresponds to some effective $\mathbb{Q}$ -divisor Δ on X such that $(p^e - 1)(K_X + \Delta)$ is Cartier, then ϕ^l corresponds to the same divisor Δ for every $l \in \mathbb{N}$. This is equivalent to saying that if $\operatorname{Hom}_R(F_*^e R((p^e - 1)\Delta), R)$ is a free $F_*^e R$ -module generated by ϕ , then $\operatorname{Hom}_R(F_*^{le} R((p^{le} - 1)\Delta), R)$ is a free $F_*^{le} R$ -module generated by ϕ^l for every $l \in \mathbb{N}$.

Lemma 16.1. *Let $(R, \mathfrak{m})$ be an F -finite normal local ring of characteristic $p > 0$ and Δ be an effective $\mathbb{Q}$ -divisor on $X := \operatorname{Spec} R$ such that $(p^{e_0} - 1)(K_X + \Delta)$ is Cartier for some $e_0 \in \mathbb{N}$. Let $\phi_{e_0} : F_*^{e_0} R \rightarrow R$ be the R -linear map corresponding to Δ . Then for all sufficiently large $l \in \mathbb{N}$, one has*

$$\sigma(R, \Delta) = \phi_{e_0}^l(F_*^{le_0} R).$$

Proof. Since $\phi_{e_0}(F_*^{e_0} R) \subseteq \sigma_1(R, \Delta)$, we have $\phi_{e_0}^2(F_*^{2e_0} R) \subseteq \sigma_2(R, \Delta)$ by the definition of $\sigma_2(R, \Delta)$. Inductively we have $\phi_{e_0}^n(F_*^{ne_0} R) \subseteq \sigma_n(R, \Delta)$ for all $n \in \mathbb{N}$. It then follows from Lemma 14.8 that for all sufficiently large $l \in \mathbb{N}$,

$$\phi_{e_0}^l(F_*^{le_0} R) \subseteq \sigma(R, \Delta) \subseteq \sum_{e \geq le_0} \sum_{\varphi} \varphi(F_*^e R),$$

where φ ranges over all elements of $\operatorname{Hom}_R(F_*^e R(\lceil (p^e - 1)\Delta \rceil), R)$. Hence, it suffices to show that $\sum_{e \geq le_0} \sum_{\varphi} \varphi(F_*^e R) \subseteq \phi_{e_0}^l(F_*^{le_0} R)$. Let $\varphi \in \operatorname{Hom}_R(F_*^e R(\lceil (p^e - 1)\Delta \rceil), R)$ with $e \geq le_0$. Then by [Sc4, Corollary 3.10], there exists an R -linear map $\psi_i : F_*^i R(\lceil (p^i - 1)\Delta \rceil) \rightarrow R$ such that $\varphi = \phi_{e_0}^l \circ (F_*^{le_0} \psi_i)$ with $i = e - le_0$. Thus, one has

$$\varphi(F_*^e R) = \phi_{e_0}^l((F_*^{le_0} \psi_i)(F_*^e R)) \subseteq \phi_{e_0}^l(F_*^{le_0} R).$$

□

Remark 16.2. In the case of a regular ring R , we have the following description of non-F-pure ideals: let $\Delta = t \cdot \operatorname{div}(f)$ be an effective $\mathbb{Q}$ -divisor on $\operatorname{Spec} R$ such that the denominator of t is not divisible by p . If $J \subseteq R$ is an ideal, one defines $J^{[1/p^e]}$ to be the smallest ideal I such that $I^{[p^e]} \supseteq J$. Then by Lemma 16.1 and [BSTZ, Proposition 3.10], one has

$$\sigma(R, \Delta) = (f^{t(p^e-1)})^{[1/p^e]}$$

for sufficiently large and divisible e such that $t(p^e - 1)$ is an integer. For example, let $R = \mathbb{F}_p[x, y]$ be the two-dimensional polynomial ring over $\mathbb{F}_p$ and let $\Delta = \operatorname{div}(x^3 - y^2)$. Note that $\{x^i y^j\}_{p^e-1 \geq i, j \geq 0}$ is a basis of R over R^{p^e} for all $e \in \mathbb{N}$. Then by [BMS, Proposition 2.5], taking a sufficiently large e , one has

$$\sigma(R, \Delta) = ((x^3 - y^2)^{p^e-1})^{[1/p^e]} = (x, y).$$

Proposition-Definition 16.3 ([Sc4, Theorem 5.2]). Let R be an F -finite normal local ring of characteristic $p > 0$ and $D+B$ be an effective $\mathbb{Q}$ -divisor on $X := \operatorname{Spec} R$ such that D is a normal prime divisor with defining ideal $Q \subseteq R$ and that D is not contained in $\operatorname{Supp} B$. Assume that there exists $e \in \mathbb{N}$ such that $(p^e - 1)(K_X + D + B)$ is Cartier.

Let $\phi : F_*^e R \rightarrow R$ be the R -linear map corresponding to $D+B$. Since the localized ring R_Q is a DVR, $\phi(F_*^e Q) \subseteq Q$ (that is, Q is an F-pure center of $(R, D+B)$). See Definition 16.8 for the definition of F-pure centers). Then we have the following commutative diagram:

$$\begin{array}{ccc} F_*^e R & \xrightarrow{\phi} & R \\ \downarrow & & \downarrow \\ F_*^e(R/Q) & \xrightarrow{\phi_Q} & R/Q, \end{array}$$

where the vertical maps are the natural surjections. We denote by $B_{R/Q}$ the effective $\mathbb{Q}$ -divisor on D corresponding to ϕ_Q . It is easy to check the following properties:

- (i) $(p^e - 1)(K_D + B_{R/Q})$ is a Cartier divisor.
- (i') $\operatorname{Hom}_{R/Q}(F_*^e(R/Q)((p^e - 1)B_{R/Q}), R/Q)$ is a free $F_*^e(R/Q)$ -module generated by ϕ_Q .
- (ii) $(R, D+B)$ is sharply F-pure if and only if $(R/Q, B_{R/Q})$ is sharply F-pure.

Remark 16.4. The divisor $B_{R/Q}$ defined above is canonically determined and exists even outside the local setting. Explicitly, if X is an F -finite normal irreducible scheme and D and B are as above, then there exists a divisor B_D on D (replacing $B_{R/Q}$) satisfying the properties (i), (i') and (ii) above locally and also satisfying the condition that $(K_X + D + B)|_D \sim_{\mathbb{Q}} B_D$. See [Sc4, Remark 9.5] for details.

Conjecture 16.5 (cf. [Sc4, Remark 7.6]). Let the notation be the same as in Definition 16.3. Then $B_{R/Q}$ coincides with the different B_D of $(X, D+B)$ on D (see Section 11 for the definition of differentials).

Remark 16.6. Conjecture 16.5 holds true if D is Cartier in codimension two. The reader is referred to [Sc4, Section 7] for details.

Now we state our restriction theorem for non-F-pure ideals.

Theorem 16.7. *Let R be an F -finite normal domain of characteristic $p > 0$ and $D + B$ be an effective $\mathbb{Q}$ -divisor on $X := \operatorname{Spec} R$ such that D is a normal prime divisor with defining ideal $Q \subseteq R$ and that D is not contained in $\operatorname{Supp} B$. Assume that $K_X + D + B$ is $\mathbb{Q}$ -Cartier with index not divisible by p . Then*

$$\sigma(R, D + B)|_D = \sigma(R/Q, B_{R/Q}).$$

Proof. The statement is local, so we may assume without loss of generality that R is also local. Since $K_X + D + B$ is $\mathbb{Q}$ -Cartier with index not divisible by p , there exist infinitely many $e \in \mathbb{N}$ such that $(p^e - 1)(K_X + D + B)$ is a Cartier divisor. We fix one of such e . Let $\phi \in \operatorname{Hom}_R(F_*^e R, R)$ be the R -linear map corresponding to $D + B$. By the definition of $B_{R/Q}$, there exists $\phi_Q \in \operatorname{Hom}_{R/Q}(F_*^e(R/Q), R/Q)$ corresponding to $B_{R/Q}$ such that we have the following commutative diagram for each $l \in \mathbb{N}$:

$$\begin{array}{ccc} F_*^{le} R & \xrightarrow{\phi^l} & R \\ \downarrow & & \downarrow \\ F_*^{le}(R/Q) & \xrightarrow{\phi_Q^l} & R/Q, \end{array}$$

where the vertical maps are natural quotient maps. Thus, it follows from Lemma 16.1 that for a sufficiently large l ,

$$\sigma(R, D + B)|_D = \phi^l(F_*^{le} R)/R/Q = \phi_Q^l(F_*^{le}(R/Q)) = \sigma(R/Q, B_{R/Q}).$$

□

In fact, the previous restriction even holds when restricting to an F -pure center of arbitrary codimension.

Definition 16.8 ([Sc3]). Suppose that (X, Δ) is a pair such that $K_X + \Delta$ is $\mathbb{Q}$ -Cartier with index not divisible by p . We say that a subvariety $W \subseteq X$ is a *center of sharp F -purity* for (X, Δ) if, after localizing at each point $x \in X$, any (equivalently, some) map $\phi : F_*^e \mathcal{O}_{X,x} \rightarrow \mathcal{O}_{X,x}$ corresponding to Δ (as at the start of this section) satisfies the property that

$$\phi(F_*^e \mathcal{I}_{W,x}) \subseteq \mathcal{I}_{W,x}.$$

Here $\mathcal{I}_W$ is the ideal sheaf defining W and $\mathcal{I}_{W,x}$ is its stalk at $x \in X$. We simply call it an F -pure center of (R, Δ) if the context is clear.

Given a pair $(X := \operatorname{Spec} R, \Delta)$ and a normal F -pure center $W \subseteq X$ with defining ideal $Q \subseteq R$ such that (X, Δ) is sharply F -pure at Q , then there exists a canonically determined $\mathbb{Q}$ -divisor $\Delta_{R/Q}$ on W satisfying the properties (i), (i') and (ii) from Proposition-Definition 16.3. The proof (and reference) are the same.

Theorem 16.9. *Let R be an F -finite normal ring of characteristic $p > 0$ and Δ be an effective $\mathbb{Q}$ -divisor on $X := \operatorname{Spec} R$ such that $K_X + \Delta$ is $\mathbb{Q}$ -Cartier with index not divisible by p . Suppose that $W \subseteq X$ is an F -pure center of (X, Δ) and also that (X, Δ) is sharply F -pure at the generic point of W . Let us use Q to denote the ideal of W . Then*

$$\sigma(R, \Delta)|_W = \sigma(R/Q, \Delta_{R/Q}).$$

Proof. The proof is the same as in Theorem 16.7. The assumption that (X, Δ) is sharply F -pure at the generic point of W is needed to define the $\mathbb{Q}$ -divisor $\Delta_{R/Q}$. $\square$

Compare the following example with Example 12.9.

Example 16.10. Let $R = k[x, y]$ be the two-dimensional polynomial ring over an F -finite field k . Set $D = \operatorname{div}(x)$ and $B = \operatorname{div}(x^3 - y^2)$. It then follows from Remark 16.2 that

$$\begin{aligned}\sigma(R, D + B) &= (x^2, y), \\ \sigma(R/(x), B|_D) &= (y).\end{aligned}$$

Hence the restriction theorem holds in this case.

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