

Fourier methods for fractional-order operators

By

GERD GRUBB*

Abstract

This is the written material for two lectures given at the conference at RIMS, Kyoto: "Harmonic Analysis and Nonlinear Partial Differential equations", July 11-13, 2022.

The intention was to explain how methods using Fourier transformation and complex analysis lead to sharp regularity results in the study of fractional-order operators such as $(-\Delta)^a$, the fractional Laplacian ($0 < a < 1$), and to give an overview of the results. As required by the organizers, we start at a fairly elementary level, introducing the role of function spaces and linear operators. In the later text we explain two important points in detail, with an elementary argumentation: How the exact solution spaces (the a -transmission spaces) come into the picture, and why a locally defined Dirichlet boundary value is relevant.

§ 1. The homogeneous Dirichlet problem

§ 1.1. Introduction, the Fourier transform

We start by recalling the basic notions of *function spaces* and *operators*:

Function spaces. When $f(x)$ is a function on Euclidean space $\mathbb{R}^n$, with points denoted $x = (x_1, x_2, \dots, x_n)$, differentiation with respect to each variable x_k gives the *partial derivative*

$$\frac{\partial f(x_1, \dots, x_n)}{\partial x_k}, \text{ also denoted } \partial_k f.$$

Here are some examples of spaces of functions with derivatives:

- $C^0(\mathbb{R}^n)$ consists of the bounded continuous functions on $\mathbb{R}^n$.

Received Aug 9, 2022. Revised June 6, 2023.

2020 Mathematics Subject Classification(s): 35S15; 47G30; 35J25; 35K05; 60G51

Key Words: Fractional Laplacian, pseudodifferential operators, regularity of Dirichlet solutions, a -transmission spaces, local Dirichlet and Neumann conditions, fractional heat equation, Green's formula.

*Copenhagen University Mathematics Department, 2100 Copenhagen, Denmark.

e-mail: grubb@math.ku.dk

- $C^m(\mathbb{R}^n)$ ($m \in \mathbb{N}$) consists of those bounded continuous functions f that allow taking partial derivatives up to m times giving bounded continuous function.
- $L_2(\mathbb{R}^n)$ consists of functions f such that $\int_{\mathbb{R}^n} |f(x)|^2 dx$ exists. (Here one uses Lebesgue's measure theory, identifying functions that coincide outside a null-set.)
- **Sobolev spaces** $H^m(\mathbb{R}^n)$ consist of the functions f in $L_2(\mathbb{R}^n)$ that have partial derivatives up to order m in $L_2(\mathbb{R}^n)$ (in a generalized sense).

Each of these function spaces is a linear infinite dimensional vector space (when f and g are there, the sum $c_1 f(x) + c_2 g(x)$ is likewise there). They are normed spaces.

Operators. Linear operators are mappings from one function space to another, preserving the vector space structure.

For example, ∂_k defines a linear operator going from $C^m(\mathbb{R}^n)$ to $C^{m-1}(\mathbb{R}^n)$, and from $H^m(\mathbb{R}^n)$ to $H^{m-1}(\mathbb{R}^n)$, when $m \geq 1$. More generally, a **partial differential operator** is a sum of composed derivatives multiplied by functions, $A = \sum_{|\alpha| \leq k} a_\alpha(x) \partial^\alpha$. (Here we use the multi-index notation: Let $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}_0^n$; $\mathbb{N}_0 = \{0, 1, 2, \dots\}$. Then $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$, $\partial^\alpha = \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n}$, $|\alpha| = \alpha_1 + \cdots + \alpha_n$.) A goes from $C^m(\mathbb{R}^n)$ to $C^{m-k}(\mathbb{R}^n)$, and from $H^m(\mathbb{R}^n)$ to $H^{m-k}(\mathbb{R}^n)$, when $m \geq k$ and the coefficients a_α are smooth and bounded with bounded derivatives.

A very important example is **the Laplace operator**

$$\Delta: u \mapsto \Delta u = \partial_1^2 u + \cdots + \partial_n^2 u.$$

It enters in three basic equations (two of them has an extra variable t):

$$\begin{aligned} -\Delta u(x) &= f(x) \text{ on } \Omega, \text{ the Laplace equation,} \\ \partial_t u(x, t) - \Delta u(x, t) &= f(x, t) \text{ on } \Omega \times \mathbb{R}, \text{ the heat equation,} \\ \partial_t^2 u(x, t) - \Delta u(x, t) &= f(x, t) \text{ on } \Omega \times \mathbb{R}, \text{ the wave equation,} \end{aligned}$$

describing physical problems. Here Ω is an open subset of $\mathbb{R}^n$, and one wants to *find solutions* u for given f . For example, the heat equation describes how the temperature develops in a container. And it is also used in financial theory, wrapped up in a stochastic formulation.

Another type of examples of operators are **integral operators**, such as

$$(\mathcal{K}u)(x) = \int_{\mathbb{R}^n} K(x, y) u(y) dy.$$

Differential equations often have integral operators as solution operators.

A very important special operator is the **Fourier transformation** $\mathcal{F}$, it is an integral operator:

$$\mathcal{F}u = \hat{u}(\xi) = \int_{\mathbb{R}^n} e^{-ix \cdot \xi} u(x) dx.$$

It is invertible (in fact isometric times a constant) from the space $L_2(\mathbb{R}^n)$ onto $L_2(\mathbb{R}^n)$, and the inverse operator looks similar: $\mathcal{F}^{-1}v = c \int_{\mathbb{R}^n} e^{+ix \cdot \xi} v(\xi) d\xi$, $c = (2\pi)^{-n}$. It has been used much in physics and mathematics, and the mathematical rigor was perfected with Schwartz' Distribution Theory around 1950, defining the rapidly decreasing functions $\mathcal{S}(\mathbb{R}^n)$ and temperate distributions $\mathcal{S}'(\mathbb{R}^n)$. (A detailed presentation is given e.g. in [29].)

The success of $\mathcal{F}$ comes from the fact that *it turns differential operators into multiplication operators*: The differential operator ∂_k is turned into multiplication by $i\xi_k$:

$$\mathcal{F}(\partial_k u) = i\xi_k \hat{u}(\xi) \quad (\text{here } i = \sqrt{-1}).$$

For example, $\mathcal{F}(\Delta u) = -(\xi_1^2 + \dots + \xi_n^2)\hat{u}(\xi) = -|\xi|^2 \hat{u}$, and therefore the equation $-\Delta u = f$ is turned into $|\xi|^2 \hat{u} = \hat{f}$.

For a particularly simple example, consider the operator $1 - \Delta$ on $\mathbb{R}^n$.

$$(1 - \Delta)u = f \text{ is transformed to } (1 + |\xi|^2)\hat{u} = \hat{f},$$

which has the unique solution $u = \mathcal{F}^{-1}(\frac{1}{1+|\xi|^2} \hat{f})$.

Pseudodifferential operators. Now we generalize the above idea: Take a function $p(\xi)$, the *symbol*, and define the *pseudodifferential operator* (ψ do) $P = \text{Op}(p)$ by

$$\text{Op}(p)(u) = \mathcal{F}^{-1}(p(\xi)\hat{u}(\xi)) = \mathcal{F}^{-1}p(\xi)\mathcal{F}u.$$

Then if p has an inverse $1/p$, $\text{Op}(p) \text{Op}(1/p) = \text{Op}(p \cdot 1/p) = I$, so $P = \text{Op}(p)$ has the inverse $P^{-1} = \text{Op}(1/p)$. This is the simple basic idea.

We often need to let the symbol p depend on x also. This is natural, since differential operators in general have x -dependent coefficients, but it gives more difficult composition rules. The definition is

$$\text{Op}(p(x, \xi))u(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{ix \cdot \xi} p(x, \xi) \hat{u}(\xi) d\xi$$

under suitable requirements on $p(x, \xi)$. We say that p is *of order* m when $\partial_x^\beta \partial_\xi^\alpha p$ is $O((1 + |\xi|)^{m-|\alpha|})$ for all multi-indices α, β .

For x -independent symbols there is the simple composition rule

$$\text{Op}(a(\xi)) \text{Op}(b(\xi))u = \mathcal{F}^{-1}a(\xi)\mathcal{F}\mathcal{F}^{-1}b(\xi)\mathcal{F}u = \text{Op}(a(\xi)b(\xi))u;$$

in other words, the symbol of $\text{Op}(a) \text{Op}(b)$ is ab . For x -dependent symbols there is, just like for differential operators, a more complicated composition formula with lower-order terms.

$$\text{Op}(a(x, \xi)) \text{Op}(b(x, \xi)) = \text{Op}(a(x, \xi)b(x, \xi)) + \mathcal{R},$$

where the order of $\mathcal{R}$ is 1 step lower than that of ab ($\mathcal{R}$ can be described in more detail).

The case of x -independent symbols can often be used as a model for the general case.

The theory was built up in the 1960's (by Kohn and Nirenberg, Hörmander, Seeley, with preceding insights by Mihlin, Calderón, Zygmund and others), and further developed through the rest of the century and beyond.

§ 1.2. The fractional Laplacian

The operator we shall be concerned with here is the fractional Laplacian $(-\Delta)^a$, $0 < a < 1$. It can be defined by spectral theory in functional analysis, since $-\Delta$ is a selfadjoint nonnegative operator (unbounded) in the Hilbert space $L_2(\mathbb{R}^n)$. It is currently of great interest in probability theory and finance, and also in mathematical physics and differential geometry.

Structurally, it is a **pseudodifferential operator**,

$$(1.1) \quad (-\Delta)^a u = \text{Op}(|\xi|^{2a})u.$$

It can also be written as a **singular integral operator**:

$$(1.2) \quad (-\Delta)^a u(x) = c_{n,a} PV \int_{\mathbb{R}^n} \frac{u(x) - u(x+y)}{|y|^{n+2a}} dy;$$

here $c_{n,a}|y|^{-n-2a} = \mathcal{F}^{-1}|\xi|^{2a}$, and PV stands for “principal value”.

Why in finance? The heat equation $\partial_t u - \Delta u = f$ is the central ingredient in the Black-Scholes equation, that has been used to predict the development of stock options. But the prediction was not optimal, here it was found that the *fractional heat equation* $\partial_t u + (-\Delta)^a u = f$ was better, as I have been told, because solution curves had a more flat profile matching better the empirical results.

Formula (1.1) has natural generalizations to x -dependent symbols $p(x, \xi)$, allowing “variable-coefficient” operators.

Formula (1.2) is often used in probability and nonlinear analysis, with generalizations to expressions with other kernel functions than $|y|^{-n-2a}$, e.g. $|y|^{-n-2a} K(y/|y|)$, K positive, and *even*: $K(-y) = K(y)$ (possibly with less smoothness). They generate Lévy processes. Here calculations are often made considering only real functions, whereas the Fourier transform of course involves complex functions.

In contrast to $-\Delta$, $(-\Delta)^a$ is a *nonlocal* operator on $\mathbb{R}^n$: When $u = 0$ in an open set ω , then $\Delta u = 0$ on ω but usually $(-\Delta)^a u \neq 0$ there; this gives substantial difficulties. To study functions u on a given open subset Ω of $\mathbb{R}^n$, we can define $(-\Delta)^a u$ by letting u be zero on $\mathbb{R}^n \setminus \overline{\Omega}$ (i.e., $\text{supp } u \subset \overline{\Omega}$), and map it to $r^+(-\Delta)^a u$, where r^+ denotes restriction to Ω . (The support $\text{supp } v$ of a distribution v is the complement of the largest open set where it is zero.)

The *homogeneous Dirichlet problem* for $P = (-\Delta)^a$ on Ω is then defined as follows: For a given function f on Ω , find a function u on $\mathbb{R}^n$ such that

$$(1.3) \quad r^+ Pu = f \text{ in } \Omega, \quad \text{supp } u \subset \overline{\Omega}.$$

We now need to introduce Sobolev spaces over Ω . Denote $\langle \xi \rangle = (1 + |\xi|^2)^{\frac{1}{2}}$; then

$$H^s(\mathbb{R}^n) = \{u \in \mathcal{S}'(\mathbb{R}^n) \mid \langle \xi \rangle^s \hat{u} \in L_2(\mathbb{R}^n)\}, \text{ with norm } \|\mathcal{F}^{-1}(\langle \xi \rangle^s \hat{u})\|_{L_2},$$

$$\overline{H}^s(\Omega) = r^+ H^s(\mathbb{R}^n), \text{ the restricted space,}$$

$$\dot{H}^s(\overline{\Omega}) = \{u \in H^s(\mathbb{R}^n) \mid \text{supp } u \subset \overline{\Omega}\}, \text{ the supported space,}$$

for $s \in \mathbb{R}$. (The dot and overline notation stems from Hörmander's books. In other works, the dot is replaced by a twiddle or an index 0.) When Ω is suitably regular, $\overline{H}^s(\Omega)$ and $\dot{H}^{-s}(\overline{\Omega})$ are dual spaces, with a duality consistent with the L_2 -scalar product. The space $C_0^\infty(\Omega)$ of smooth functions with compact support in Ω is dense in $\dot{H}^s(\overline{\Omega})$ for all s . The space $\overline{H}^s(\Omega)$ coincides with $\dot{H}^s(\overline{\Omega})$ when $|s| < \frac{1}{2}$.

There is a variational formulation that gives unique solvability of (1.3) in low-order Sobolev spaces. Let $P = (-\Delta)^a$, $0 < a < 1$. First note that P maps $H^s(\mathbb{R}^n) \rightarrow H^{s-2a}(\mathbb{R}^n)$ continuously, since

$$\|Pu\|_{H^{s-2a}}^2 = c \int (\langle \xi \rangle^{s-2a} |\xi|^{2a} |\hat{u}|)^2 d\xi \leq c \int (\langle \xi \rangle^s |\hat{u}|)^2 d\xi = \|u\|_{H^s}^2.$$

In particular, $P: H^a(\mathbb{R}^n) \rightarrow H^{-a}(\mathbb{R}^n)$, and hence $r^+ P: \dot{H}^a(\overline{\Omega}) \rightarrow \overline{H}^{-a}(\Omega)$.

The sesquilinear form Q_0 on $\dot{H}^a(\overline{\Omega})$ obtained by closure of

$$Q_0(u, v) = \int_{\Omega} Pu \bar{v} dx \text{ for } u, v \in C_0^\infty(\Omega),$$

satisfies $Q_0(u, u) \geq 0$, and equals $\langle r^+ Pu, v \rangle_{\overline{H}^{-a}, \dot{H}^a}$. The L_2 Dirichlet realization P_D is defined as the operator acting like $r^+ P$ and having domain

$$D(P_D) = \{u \in \dot{H}^a(\overline{\Omega}) \mid r^+ Pu \in L_2(\Omega)\}.$$

By variational theory (the Lax-Milgram lemma, cf. e.g. [29], Sect. 12.4), this operator is selfadjoint in $L_2(\Omega)$ and ≥ 0 .

When Ω is bounded, there is a Poincaré inequality assuring that P_D has positive lower bound, hence it is *bijective* from $D(P_D)$ onto $L_2(\Omega)$. So there *exists* a bijective solution operator for (1.3), but what more can we say about u , when $f \in L_2(\Omega)$, or lies in better spaces $\overline{H}^s(\Omega)$?

It has been known since the 1960's that $D(P_D) = \dot{H}^{2a}(\overline{\Omega})$ if $a < \frac{1}{2}$, and $D(P_D) \subset \dot{H}^{a+\frac{1}{2}-\varepsilon}(\overline{\Omega})$ if $a \geq \frac{1}{2}$, [23], but more precise information has been obtained in recent years.

The new knowledge is that under some regularity assumptions, the solution u has a factor d^a , where $d(x) = \text{dist}(x, \partial\Omega)$. We shall quote the detailed results later, but will now show how the factor d^a comes in via Fourier transformation methods. For this, we recall some important formulas for the Fourier transform from functions of $x_n \in \mathbb{R}$ to functions of $\xi_n \in \mathbb{R}$: Denoting $1|_{\mathbb{R}_+} = H(x_n)$ (the Heaviside function), we have for $\sigma > 0$, $a > -1$,

$$(1.4) \quad \begin{aligned} \mathcal{F}_{x_n \rightarrow \xi_n}(H(x_n)e^{-\sigma x_n}) &= \frac{1}{\sigma + i\xi_n}, \\ \mathcal{F}_{x_n \rightarrow \xi_n}(H(x_n)x_n^a e^{-\sigma x_n}) &= \frac{c}{(\sigma + i\xi_n)^{a+1}}, \quad c = \Gamma(a+1). \end{aligned}$$

The complex number $\sigma + i\xi_n$ has real part $\sigma > 0$, so its noninteger powers (defined with a cut along the negative axis $\mathbb{R}_-$) make good sense. The first formula is elementary; proofs of the second formula are found e.g. in Schwartz [52], (V,1;44), and in the lines after Example 7.1.17 in [43] (with different conventions). (Using (1.4), we can avoid going in detail with homogeneous distributions, limits for $\sigma \rightarrow 0$.)

§ 1.3. Model Dirichlet problems (detailed)

We shall now study the Dirichlet problem (1.3) in the simplest possible case where P is the invertible ψ do $(1 - \Delta)^a$ and $\Omega = \mathbb{R}_+^n$.

Example 1.1. First we make some remarks on the Dirichlet problem for $1 - \Delta$ on $\mathbb{R}_+^n$ ($= \{(x', x_n) \mid x_n > 0\}$; $x' = (x_1, \dots, x_{n-1})$), denoting $u(x', 0) = \gamma_0 u$:

$$(1.5) \quad (1 - \Delta)u = 0 \text{ on } \mathbb{R}_+^n, \quad \gamma_0 u = \varphi \text{ on } \mathbb{R}^{n-1}.$$

Fourier transformation in x' turns the operator into $1 + |\xi'|^2 - \partial_n^2$, so denoting $\mathcal{F}_{x' \rightarrow \xi'} u = \acute{u}(\xi', x_n)$, (1.5) becomes an ODE problem for each ξ' :

$$(\langle \xi' \rangle^2 - \partial_n^2) \acute{u}(\xi', x_n) = 0 \text{ on } \mathbb{R}_+^n, \quad \acute{u}(\xi', 0) = \hat{\varphi}(\xi').$$

This has the unique bounded solution $\acute{u}(\xi', x_n) = \hat{\varphi}(\xi') e^{-\langle \xi' \rangle x_n}$ on $\mathbb{R}_+$. Inverse Fourier transformation from ξ' to x' gives that (1.5) is solved by $u = K_0 \varphi$, where K_0 is the *Poisson operator* defined by

$$K_0 \varphi = \mathcal{F}_{\xi' \rightarrow x'}^{-1} (H(x_n) e^{-\langle \xi' \rangle x_n} \hat{\varphi}(\xi')) = \mathcal{F}_{\xi \rightarrow x}^{-1} \left(\frac{1}{\langle \xi' \rangle + i\xi_n} \hat{\varphi}(\xi') \right),$$

using (1.4). Here K_0 maps continuously $K_0: H^{s-\frac{1}{2}}(\mathbb{R}^{n-1}) \rightarrow e^+ \overline{H}^s(\mathbb{R}_+^n)$, for all $s \in \mathbb{R}$.¹

¹Strictly speaking, the standard Poisson operator is $r^+ K_0$, but this distinction is often left out, since $K_0 \varphi$ is 0 on $\mathbb{R}_-^n$. More comments in [30], (A-13)-(A-14), [36], Remark 3.2.

Example 1.2. Now turn to the model Dirichlet problem for the fractional Laplacian (recall $0 < a < 1$):

$$(1.6) \quad r^+(1 - \Delta)^a u = f \text{ on } \mathbb{R}_+^n, \quad \text{supp } u \subset \overline{\mathbb{R}_+^n}.$$

The variational solution method applies straightforwardly to $P = (1 - \Delta)^a$, showing that for $f \in L_2(\mathbb{R}_+^n)$ there is a unique solution $u \in \dot{H}^a(\overline{\mathbb{R}_+^n})$. It will now be examined.

The symbol $p(\xi) = (1 + |\xi|^2)^a$ of P has the factorization:

$$(1 + |\xi|^2)^a = (\langle \xi' \rangle^2 + \xi_n^2)^a = (\langle \xi' \rangle - i\xi_n)^a (\langle \xi' \rangle + i\xi_n)^a.$$

Introduce for general $t \in \mathbb{R}$ the *order-reducing operators*:

$$\Xi_\pm^t = \text{Op}(\langle \xi' \rangle \pm i\xi_n)^t.$$

They are invertible, mapping for all $s \in \mathbb{R}$:

$$\Xi_\pm^t : H^s(\mathbb{R}^n) \xrightarrow{\sim} H^{s-t}(\mathbb{R}^n), \text{ with inverse } \Xi_\pm^{-t}.$$

Using these, P has the factorization

$$(1 - \Delta)^a = \Xi_-^a \Xi_+^a, \text{ with inverse } (1 - \Delta)^{-a} = \Xi_+^{-a} \Xi_-^{-a}.$$

The operators $\Xi_\pm^t$ have special roles relative to $\mathbb{R}_+^n$. The *plus-family* Ξ_+^t has symbols that extend analytically in ξ_n to the lower complex halfplane $\mathbb{C}_- = \{\text{Im } \xi_n < 0\}$; then by the Paley-Wiener theorem, Ξ_+^t *preserves support in* $\overline{\mathbb{R}_+^n}$. The inverse is Ξ_+^{-t} . Thus for all $s \in \mathbb{R}$,

$$(1.7) \quad \Xi_+^t : \dot{H}^s(\overline{\mathbb{R}_+^n}) \xrightarrow{\sim} \dot{H}^{s-t}(\overline{\mathbb{R}_+^n}).$$

The *minus-family* Ξ_-^t behaves in a similar way with respect to $\overline{\mathbb{R}_-^n}$. Since $\Xi_-^t = (\Xi_+^t)^*$, we have moreover, in view of the duality between $\dot{H}^s(\overline{\mathbb{R}_+^n})$ and $\overline{H}^{-s}(\mathbb{R}_+^n)$, that

$$(1.8) \quad r^+ \Xi_-^t e^+ : \overline{H}^s(\mathbb{R}_+^n) \xrightarrow{\sim} \overline{H}^{s-t}(\mathbb{R}_+^n),$$

with inverse $(r^+ \Xi_-^t e^+)^{-1} = r^+ \Xi_-^{-t} e^+$; here e^+ indicates “extension by zero” from $\mathbb{R}_+^n$ to $\mathbb{R}^n$. (For negative values of s , there is a distributional interpretation of (1.8), cf. [31], Remark 1.1.)

Note also that for $v \in L_2(\mathbb{R}^n)$, $v = e^+ r^+ v + e^- r^- v$, by the identification of $L_2(\mathbb{R}^n)$ with $e^+ L_2(\mathbb{R}_+^n) \dot{+} e^- L_2(\mathbb{R}_-^n)$.

Let $u \in \dot{H}^a(\overline{\mathbb{R}_+^n})$; then Ξ_+^a maps it into $e^+ L_2(\mathbb{R}_+^n)$. Now since $P = \Xi_-^a \Xi_+^a$, we may write

$$r^+ P u = r^+ \Xi_-^a \Xi_+^a u = r^+ \Xi_-^a (e^+ r^+ + e^- r^-) \Xi_+^a u = r^+ \Xi_-^a e^+ r^+ \Xi_+^a u,$$

where we used that $r^-\Xi_+^a u = 0$. In a diagram, r^+P is the composition

$$\dot{H}^a(\overline{\mathbb{R}}_+^n) \xrightarrow{r^+\Xi_+^a} L_2(\mathbb{R}_+^n) \xrightarrow{r^+\Xi_-^a e^+} \overline{H}^{-a}(\mathbb{R}_+^n).$$

Here both factors are bijections, in view of (1.7) and (1.8). Hence the inverse R , the solution operator for (1.6) with $f \in \overline{H}^{-a}(\mathbb{R}_+^n)$, is the composed operator $R = \Xi_+^{-a} e^+ (r^+ \Xi_-^a e^+)$;

$$(1.9) \quad \overline{H}^{-a}(\mathbb{R}_+^n) \xrightarrow{r^+\Xi_-^a e^+} L_2(\mathbb{R}_+^n) \xrightarrow{\Xi_+^{-a} e^+} \dot{H}^a(\overline{\mathbb{R}}_+^n).$$

To find the solutions of (1.6) with $f \in L_2(\mathbb{R}_+^n)$, we restrict the operator R to $L_2(\mathbb{R}_+^n)$; this is expressed in the diagram

$$(1.10) \quad L_2(\mathbb{R}_+^n) \xrightarrow{r^+\Xi_-^a e^+} \overline{H}^a(\mathbb{R}_+^n) \xrightarrow{\Xi_+^{-a} e^+} \Xi_+^{-a} e^+ \overline{H}^a(\mathbb{R}_+^n) = D(P_D).$$

Property (1.8) is used in the first mapping, but in the second mapping there can be a mismatch; property (1.7) may not be applicable. The space at the right end is the so-called *a-transmission space* $H^{a(2a)}(\overline{\mathbb{R}}_+^n)$, generally defined for $t > a - \frac{1}{2}$ by

$$(1.11) \quad H^{a(t)}(\overline{\mathbb{R}}_+^n) = \Xi_+^{-a} e^+ \overline{H}^{t-a}(\mathbb{R}_+^n).$$

The idea can also be applied starting with f given in a space $\overline{H}^s(\mathbb{R}_+^n)$ with $s \geq -a$:

$$(1.12) \quad \overline{H}^s(\mathbb{R}_+^n) \xrightarrow{r^+\Xi_-^a e^+} \overline{H}^{s+a}(\mathbb{R}_+^n) \xrightarrow{\Xi_+^{-a} e^+} \Xi_+^{-a} e^+ \overline{H}^{s+a}(\mathbb{R}_+^n) \equiv H^{a(s+2a)}(\overline{\mathbb{R}}_+^n),$$

with bijective mappings. This proves

Theorem 1.3. *Let $s \geq -a$. The solution u of the Dirichlet problem (1.6) satisfies*

$$f \in \overline{H}^s(\mathbb{R}_+^n) \iff u \in H^{a(s+2a)}(\overline{\mathbb{R}}_+^n).$$

In particular, $D(P_D) = H^{a(2a)}(\overline{\mathbb{R}}_+^n)$; the case $s = 0$.

What are these transmission spaces? Note that they decrease with increasing s , and $H^{a(a)}(\overline{\mathbb{R}}_+^n) = \dot{H}^a(\overline{\mathbb{R}}_+^n)$. We will study $D(P_D) = \Xi_+^{-a} e^+ \overline{H}^a(\mathbb{R}_+^n) \equiv H^{a(2a)}(\overline{\mathbb{R}}_+^n)$ more closely:

For $a < \frac{1}{2}$, $\overline{H}^a(\mathbb{R}_+^n)$ identifies with $\dot{H}^a(\overline{\mathbb{R}}_+^n)$ so here (1.7) can be applied and gives that $D(P_D) = \dot{H}^{2a}(\overline{\mathbb{R}}_+^n)$.

For $\frac{1}{2} \leq a < 1$, we are in a new situation. Since $\overline{H}^a(\mathbb{R}_+^n) \subset \overline{H}^{\frac{1}{2}-\varepsilon}(\mathbb{R}_+^n) = \dot{H}^{\frac{1}{2}-\varepsilon}(\overline{\mathbb{R}}_+^n)$ (small $\varepsilon > 0$), we have at least $D(P_D) \subset \dot{H}^{a+\frac{1}{2}-\varepsilon}(\overline{\mathbb{R}}_+^n)$, by (1.7). For $a = \frac{1}{2}$, the sharpest information is $H^{\frac{1}{2}(1)}(\overline{\mathbb{R}}_+^n) = \Xi_+^{-\frac{1}{2}} e^+ \overline{H}^{\frac{1}{2}}(\mathbb{R}_+^n)$. For $a > \frac{1}{2}$ we can analyze more:

Formula (1.4) with $\sigma = \langle \xi' \rangle$ shows:

$$(1.13) \quad \mathcal{F}_{\xi_n \rightarrow x_n}^{-1} \left(\frac{1}{(\langle \xi' \rangle + i\xi_n)^{a+1}} \right) = \frac{1}{\Gamma(a+1)} H(x_n) x_n^a e^{-\langle \xi' \rangle x_n}.$$

Recall from Example 1.1 the Poisson operator $K_0: \varphi \mapsto u = \mathcal{F}_{\xi \rightarrow x}^{-1} \left(\frac{1}{\langle \xi' \rangle + i\xi_n} \hat{\varphi}(\xi') \right)$ solving $(1 - \Delta)u = 0$ on $\mathbb{R}_+^n$, $\gamma_0 u = \varphi$. We can show:

Theorem 1.4. 1° For $\frac{1}{2} < a < 1$, $u \in H^{a(2a)}(\overline{\mathbb{R}_+^n})$ if and only if

$$(1.14) \quad u = v + w, \text{ where } v \in \dot{H}^{2a}(\overline{\mathbb{R}_+^n}), \ w = x_n^a K_0 \psi \text{ with } \psi \in H^{a-\frac{1}{2}}(\mathbb{R}^{n-1}).$$

2° With $\mathcal{S}(\overline{\mathbb{R}_+^n}) = r^+ \mathcal{S}(\mathbb{R}^n)$, $x_n^a \mathcal{S}(\overline{\mathbb{R}_+^n})$ is dense in $H^{a(t)}(\overline{\mathbb{R}_+^n})$. The mapping $\gamma_0^a: u \mapsto \gamma_0(u/x_n^a)$ from $x_n^a \mathcal{S}(\overline{\mathbb{R}_+^n})$ to $\mathcal{S}(\mathbb{R}^{n-1})$ extends by continuity to a mapping (when $t > a + \frac{1}{2}$),

$$\gamma_0^a: H^{a(t)}(\overline{\mathbb{R}_+^n}) \rightarrow H^{t-a-\frac{1}{2}}(\mathbb{R}^{n-1}).$$

Here ψ in (1.14) equals $\gamma_0^a u$.

Proof. 1°. Let $u \in H^{a(2a)}(\overline{\mathbb{R}_+^n})$, that is, $u = \Xi_+^{-a} e^+ f$ for some $f \in \overline{H}^a(\mathbb{R}_+^n)$. Since $a > \frac{1}{2}$, f has a boundary value $\varphi = \gamma_0 f \in H^{a-\frac{1}{2}}(\mathbb{R}^{n-1})$, and there is a decomposition $f = g + h$, where

$$h = K_0 \varphi \in K_0 H^{a-\frac{1}{2}}(\mathbb{R}^{n-1}), \quad g = f - K_0 \varphi \in \dot{H}^a(\overline{\mathbb{R}_+^n}),$$

since $\gamma_0 g = \gamma_0 f - \varphi = 0$. Going back to u by applying Ξ_+^{-a} , we find

$$u = v + w, \quad v = \Xi_+^{-a} g, \quad w = \Xi_+^{-a} K_0 \varphi.$$

By the mapping property (1.7), $v \in \dot{H}^{2a}(\overline{\mathbb{R}_+^n})$. For w , we find:

$$\begin{aligned} w &= \Xi_+^{-a} K_0 \varphi = \mathcal{F}_{\xi \rightarrow x}^{-1} \left(\frac{1}{(\langle \xi' \rangle + i\xi_n)^a} \frac{1}{\langle \xi' \rangle + i\xi_n} \hat{\varphi}(\xi') \right) \\ &= \mathcal{F}_{\xi \rightarrow x}^{-1} \left(\frac{1}{(\langle \xi' \rangle + i\xi_n)^{a+1}} \hat{\varphi}(\xi') \right) \\ &= \frac{1}{\Gamma(a+1)} \mathcal{F}_{\xi' \rightarrow x'}^{-1} (x_n^a e^{-\langle \xi' \rangle x_n} H(x_n) \hat{\varphi}(\xi')) = \frac{1}{\Gamma(a+1)} x_n^a K_0 \varphi, \end{aligned}$$

where we have used formula (1.13). This shows the asserted form of w , with $\psi = \frac{1}{\Gamma(a+1)} \varphi$. (We have omitted a constant entering in the definition of γ_0^a in the literature.)

2°. The properties of γ_0^a are known from [31] (with the notation $\gamma_{0,a} u = \Gamma(a+1) \gamma_0(u/x_n^a)$); we shall just indicate a quick way to obtain the mentioned statements. The space $\mathcal{S}(\overline{\mathbb{R}_+^n}) = r^+ \mathcal{S}(\mathbb{R}^n)$ is dense in $\overline{H}^r(\mathbb{R}_+^n)$ for all $r \in \mathbb{R}$. There is an elementary proof in [38], Sect. 6, of the identity

$$(1.15) \quad e^+ x_n^a \mathcal{S}(\overline{\mathbb{R}_+^n}) = \Xi_+^{-a} e^+ \mathcal{S}(\overline{\mathbb{R}_+^n}),$$

(based on Taylor expansion in x_n of $e^{-\langle \xi' \rangle x_n} \mathcal{F}_{x' \rightarrow \xi'} u$ and formulas like (1.13)); here when $u \in e^+ x_n^a \mathcal{S}(\overline{\mathbb{R}}_+^n)$,

$$(1.16) \quad \gamma_0(u/x_n^a) = c^{-1} \gamma_0(\Xi_+^a u), \quad c = \Gamma(a+1).$$

(Lemma 6.1 and (6.9) in [38].)

By (1.15), the denseness of $\mathcal{S}(\overline{\mathbb{R}}_+^n)$ in $\overline{H}^{t-a}(\mathbb{R}_+^n)$ implies the denseness of $e^+ x_n^a \mathcal{S}(\overline{\mathbb{R}}_+^n)$ in $\Xi_+^{-a} e^+ \overline{H}^{t-a}(\mathbb{R}_+^n) = H^{a(t)}(\overline{\mathbb{R}}_+^n)$. It is well-known that for $t-a > \frac{1}{2}$, γ_0 defined on $\mathcal{S}(\overline{\mathbb{R}}_+^n)$ extends by continuity to the map $\gamma_0: \overline{H}^{t-a}(\mathbb{R}_+^n) \rightarrow H^{t-a-\frac{1}{2}}(\mathbb{R}^{n-1})$. Then the map $u \rightarrow \gamma_0(u/x_n^a)$ from $x_n^a e^+ \mathcal{S}(\overline{\mathbb{R}}_+^n)$ to $\mathcal{S}(\mathbb{R}^{n-1})$ extends by continuity to a map from $H^{a(t)}(\overline{\mathbb{R}}_+^n)$ to $H^{t-a-\frac{1}{2}}(\mathbb{R}^{n-1})$ in view of (1.16). $\square$ $\square$

Summing up, we conclude that for $\frac{1}{2} < a < 1$, $D(P_D) = H^{a(2a)}(\overline{\mathbb{R}}_+^n)$ is the set of functions u of the form

$$(1.17) \quad u = v + x_n^a K_0 \psi, \quad \text{where } \psi = \gamma_0(u/x_n^a);$$

here v and ψ run through $\dot{H}^{2a}(\overline{\mathbb{R}}_+^n)$ resp. $H^{a-\frac{1}{2}}(\mathbb{R}^{n-1})$.

§ 1.4. The Dirichlet problem for curved domains

For general domains Ω , we shall list some recent regularity and solvability results in a brief formulation. First we recall the definitions of some more function spaces:

The *Bessel-potential spaces* $H_q^s(\mathbb{R}^n)$, $1 < q < \infty$, $s \in \mathbb{R}$, extend the Sobolev spaces $H^s(\mathbb{R}^n)$ to $q \neq 2$:

$$H_q^s(\mathbb{R}^n) = \{u \in \mathcal{S}'(\mathbb{R}^n) \mid \mathcal{F}^{-1}(\langle \xi \rangle^s \hat{u}) \in L_q(\mathbb{R}^n)\}.$$

The *Hölder-Zygmund spaces* $C_*^s(\mathbb{R}^n)$, $s \in \mathbb{R}$, generalize the Hölder spaces $C^s(\mathbb{R}^n)$ with $s \in \mathbb{R}_+ \setminus \mathbb{N}$, to all s . (For $0 < \sigma < 1$, $u \in C^{k+\sigma}(\mathbb{R}^n)$ with $k \in \mathbb{N}_0$ and $0 < \sigma < 1$, when u and its derivatives up to order k satisfy $|u(x)| + \frac{|u(x)-u(y)|}{|x-y|^\sigma} \leq C$ on $\mathbb{R}^n$.) The cases $s \in \mathbb{N}$ are interpolation spaces between noninteger cases.

For an open subset $\Omega \subset \mathbb{R}^n$, we define the scales of *restricted* spaces:

$$\overline{H}_q^s(\Omega) = r^+ H_q^s(\mathbb{R}^n), \quad \overline{C}_*^s(\Omega) = r^+ C_*^s(\mathbb{R}^n),$$

and the scales of *supported* spaces:

$$\dot{H}_q^s(\overline{\Omega}) = \{u \in H_q^s(\mathbb{R}^n) \mid \text{supp } u \subset \overline{\Omega}\}, \quad \dot{C}_*^s(\overline{\Omega}) = \{u \in C_*^s(\mathbb{R}^n) \mid \text{supp } u \subset \overline{\Omega}\}.$$

For $\Omega = \mathbb{R}_+^n$, we define the **a -transmission spaces** $H_q^{a(t)}(\overline{\mathbb{R}}_+^n)$ and $C_*^{a(t)}(\overline{\mathbb{R}}_+^n)$ as follows ($\frac{1}{q'} = 1 - \frac{1}{q}$):

$$\begin{aligned} H_q^{a(t)}(\overline{\mathbb{R}}_+^n) &= \Xi_+^{-a} e^+ \overline{H}_q^{t-a}(\mathbb{R}_+^n), \quad \text{for } t-a > -\frac{1}{q'}, \\ C_*^{a(t)}(\overline{\mathbb{R}}_+^n) &= \Xi_+^{-a} e^+ \overline{C}_*^{t-a}(\mathbb{R}_+^n), \quad \text{for } t-a > -1. \end{aligned}$$

The a -transmission spaces are defined over Ω by localization. When Ω is a bounded $C^{1+\tau}$ -domain ($\tau > 0$), $u \in H_q^{a(t)}(\overline{\Omega})$ is defined for $t < 1 + \tau$ to mean that 1.–2. hold:

1. u is in H_q^t on compact subsets of Ω .
2. Every $x_0 \in \partial\Omega$ has an open neighborhood U and a $C^{1+\tau}$ -diffeomorphism in $\mathbb{R}^n$ mapping U' to U such that $U' \cap \overline{\mathbb{R}_+^n}$ is mapped to $U \cap \overline{\Omega}$, and u is pulled back to a function u' in $H_q^{a(t)}(\overline{\mathbb{R}_+^n})$ locally (i.e., $\varphi u' \in H^{a(t)}(\overline{\mathbb{R}_+^n})$ when $\varphi \in C_0^\infty(U')$).

There is a similar definition of $C_*^{a(t)}(\overline{\Omega})$.

A structural analysis as in Theorem 1.2 is valid also for these a -transmission spaces. Generally, $\dot{H}_q^t(\overline{\Omega}) \subset H_q^{a(t)}(\overline{\Omega}) \subset \dot{H}_q^a(\overline{\Omega})$ for $t \geq a$, and there holds:

$$(1.18) \quad H_q^{a(t)}(\overline{\Omega}) \begin{cases} = \dot{H}_q^t(\overline{\Omega}) & \text{when } -\frac{1}{q'} < t - a < \frac{1}{q}, \\ \subset \dot{H}_q^{t(-\varepsilon)}(\overline{\Omega}) + d^a e^+ \overline{H}_q^{t-a}(\Omega), & \text{when } t - a > \frac{1}{q}, \end{cases}$$

where $d(x) = \text{dist}(x, \partial\Omega)$ near $\partial\Omega$ (extended positively to Ω), and $(-\varepsilon)$ is active if $t - a - \frac{1}{q}$ is integer. The d^a -contribution can be described more exactly for specific values of t . (Cf. [31], [36], [4].) Moreover, $\dot{H}_q^t(\overline{\Omega}) \subset d^t L_q(\Omega)$ for $t \geq 0$, Ω smooth (as kindly told us by Triebel, cf. e.g. [54], Prop. 5.7). We also note that $H_q^{a(2a)}(\overline{\Omega}) \subset \dot{H}_q^{a+\frac{1}{q}-\varepsilon}(\overline{\Omega})$ (any $\varepsilon > 0$).

There are similar statements for the C_*^t -scale; moreover, $C_*^{a(s+2a)}(\overline{\Omega}) \subset d^a C^{s+a}(\overline{\Omega})$ when $s + 2a, s + a \notin \mathbb{N}$, $s + a > 0$, cf. [40].

Note that in all the mentioned spaces with $t \geq a$, resp. $s > -a$, the functions have a factor d^a at the boundary.

Let $P = (-\Delta)^a$ (or a suitable pseudodifferential generalization explained further below), and let $\Omega \subset \mathbb{R}^n$ be a bounded open set with some regularity. Recall that the homogeneous Dirichlet problem (1.3) is:

$$r^+ P u = f \text{ in } \Omega, \quad \text{supp } u \subset \overline{\Omega}.$$

We know from the variational theory that the Dirichlet realization P_D in $L_2(\Omega)$ is bijective from $D(P_D)$ to $L_2(\Omega)$, and ask now what can be said about u when f has some regularity. Modern results:

Ros-Oton and Serra showed in [49] by potential-theoretic methods, when Ω is $C^{1,1}$:

$$(1.19) \quad f \in L_\infty(\Omega) \implies u \in d^a \overline{C}^t(\Omega), \text{ for small } t > 0.$$

The result was extended later to t up to a . It was lifted to higher-order Hölder spaces by Abatangelo and Ros-Oton in [3].

The present author showed in [30],[31] by pseudodifferential methods, when Ω is C^∞ , $1 < q < \infty$:

$$(1.20) \quad f \in \overline{H}_q^s(\Omega) \iff u \in H_q^{a(s+2a)}(\overline{\Omega}), \text{ when } s > -a - 1/q',$$

$$(1.21) \quad f \in \overline{C}_*^s(\Omega) \iff u \in C_*^{a(s+2a)}(\overline{\Omega}), \text{ when } s > -a - 1,$$

$$(1.22) \quad f \in C^\infty(\overline{\Omega}) \iff u \in d^a C^\infty(\overline{\Omega}).$$

This theory initiated in an unpublished (and on some points sketchy) lecture note of Hörmander [42] (with $q = 2$); (1.22) was obtained there.

(1.20) is extended to $C^{1+\tau}$ -domains ($\tau > 2a$) in a joint work with Abels [4], then valid for $0 \leq s < \tau - 2a$. The part $\implies$ in (1.21) is also obtained there with $s + 2a$ replaced by $s + 2a - \varepsilon$.

Note the *sharpness* in (1.20)–(1.22); they exhibit *the exact solution space* for (1.3). As pointed out above, the functions there all have a factor d^a near the boundary.

Remark. An advantage of viewing P as an elliptic pseudodifferential operator is that we get *interior regularity* for free: When f is locally in H_q^s (or C_*^s) in Ω , then any solution of (1.3) is locally in H_q^{s+2a} (resp. C_*^{s+2a}) in Ω . This has been known since the advent of ψ do methods in the 1960's.

Now let us list the hypotheses on general ψ do's $P = \text{Op}(p(x, \xi))$, under which our results hold.

Assumption 1.5. $P = \text{Op}(p(x, \xi))$ satisfies:

1° p is **classical** of order $2a > 0$, i.e., $p \sim \sum_{j \in \mathbb{N}_0} p_j$ with $p_j(x, t\xi) = t^{2a-j} p_j(x, \xi)$ for $|\xi| \geq 1$. The sign $\sim$ means that for all J , $\partial_x^\beta \partial_\xi^\alpha [p - \sum_{j < J} p_j]$ is $O(\langle \xi \rangle^{2a-J-|\alpha|})$, for all multi-indices α, β .

2° p is **strongly elliptic**: $\text{Re } p_0(x, \xi) \geq c|\xi|^{2a}$ for $|\xi| \geq 1$, with $c > 0$.

3° p is **even**: $p_j(x, -\xi) = (-1)^j p_j(x, \xi)$, all j , $|\xi| \geq 1$.

Assumption 1.5 is satisfied e.g. by L^a when L is a $2'$ order strongly elliptic differential operator, and the a 'th power is constructed as in Seeley [53], but also cases not stemming from differential operators are included.

For a given smooth Ω , we need for the results (1.20)–(1.22) only in 3° to assume that it holds for p and derivatives $\partial_x^\beta \partial_\xi^\alpha p$ at the points $x \in \partial\Omega$, with ξ just taken equal to the interior normal $\nu(x)$; this is the so-called **a -transmission condition** introduced by Hörmander [42], [44], also explained in [31].

In [4], the hypotheses were generalized to allow symbols that are only C^τ with respect to x , coupled with domains Ω that are only $C^{1+\tau}$; in this case (1.20) (and part of (1.21)) was obtained for $0 \leq s < \tau - 2a$.

Here are some words on the proof of (1.20), in the case where Ω is C^∞ . Roughly speaking, we perform two steps:

Step 1. Reduce, by cut-downs and change-of-variables, to situations where Ω is replaced by $\mathbb{R}_+^n$. Then P is also modified.

Step 2. For the resulting P , let $Q = \Xi_-^{-a} P \Xi_+^{-a}$, so that

$$(1.23) \quad P = \Xi_-^a Q \Xi_+^a.$$

Here Q is of order 0, and has some bijectivity properties (as a special case of an operator in the calculus of Boutet de Monvel [13]). Namely. $r^+ Q e^+$ is essentially bijective from $\overline{H}_q^t(\mathbb{R}_+^n)$ to itself for all $t \geq 0$. Then we find a solution operator $R = \Xi_+^{-a} e^+ (r^+ Q e^+)^{-1} (r^+ \Xi_-^{-a} e^+)$,

$$\overline{H}_q^s(\mathbb{R}_+^n) \xrightarrow{r^+ \Xi_-^{-a} e^+} \overline{H}_q^{s+a}(\mathbb{R}_+^n) \xrightarrow{(r^+ Q e^+)^{-1}} \overline{H}_q^{s+a}(\mathbb{R}_+^n) \xrightarrow{\Xi_+^{-a} e^+} \Xi_+^{-a} e^+ \overline{H}_q^{s+a}(\mathbb{R}_+^n),$$

where the last space is the a -transmission space $H_q^{a(s+2a)}(\overline{\mathbb{R}_+^n})$.

The above explanation has been simplified in particular on two points: 1) The $\Xi_\pm^t$ should actually be replaced by a refined family $\Lambda_\pm^t$ with better pseudodifferential properties. 2) In some of the calculations, there is an error term of order $-\infty$ that has to be dealt with (a common feature of pseudodifferential calculations).

Our proof for Hölder-Zygmund spaces follows the same lines, using that the pseudodifferential theory extends to such spaces. It also works for a wealth of other Besov- and Triebel-Lizorkin spaces, cf. [30].

In the case of domains with finite smoothness, there was a need to expand the (complicated) tools that exist for ψ do's with nonsmooth x -dependence, cf. [4].

For Lipschitz domains (where the boundary is only $C^{0,1}$), there are results about regularity and numerical methods e.g. by Acosta, Borthagaray and Nochetto [8], [12], in basic spaces of Sobolev and Besov types. There also exist studies where f is given in spaces with powers of d as weights.

For orientation, we end this section with a list of significant contributions to the field through the times. Further contributions are mentioned in the text when needed; the publication details are collected in the reference list at the end. The listed works give references to many other remarkable contributions.

A small selection of the many contributors to the field: Blumenthal and Getoor [9], Vishik and Eskin '60s (presented in [23]), Hoh and Jacob [41], Kulczycki [46], Chen and Song [16], Jakubowski [45], Bogdan, Burdzy and Chen [10], Cont and Tankov [18], Caffarelli and Silvestre [14], Gonzales, Mazzeo and Sire [28], Ros-Oton and Serra [49], Grubb [31], Abatangelo [1], Felsinger, Kassmann and Voigt [24], Bonforte, Sire and Vazquez [11], Dipierro, Ros-Oton and Valdinoci [20], Dyda, Kuznetsov and Kvas-

nicki [22], Abatangelo, Jarohs and Saldana [2], Chan, Gomez-Castro and Vazquez [15], Borthagaray and Nochetto [12].

§ 2. Further developments

§ 2.1. Evolution problems and resolvents

First we give a quick review of consequences of the analysis of P_D for evolution problems (heat equations) with homogeneous Dirichlet condition. The basic problem is;

$$(2.1) \quad \begin{aligned} \partial_t u + r^+ P u &= f \text{ on } \Omega \times I, \\ u &= 0 \text{ on } (\mathbb{R}^n \setminus \Omega) \times I, \\ u|_{t=0} &= 0; \end{aligned}$$

where u and f depend on (x, t) . Here $I =]0, T[$ and Ω is bounded, open and $C^{1+\tau}$ for suitable $\tau > 0$; for simplicity we take zero initial data.

By Laplace transformation, the evolution problem is closely connected with the stationary problem for $P - \lambda I$, where $\lambda \in \mathbb{C}$.

There is an easy result in the L_2 -framework: Here P_D is positive selfadjoint when $P = (-\Delta)^a$, and for more general P satisfying Assumption 1.5, P_D is lower semibounded with its discrete spectrum and numerical range contained in a sectorial region

$$M = \{\lambda \in \mathbb{C} \mid \operatorname{Re} \lambda + \beta \geq c_1 > 0, |\operatorname{Im} \lambda| \leq c_2(\operatorname{Re} \lambda + \beta)\}.$$

In particular, $\mathbb{C} \setminus M$ is in the resolvent set, and there is a resolvent estimate

$$(2.2) \quad \|(P_D - \lambda)^{-1}\|_{\mathcal{L}(L_2(\Omega))} \leq c_3 \langle \lambda \rangle^{-1} \text{ for } \operatorname{Re} \lambda \leq -\beta.$$

Then standard old techniques show existence and uniqueness of a solution of (2.1) for $f \in L_2(\Omega \times I)$, and

$$(2.3) \quad f \in L_2(\Omega \times I) \iff u \in L_2(I; D(P_D)) \cap \overline{H}^1(I; L_2(\Omega)) \text{ with } u(x, 0) = 0.$$

Thanks to the analysis of P_D , we can in the right-hand side replace $D(P_D)$ by $H^{a(2a)}(\overline{\Omega})$, giving a precise result. It is interesting that it only depends on a , not on the value of the symbol p . (More details in [34],[35] for $\tau = \infty$, [40] for finite τ .)

Now one can ask what happens if f is in other spaces?

In the L_2 -setting there is a functional analytic result from Lions and Magenes' book [48] that can be applied to lift (2.3) a small step in x and a large step in t [34],[35],[40]:

- For $k \in \mathbb{N}$, $r = \min\{2a, a + \frac{1}{2} - \varepsilon\}$,

$$(2.4) \quad \begin{aligned} f \in L_2(I; \overline{H}^r(\Omega)) \cap \dot{H}^k(\overline{I}; L_2(\Omega)) &\implies \\ u \in L_2(I; H^{a(2a+r)}(\overline{\Omega})) \cap \overline{H}^{k+1}(I; L_2(\Omega)). \end{aligned}$$

In L_q -spaces other techniques are needed. Here we have shown in [34],[35],[40]:

- When P satisfies Assumption 1.5 and is x -independent and symmetric, then for $1 < q < \infty$,

$$(2.5) \quad f \in L_q(\Omega \times I) \iff u \in L_q(I; H_q^{a(2a)}(\overline{\Omega})) \cap \overline{H}_q^1(I; L_q(\Omega)) \text{ with } u(x, 0) = 0.$$

This is based on the fact that the L_q -Dirichlet realization $P_{D,q}$ (whose domain satisfies $D(P_{D,q}) = H_q^{a(2a)}(\overline{\Omega})$) is defined from a Dirichlet form in the sense of Fukushima, Oshima and Takeda [27] (also called sub-Markovian), allowing application of a result of Lamberton [47]. This also implies an estimate like (2.2) with L_2 replaced by L_q . The time-regularity can moreover be lifted by use of general techniques of Amann [6], and there are results for other regularity classes with respect to x .

This type of solvability result is often called *maximal L_q -regularity*, cf. e.g. Denk and Seiler [19]. We expect that perturbation methods would allow x -dependent symbols to some extent; there is work in progress investigating this².

• In anisotropic Hölder spaces $\overline{C}^{s,r}(\Omega \times I) = L_\infty(I; \overline{C}^s(\Omega)) \cap L_\infty(\Omega; \overline{C}^r(I))$, Ros-Oton with coauthors Fernandez-Real and Vivas [25], [51] have shown for x -independent symmetric operators, that the regularity can be lifted as follows:

$$f \in \overline{C}^{\gamma, \gamma/2a}(\Omega \times I) \implies \partial_t u \in \overline{C}^{\gamma, \gamma/2a}(\Omega \times I'), \quad u/d^a \in \overline{C}^{a+\gamma, (a+\gamma)/2a}(\Omega \times I'),$$

when $\overline{I}' \subset I$; here Ω is assumed $C^{2+\gamma}$, and $0 < \gamma < a$ with $a + \gamma \notin \mathbb{N}$.

There have also been studies of evolution problems in numerical analysis, e.g. by Acosta, Bersetche and Borthagaray [7] in L_2 -Sobolev spaces over Lipschitz domains. There is a very recent posting on results in L_q -Sobolev spaces weighted by powers of the distance $d(x)$ and other functions, by Choi, Kim and Ryu [17].

As another aspect, we mention that there is an analysis (in C^∞ -domains) [36] showing that the regularity of u *cannot be lifted all the way to $C^\infty(\overline{\Omega} \times \overline{I})$ or $d^a C^\infty(\overline{\Omega} \times \overline{I})$* when $f \in C^\infty(\overline{\Omega} \times \overline{I})$. This is in contrast with heat problems for the local operator Δ .

§ 2.2. Motivation for local nonhomogeneous boundary conditions (detailed)

Now we turn to nonhomogeneous Dirichlet conditions [31], which will be explained in detail.

²Added during typesetting June 2025: Results have later been obtained for a large class of x -dependent operators, cf. Abels and Grubb [5].

As a nonhomogeneous Dirichlet problem, much of the literature considers the problem

$$(2.6) \quad r^+ Pu = f \text{ in } \Omega, \quad u = g \text{ on } \mathbb{R}^n \setminus \Omega,$$

where the difference from (1.3) is that u may take a nonzero value g outside of Ω . There is an easy reduction of this problem to the homogeneous case, namely: Let G be a function extending g to $\mathbb{R}^n$, then the problem (2.6) can be turned into the homogeneous problem

$$(2.7) \quad r^+ Pu' = f' \text{ in } \Omega, \quad u' = 0 \text{ on } \mathbb{R}^n \setminus \Omega,$$

where $u' = u - G$, $f' = f - r^+ PG$. The discussion of regularity of solutions then involves how the extension from g to G is performed and how it influences $r^+ PG$.

We shall here discuss another Dirichlet condition that involves a boundary value on $\partial\Omega$ and is *local*. For the motivation, consider C^∞ -results. Define for any $\mu > -1$:

$$\mathcal{E}_\mu(\overline{\Omega}) = e^+ d^\mu C^\infty(\overline{\Omega}).$$

(As usual, e^+ means extension by zero.) Here $\mathcal{E}_0(\overline{\Omega}) \simeq C^\infty(\overline{\Omega})$.

With this notation, the regularity result (1.22) for $(-\Delta)^a$ and for the generalizations P satisfying Assumption 1.5 states that

$$(2.8) \quad f \in C^\infty(\overline{\Omega}) \iff u \in \mathcal{E}_a(\overline{\Omega}).$$

Moreover, one can show the forward mapping property for all integers $k \geq -1$ [31]

$$r^+ P: \mathcal{E}_{a+k}(\overline{\Omega}) \rightarrow C^\infty(\overline{\Omega}).$$

There are Taylor expansions at the boundary, in local coordinates where Ω is replaced by $\mathbb{R}_+^n = \{x = (x', x_n) \mid x_n > 0\}$ so that $d(x) = x_n$:

$$\text{In } \mathcal{E}_0 : u(x) \sim v_0(x') + v_1(x')x_n + v_2(x')x_n^2 + \dots, \quad \text{when } x_n > 0.$$

$$\text{In } \mathcal{E}_1 : u(x) \sim v_0(x')x_n + v_1(x')x_n^2 + v_2(x')x_n^3 + \dots$$

$$\text{In } \mathcal{E}_a : u(x) \sim v_0(x')x_n^a + v_1(x')x_n^{a+1} + v_2(x')x_n^{a+2} + \dots$$

$$\text{In } \mathcal{E}_{a-1} : u(x) \sim v_0(x')x_n^{a-1} + v_1(x')x_n^a + v_2(x')x_n^{a+1} + \dots$$

Recall the notation $u|_{\partial\Omega} = \gamma_0 u$. Note that the expansions of functions in $\mathcal{E}_{a-1}$ only differ from those in $\mathcal{E}_a$ by having a term $v_0(x')x_n^{a-1}$; i.e., $\gamma_0(u/x_n^{a-1})$ can be nontrivial. This leads to the important observation:

$$(2.9) \quad \mathcal{E}_a \text{ is the subset of } \mathcal{E}_{a-1} \text{ where } \gamma_0(u/d^{a-1}) = 0.$$

(It also holds when $a \geq 1$.)

Let $f \in C^\infty(\overline{\Omega})$, $\varphi \in C^\infty(\partial\Omega)$, for a bounded C^∞ -domain Ω , and let us compare boundary value problems for Δ and $(-\Delta)^a$:

Old fact: The **nonhomogeneous Dirichlet problem for Δ :**

$$(2.10) \quad \begin{aligned} \Delta u &= f \text{ on } \Omega, \\ \gamma_0 u &= \varphi \text{ on } \partial\Omega, \end{aligned}$$

is uniquely solvable in $C^\infty(\overline{\Omega}) \simeq \mathcal{E}_0(\overline{\Omega})$.

As a special case, the **homogeneous Dirichlet problem for Δ :**

$$(2.11) \quad \begin{aligned} \Delta u &= f \text{ on } \Omega, \\ \gamma_0 u &= 0 \text{ on } \partial\Omega, \end{aligned}$$

is uniquely solvable in $\{u \in C^\infty(\overline{\Omega}) \mid \gamma_0 u = 0\} \simeq \mathcal{E}_1(\overline{\Omega})$, cf. also (2.9).

Modern result: The **homogeneous Dirichlet problem for $(-\Delta)^a$**

$$\begin{aligned} (-\Delta)^a u &= f \text{ on } \Omega, \\ \text{supp } u &\subset \overline{\Omega}, \end{aligned}$$

is uniquely solvable in $\mathcal{E}_a(\overline{\Omega})$ (as already stated in (1.22) and (2.8)). Here $\mathcal{E}_a(\overline{\Omega})$ has a role like the one $\mathcal{E}_1(\overline{\Omega})$ has for Δ .

Now it is natural to define a **nonhomogeneous Dirichlet problem for $(-\Delta)^a$** by going out to the larger space $\mathcal{E}_{a-1}(\overline{\Omega})$. The problem

$$(2.12) \quad \begin{aligned} (-\Delta)^a u &= f \text{ on } \Omega, \\ \gamma_0(u/d^{a-1}) &= \varphi \text{ on } \partial\Omega, \\ \text{supp } u &\subset \overline{\Omega}, \end{aligned}$$

is uniquely solvable in $\mathcal{E}_{a-1}(\overline{\Omega})$. (*Proof:* subtract a function $w \in \mathcal{E}_{a-1}$ with $\gamma_0(w/d^{a-1}) = \varphi$, then $v = u - w$ solves a homogeneous Dirichlet problem, cf. (2.8), (2.9).)

This is surprisingly simple! It can be generalized to solvability statements in Sobolev spaces after some more work; see later.

The interest of the nonhomogeneous Dirichlet problem (2.12) was also pointed out by Abatangelo [1], from a very different viewpoint: He started with a Green's function $G_\Omega(x, y)$ for the homogeneous Dirichlet problem for $(-\Delta)^a$, and developed integral representation formulas imitating the formulas known for Δ , arriving at a strange boundary operator $u \mapsto Eu$, that he showed was proportional to $\gamma_0(u/d^{a-1})$ in the case where Ω is a ball.

E is defined by an integral formula; a proof that $Eu = c_0\gamma_0(u/d^{a-1})$ for more general Ω is given in [40] (the constant c_0 equals $\Gamma(a)\Gamma(a+1)$). This boundary operator also enters in other studies, e.g. by Chan, Gomez-Castro and Vazquez [15], and by Fernandez-Real and Ros-Oton [26].

The solutions in $\mathcal{E}_{a-1}$ are generally unbounded on Ω , since u behaves like the unbounded function d^{a-1} near $\partial\Omega$ (when $\varphi \neq 0$). They are therefore often called “blow-up solutions”. They are in $L_q(\Omega)$ for $q < (1-a)^{-1}$.

There is also a local *Neumann condition* $\gamma_1(u/d^{a-1}) = \psi$, which has a good solvability theory [30],[33]; here $\gamma_1 v = \gamma_0 \partial_\nu v$, the normal derivative.

For solvability results in general Sobolev spaces, the role of $\mathcal{E}_{a-1}$ will for $\mathbb{R}_+^n$ be taken over by the $(a-1)$ -transmission spaces defined by

$$H^{(a-1)(t)}(\overline{\mathbb{R}_+^n}) = \Xi_+^{-a+1} e^+ \overline{H}^{t-a+1}(\mathbb{R}_+^n),$$

and L_q -variants with $q \neq 2$. The model problem (as in Example 1.2 above) is now:

$$(2.13) \quad \begin{aligned} r^+(1-\Delta)^a u &= f \text{ in } \mathbb{R}_+^n, \\ \gamma_0(u/x_n^{a-1}) &= \varphi \text{ on } \mathbb{R}^{n-1}, \\ \text{supp } u &\subset \overline{\mathbb{R}_+^n}. \end{aligned}$$

First we observe that there is a result on boundary values like in Theorem 1.4 but with a replaced by $a-1$:

Theorem 2.1. *The mapping $\gamma_0^{a-1}: u \mapsto \gamma_0(u/x_n^{a-1})$ from $x_n^{a-1}\mathcal{S}(\overline{\mathbb{R}_+^n})$ to $\mathcal{S}(\mathbb{R}^{n-1})$ extends to a continuous surjective mapping (when $t > a - \frac{1}{2}$),*

$$\gamma_0^{a-1}: H^{(a-1)(t)}(\overline{\mathbb{R}_+^n}) \rightarrow H^{t-a+\frac{1}{2}}(\mathbb{R}^{n-1}).$$

Here $H^{a(t)}(\overline{\mathbb{R}_+^n})$ is a closed subspace of $H^{(a-1)(t)}(\overline{\mathbb{R}_+^n})$, equal to the set where $\gamma_0^{a-1}u = 0$.

The last line comes from (2.9).

Then we solve (2.13) by subtracting from u a term w with $\gamma_0^{a-1}w = \varphi$, reducing to the homogeneous Dirichlet problem. As a result (note that $s+2a$ plays the role of t):

Theorem 2.2. *The nonhomogeneous Dirichlet problem (2.13) with given $f \in \overline{H}^s(\mathbb{R}_+^n)$, $\varphi \in H^{s+a+\frac{1}{2}}(\mathbb{R}^{n-1})$, $s \geq 0$, is uniquely solvable with a solution $u \in H^{(a-1)(s+2a)}(\overline{\mathbb{R}_+^n})$.*

§ 2.3. Nonhomogeneous Dirichlet conditions over curved domains

For curved domains Ω , the $(a-1)$ -transmission spaces are defined by use of local coordinates as in §1.4. For the H_q^s -scales with $q \neq 2$, the correct spaces over the boundary are Besov spaces B_q^t (also denoted $B_{q,q}^t$). Here the trace map $\gamma_0^{a-1}u = \gamma_0(u/d^{a-1})$ satisfies that

$$\gamma_0^{a-1}: H_q^{(a-1)(t)}(\overline{\Omega}) \rightarrow B_q^{t-a+\frac{1}{q'}}(\partial\Omega)$$

is continuous and surjective for $t > a - \frac{1}{q'}$, with kernel $H_q^{a(t)}(\overline{\Omega})$. One finds:

Theorem 2.3. *There is unique solvability of the nonhomogeneous Dirichlet problem*

$$(2.14) \quad \begin{aligned} Pu &= f \text{ in } \Omega, \\ \gamma_0^{a-1}u &= \varphi \text{ on } \partial\Omega, \\ \text{supp } u &\subset \overline{\Omega}, \end{aligned}$$

for given $f \in \overline{H}_q^s(\Omega)$, $\varphi \in B_q^{s+a+1/q'}(\partial\Omega)$, $s \geq 0$, with solution $u \in H_q^{(a-1)(s+2a)}(\overline{\Omega})$.

This is shown in [31] for bounded smooth Ω , under Assumption 1.5. (More precisely, if $P \neq (-\Delta)^a$, 0 can be an eigenvalue of the homogeneous Dirichlet problem, and in that case, there is only a Fredholm solvability.) In [40] the result is generalized to $C^{1+\tau}$ -domains Ω and ψ 's P with C^τ x -dependence, when $0 \leq s < \tau - 2a - 1$.

These stationary results can be followed up with results for evolution problems (for $\partial_t + P$) and resolvent problems (for $P - \lambda$, $\lambda \in \mathbb{C}$):

For the study of (2.14) with P replaced by $P - \lambda$, we need u to be at least in $L_q(\Omega)$. The domain space $H_q^{(a-1)(s+2a)}(\overline{\Omega})$ ($s \geq 0$) is not always there. In fact, already for $s = 0$ (recall $1 < q < \infty$),

$$(2.15) \quad H_q^{(a-1)(2a)}(\overline{\Omega}) \subset L_q(\Omega) \text{ if and only if } q < (1-a)^{-1}.$$

(For $q = 2$, this holds when $a > \frac{1}{2}$.)

The evolution problem is:

$$(2.16) \quad \begin{aligned} Pu + \partial_t u &= f \text{ on } \Omega \times I, \\ u &= 0 \text{ on } (\mathbb{R}^n \setminus \Omega) \times I, \\ \gamma_0^{a-1}u &= \psi \text{ on } \partial\Omega \times I, \\ u|_{t=0} &= 0. \end{aligned}$$

Here we can show [40]:

Theorem 2.4. *Let $q < (1-a)^{-1}$. If $q \neq 2$, let P be x -independent symmetric. For $f(x, t)$ given in $L_q(\Omega \times I)$, and $\psi(x, t)$ given in $L_q(I; B_q^{a+1/q'}(\partial\Omega)) \cap \overline{H}_q^1(I; B_q^\varepsilon(\partial\Omega))$ with $\psi(x, 0) = 0$ (some $\varepsilon > 0$), there is a unique solution $u(x, t)$ of (2.16) satisfying*

$$u \in L_q(I; H_q^{(a-1)(2a)}(\overline{\Omega})) \cap \overline{H}_q^1(I; L_q(\Omega)).$$

It is shown by reduction to a problem with $\psi = 0$, where (2.3)–(2.5) can be applied. Solvability of resolvent problems is obtained in the following theorem [40], Th. 5.4:

Theorem 2.5. *Let $q < (1 - a)^{-1}$. If $q \neq 2$, let P be x -independent symmetric. Denote by Σ the spectrum of P_D (it is discrete). Consider for $\lambda \in \mathbb{C}$ the problem*

$$(2.17) \quad \begin{aligned} Pu - \lambda u &= f \text{ in } \Omega, \\ u &= 0 \text{ in } \mathbb{R}^n \setminus \Omega, \\ \gamma_0^{a-1} u &= \varphi \text{ on } \partial\Omega, \end{aligned}$$

with f given in $L_q(\Omega)$, φ given in $B_q^{a+1/q'}(\partial\Omega)$, and the solution being sought in $H_q^{(a-1)(2a)}(\overline{\Omega})$.

If $\lambda \notin \Sigma$, it is uniquely solvable.

If $\lambda \in \Sigma$, it is Fredholm solvable, with the same dimension of the kernel and cokernel of the mapping $u \rightarrow \{f, \varphi\}$.

There are related resolvent studies by Chan, Gomez-Castro and Vazques [15] in weighted L_1 -spaces, generally larger than the spaces we consider in [40]. For $f = 0$, [15] regards (2.17) as an "eigenvalue problem", and presents it as a mysterious fact that the solutions ("eigenfunctions") generally blow up at the boundary. We find this natural, since the functions in the precise domain $H_q^{(a-1)(2a)}(\overline{\Omega})$ have a factor d^{a-1} at the boundary as soon as $\gamma_0(u/d^{a-1})$ is nontrivial.

§ 2.4. Integration by parts, Green's formula

Another topic that we shall touch upon very briefly is the question of integration by parts formulas for the fractional Laplacian and its generalizations. Ros-Oton and Serra [50] started the analysis by showing a Pohozaev formula for solutions of the homogeneous Dirichlet problem, important for uniqueness questions in nonlinear variants. Their basic result is, in an equivalent version:

Theorem 2.6. *Let Ω be bounded and $C^{1,1}$. Let u and v be solutions of the homogeneous Dirichlet problem (1.3) for $(-\Delta)^a$ with real right-hand side in $L_\infty(\Omega)$, so they are in $d^a C^t(\overline{\Omega})$ (small t) by (1.19). Then for each j ,*

$$(2.18) \quad \int_{\Omega} ((-\Delta)^a u \partial_j v + \partial_j u (-\Delta)^a v) dx = \Gamma(a+1)^2 \int_{\partial\Omega} \nu_j \gamma_0\left(\frac{u}{d^a}\right) \gamma_0\left(\frac{v}{d^a}\right) d\sigma,$$

where $\nu = (\nu_1, \dots, \nu_n)$ is the interior normal.

Their proof is based on a fine analysis of the factorization $(-\Delta)^a = (-\Delta)^{a/2}(-\Delta)^{a/2}$ applied to real functions. In [32], we worked out a proof of (2.18) based on Fourier analysis and factorizations developed from (1.23), applicable to operators satisfying Assumption 1.5 and smooth domains.

Moreover, we have shown integration formulas also for solutions of nonhomogeneous boundary problems. Let us just quote the Green's formula for $(-\Delta)^a$ [33], [37]:

Theorem 2.7. *Let Ω be bounded smooth. For $u, v \in H^{(a-1)(s)}(\overline{\Omega})$ there holds when $s > a + \frac{1}{2}$:*

$$(2.19) \quad \int_{\Omega} ((-\Delta)^a u \bar{v} - u (-\Delta)^a \bar{v}) dx = c_0 \int_{\partial\Omega} \left(\gamma_1\left(\frac{u}{d^{a-1}}\right) \gamma_0\left(\frac{\bar{v}}{d^{a-1}}\right) - \gamma_0\left(\frac{u}{d^{a-1}}\right) \gamma_1\left(\frac{\bar{v}}{d^{a-1}}\right) \right) d\sigma,$$

$$c_0 = \Gamma(a)\Gamma(a+1).$$

Note that both the Dirichlet trace $\gamma_0(\frac{u}{d^{a-1}})$ and the Neumann trace $\gamma_0(\partial_\nu(\frac{u}{d^{a-1}}))$ enter in (2.19). When $\gamma_0(\frac{u}{d^{a-1}}) = 0$, the Neumann trace equals the value $\gamma_0(\frac{u}{d^a})$ entering in (2.18). For general P satisfying Assumption 1.5, there is a similar formula with an extra term $\int_{\partial\Omega} B \gamma_0^{a-1} u \gamma_0^{a-1} \bar{v} dx$, where B is a ψ do on $\partial\Omega$ of order 1.

We end this survey by some remarks on what more can be done, or needs doing, in the present context. Here are a few suggestions³:

1. More on evolution problems in L_p -Sobolev spaces, also for x -dependent operators P .
2. Development from [30] of consequences in L_1 -spaces and in general $F_{p,q}^s$ - and $B_{p,q}^s$ -spaces.
3. Extension of more results known for smooth domains (e.g. integration formulas), to nonsmooth domains.
4. Applications to problems with nonlinearity.
5. Treatment of operators without the reflection symmetries of $(-\Delta)^a$.

Ad 5.: Ros-Oton and colleagues have initiated studies of boundary value problems for operators that do not have the *evenness* property of $(-\Delta)^a$ and the operators P we have listed. For example $(-\Delta)^{\frac{1}{2}} + b \cdot \nabla$, $b \in \mathbb{R}^n$, with an even part $(-\Delta)^{\frac{1}{2}}$ and an odd part $b \cdot \nabla$. They get results by real integral operator methods (from potential theory and function theory); for a comprehensive treatment see Dipierro, Ros-Oton, Serra and Valdinoci [21].

By Fourier methods we can treat completely general strongly elliptic operators $L = \text{Op}(\ell(\xi))$, where $\ell(\xi)$ is homogeneous of order $2a$ and just satisfies $\text{Re } \ell(\xi) \geq c|\xi|^{2a}$ with $c > 0$, showing how a μ -transmission space comes in (with a possibly complex μ), and obtaining an integration by parts formula; but so far only in the model case of $\mathbb{R}_+^n$ [39]. It might be worth trying to apply the localization techniques of [21] to extend the results for L to curved domains.

³Added during typesetting June 2025: There has been substantial progress in 1. and 4., see [5].

References

- [1] N. Abatangelo, Large s -harmonic functions and boundary blow-up solutions for the fractional Laplacian *Discrete Contin. Dyn. Syst.* **35** (2015), 5555–5607.
- [2] N. Abatangelo, S. Jarohs, and A. Saldaña, Integral representation of solutions to higher-order fractional Dirichlet problems on balls, *Commun. Contemp. Math.* **36** (2018), 20(8):1850002.
- [3] N. Abatangelo and X. Ros-Oton, Obstacle problems for integro-differential operators: higher regularity of free boundaries, *Adv. Math.*, **360** (2020), 106931, 61pp.
- [4] H. Abels and G. Grubb, Fractional-order operators on nonsmooth domains, *J. Lond. Math. Soc. (2)* **107** (2023), 1297–1350.
- [5] H. Abels and G. Grubb, Maximal L_p -regularity for x -dependent fractional heat equations with Dirichlet conditions, *Math. Ann.*, (2024), DOI <https://doi.org/10.1007/s00208-024-02999-2>.
- [6] H. Amann, Operator-valued Fourier multipliers, vector-valued Besov spaces, and applications, *Math. Nachr.*, **186** (1997), 5–56.
- [7] G. Acosta, F. M. Bersetche and J. P. Borthagaray, Finite element approximations for fractional evolution problems, *Fract. Cal. Appl. Analysis* **22 (3)** (2019), 767–794.
- [8] G. Acosta and J. P. Borthagaray, A fractional Laplace equation: Regularity of solutions and finite element approximations *SIAM J. Num. Anal.* **55(2)** (2017).
- [9] R. M. Blumenthal and R. K. Gettoor, The asymptotic distribution of the eigenvalues for a class of Markov operators *Pacific J. Math.* **9** (1959), 399–408.
- [10] K. Bogdan, K. Burdzy, and Z.-Q. Chen, Censored stable processes *Probab. Theory Related Fields* **127(1)** (2003), 89–152.
- [11] M. Bonforte, Y. Sire, and J. L. Vázquez, Existence, uniqueness and asymptotic behaviour for fractional porous medium equations on bounded domains *Discrete Contin. Dyn. Syst.* **35(12)** (2015), 5725–5767.
- [12] J. P. Borthagaray and R. H. Nochetto, Besov regularity for the Dirichlet integral fractional Laplacian in Lipschitz domains, *J. Funct. Anal.* **284** (2023), 109829.
- [13] L. Boutet de Monvel, Boundary problems for pseudo-differential operators, *Acta Math.* **126** (1971), 11–51.
- [14] L. Caffarelli and L. Silvestre, An extension problem related to the fractional Laplacian, *Comm. Part. Diff. Eq.* **32** (2007), 1245–1260.
- [15] H. Chan, D. Gomez-Castro and J. L. Vazquez, Blow-up phenomena in nonlocal eigenvalue problems: when theories of L^1 and L^2 meet, *J. Funct. Anal.* **280 no. 7** (2021), 68.
- [16] Z.-Q. Chen and R. Song, Estimates on Green functions and Poisson kernels for symmetric stable processes, *Math. Ann.* **312(3)** (1998), 465–501.
- [17] J. Choi, K. Kim and J. Ryu, Sobolev regularity theory for the non-local elliptic and parabolic equations on $C^{1,1}$ open sets, *Discrete Contin. Dyn. Syst.*, **43** (2023), 3338–3377.
- [18] R. Cont and P. Tankov, *Financial modelling with jump processes*, Chapman & Hall/CRC Financial Mathematics Series. Chapman & Hall/CRC, Boca Raton, FL 2004.
- [19] R. Denk and J. Seiler, Maximal L_p -regularity of non-local boundary value problems *Monatsh. Math.* **176** (2015), 53–80.
- [20] S. Dipierro, X. Ros-Oton, and E. Valdinoci, Nonlocal problems with Neumann boundary conditions, *Rev. Mat. Iberoam.* **33(2)** (2017), 377–416.
- [21] S. Dipierro, X. Ros-Oton, J. Serra, and E. Valdinoci, Nonsymmetric stable operators, regularity theory and integration by parts, *Adv. Math.* **401** (2022), 108321.

- [22] B. Dyda, A. Kuznetsov and M. Kvasnicki, Eigenvalues of the fractional Laplace operator in the unit ball. *J. London Math. Soc.* **95** (2017), 500–518.
- [23] G. Eskin, *Boundary value problems for elliptic pseudodifferential equations*, Translation of Math. Monographs vol. 52, American Mathematical Society, Rhode Island 1981.
- [24] M. Felsinger, M. Kassmann, and P. Voigt, The Dirichlet problem for nonlocal operators, *Math. Z.* **279** (2015), 779–809.
- [25] X. Fernandez-Real and X. Ros-Oton, *Regularity theory for general stable operators: parabolic equations* *J. Funct. Anal.* **272** (2017), 4165–4221.
- [26] X. Fernandez-Real and X. Ros-Oton, Stable cones in the thin one-phase problem, *Am. J. Math.*, **146** (2024), 631–685.
- [27] M. Fukushima, Y. Oshima and M. Takeda, *Dirichlet forms and symmetric Markov processes*, De Gruyter Studies in Mathematics, **19**, Walter de Gruyter & Co., Berlin 1994.
- [28] M. d. M. González, R. Mazzeo, and Y. Sire, Singular solutions of fractional order conformal Laplacians, *J. Geom. Anal.* **22**(3) (2012), 845–863.
- [29] G. Grubb, *Distributions and operators*, Graduate Texts in Mathematics, **252**, Springer, New York, 2009.
- [30] G. Grubb, Local and nonlocal boundary conditions for μ -transmission and fractional elliptic pseudodifferential operators, *Analysis and P.D.E.* **7** (2014), 1649–1682.
- [31] G. Grubb, Fractional Laplacians on domains, a development of Hörmander’s theory of μ -transmission pseudodifferential operators, *Adv. Math.* **268** (2015), 478–528.
- [32] G. Grubb, Integration by parts and Pohozaev identities for space-dependent fractional-order operators, *J. Diff. Equ.* **261**(3) (2016), 1835–1879.
- [33] G. Grubb, Green’s formula and a Dirichlet-to-Neumann operator for fractional-order pseudodifferential operators, *Comm. Part. Diff. Equ.* **43** (2018), 750–789.
- [34] G. Grubb, Regularity in L_p Sobolev spaces of solutions to fractional heat equations, *J. Funct. Anal.* **274** (2018), 2634–2660.
- [35] G. Grubb, Fractional-order operators: boundary problems, heat equations, *Springer Proceedings in Mathematics and Statistics: "Mathematical Analysis and Applications — Plenary Lectures, ISAAC 2017, Vaxjo Sweden"*, eds. L. G. Rodino and J. Toft, Springer, Switzerland 2018, 51–81.
- [36] G. Grubb, Limited regularity of solutions to fractional heat and Schrödinger equations, *Discrete Contin. Dyn. Syst.* **39** (2019), 3609–3634.
- [37] G. Grubb, Exact Green’s formula for the fractional Laplacian and perturbations, *Math. Scand.* **126** (2020), 568–592.
- [38] G. Grubb, Integration by parts for nonsymmetric fractional-order operators on a halfspace, *J. Math. Anal. Appl.* **499** (2021), 125012.
- [39] G. Grubb, The principal transmission condition, *Math. in Engineering* (2022) **4**(4), 1–33.
- [40] G. Grubb, Resolvents for fractional-order operators with nonhomogeneous local boundary conditions, *J. Funct. Anal.* **284** (2023), 109815.
- [41] W. Hoh and N. Jacob, On the Dirichlet problem for pseudodifferential operators generating Feller semigroups, *J. Funct. Anal.* **137**(1) (1996), 19–48.
- [42] L. Hörmander, *Seminar notes on pseudo-differential operators and boundary problems*, Lectures at IAS Princeton 1965–66, available from Lund University, <https://lup.lub.lu.se/search/>
- [43] L. Hörmander, *The analysis of linear partial differential operators I*, Springer Verlag, Berlin 1983.
- [44] L. Hörmander, *The analysis of linear partial differential operators III*, Springer Verlag,

- Berlin 1985.
- [45] T. Jakubowski, The estimates for the Green function in Lipschitz domains for the symmetric stable processes, *Probab. Math. Statist.* **22** (2002), 419–441. Acta Univ. Wratislav. No. 2470.
 - [46] T. Kulczycki, Properties of Green function of symmetric stable processes *Probab. Math. Statist.* **17** (1997), 339–364. Acta Univ. Wratislav. No. 2029.
 - [47] D. Lamberton, Équations d'évolution linéaires associées à des semi-groupes de contractions dans les espaces L_p , *J. Funct. Anal.* **72** (1987), 252–262.
 - [48] J.-L. Lions and E. Magenes, *Problèmes aux limites non homogènes et applications. Vol. 2*, Editions Dunod, Paris, 1968.
 - [49] X. Ros-Oton and J. Serra, The Dirichlet problem for the fractional Laplacian: regularity up to the boundary, *J. Math. Pures Appl.* **101** (2014), 275–302.
 - [50] X. Ros-Oton and J. Serra, The Pohozaev identity for the fractional Laplacian, *Arch. Rat. Mech. Anal.* **213** (2014), 587–628.
 - [51] X. Ros-Oton and H. Vivas, Higher-order boundary regularity estimates for nonlocal parabolic equations, *Calc. Var. Partial Differential Equations* **57** (2018), Paper No. 111, 20 pp .
 - [52] L. Schwartz, *Méthodes mathématiques pour les sciences physiques*, Hermann, Paris 1961.
 - [53] R. T. Seeley, Complex powers of an elliptic operator, *Proc. Symp. Pure Math.* **10**, Amer. Math. Soc., R. I. 1967, 288–307.
 - [54] H. Triebel, *The structure of functions*, Springer, Basel AG, 2012.