

Five-dimensional AGT Relation, q - $\mathcal{W}$ Algebra and Deformed β -ensemble

By

Hidetoshi AWATA* and Yasuhiko YAMADA**

Abstract

We define a q -deformation of the β -ensemble which satisfies q - $\mathcal{W}_N$ constraint. We also show a relation with the Nekrasov partition function of 5D $SU(N)$ gauge theory with $N_f = 2N$.

§ 1. Introduction

In Ref. [1], Alday Gaiotto and Tachikawa discovered remarkable relations between the 4D $\mathcal{N} = 2$ super conformal gauge theories and the 2D Liouville conformal field theories. Some explanations have been addressed from β -ensemble (generalized matrix model) [2, 3] in Ref. [4]–[7].

In the pure $SU(2)$ case, the AGT relation [8] between the instanton part of the partition functions of the gauge theory and correlation functions of the Virasoro algebra is extended naturally to 5D in Ref. [9] (see also [10]). The instanton counting [11]–[14] of the 5D gauge theory [15] can be viewed as a q -analog of 4D cases, [16]–[18] and there also exists a natural q -deformation of the Virasoro/ $\mathcal{W}_N$ algebra. [19]–[22]

In this talk, we will study a 5D extension of the AGT relation with $N_f = 2N$ in terms of β -ensemble. The A_{N-1} type quiver matrix model (the ITEP model) [23] was generalized as a β -ensemble [2] satisfying the $\mathcal{W}_N$ constraint by Ref. [3]. The partition function of the A_{N-1} type β -ensemble is defined as the singular vector of the $\mathcal{W}_N$

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*Graduate School of Mathematics, Nagoya University, Nagoya, 464-8602, Japan.

e-mail: awata@math.nagoya-u.ac.jp

**Department of Mathematics, Faculty of Science, Kobe University, Kobe 657-8501, Japan.

e-mail: [yamaday@math.kobe-u.ac.jp](mailto:yamadaya@math.kobe-u.ac.jp)

algebra as follows[3]

$$(1.1) \quad Z_N^{c\ell} := \oint \prod_{a=1}^{N-1} \prod_{j=1}^{r_a} \frac{dz_j^a}{2\pi i} \cdot \prod_{a=1}^{N-1} \Delta^{c\ell}(z^a) e^{W^{c\ell}(z^a, z^{a+1})}$$

with $z_i^N := 0$. Here $\Delta^{c\ell}(z) := \prod_{i < j} (1 - z_j/z_i)^\beta (z_i/z_j - 1)^\beta$ is the β -deformed kernel and

$$(1.2) \quad W^{c\ell}(z^a, z^{a+1}) := \sum_{i=1}^{r_a} \left\{ \beta \sum_{n>0} \frac{1}{n} (z_i^a)^n p_n^{(a)} - \beta \sum_{j=1}^{r_{a+1}} \log(1 - z_j^{a+1}/z_i^a) - (s_a + 1) \log z_i^a \right\}$$

is the Penner type potential. The partition function $Z_N^{c\ell}$ is a function in coupling constants $p_n^{(a)}$ and is specified by a set of integers r_a and s_a ($n \in \mathbb{N}$ and $a = 1, 2, \dots, N$). Since $Z_N^{c\ell}$ is the singular vector, it satisfies the $\mathcal{W}_N$ constraint $\mathcal{W}_{c\ell n}^a Z_N^{c\ell} = 0$ ($n > 0$) with the $\mathcal{W}_N$ generators $\mathcal{W}_{c\ell n}^a$, which Virasoro central charge is $c = N - 1 - N(N^2 - 1)(\sqrt{\beta} - 1/\sqrt{\beta})^2$. Under the strategy of Ref. [3], we will introduce a q -deformed β -ensemble which automatically satisfies q - $\mathcal{W}_N$ constraint. The partition function Z_N of the A_{N-1} type q -deformed β -ensemble will be defined as the singular vector of the q - $\mathcal{W}_N$ algebra and is given by replacing $\Delta^{c\ell}(z)$ and $W^{c\ell}(z^a, z^{a+1})$ in (1.1) with

$$(1.3) \quad \Delta(z) := \prod_{i < j} (1 - z_j/z_i) \prod_{\ell \geq 0} \frac{1 - q^\ell p z_j/z_i}{1 - q^\ell t z_j/z_i} \cdot \prod_{i=1}^r z_i^{(r+1-2i)\beta},$$

$$W(z^a, z^{a+1}) := \sum_{i=1}^{r_a} \left\{ \sum_{n>0} \frac{[\beta]_{q^n}}{n} \left((z_i^a)^n p_n^{(a)} + \sum_{j=1}^{r_{a+1}} \left(p^{\frac{1}{2}} z_j^{a+1}/z_i^a \right)^n \right) - (s_a + 1) \log z_i^a \right\}.$$

Here $[\beta]_q = (q^{\frac{\beta}{2}} - q^{-\frac{\beta}{2}})/(q^{\frac{1}{2}} - q^{-\frac{1}{2}})$, $t = q^\beta$ and $p = q/t$. Then this satisfies the q - $\mathcal{W}_N$ constraint with the q - $\mathcal{W}_N$ generators defined in (3.12). If we specialize the mass parameters appropriately, the 5D Nekrasov partition function of $SU(N)$ gauge theory with $N_f = 2N$ reduces to the q -hypergeometric function. For $N = 2$, we will show that if we specialize the coupling constants appropriately, Z_2 also reduces to the q -hypergeometric function and coincides with the corresponding Nekrasov partition function.

This paper is organized as follows: In section 2, we start with recapitulating the result of the q - $\mathcal{W}_N$ algebra and also define primary fields. In section 3, we introduce q -deformed β -ensemble which automatically satisfies q - $\mathcal{W}_N$ constraint. Section 4 deals with the $N = 2$ case. Finally in section 5, we explain a reduction of the 5D Nekrasov partition function to the q -hypergeometric function and show a coincidence with the partition function of our q -deformed β -ensemble. Appendix A contains a definition of the Macdonald polynomial and several useful formulas.

Notation. Let $[n]_p := (p^{\frac{n}{2}} - p^{-\frac{n}{2}})/(p^{\frac{1}{2}} - p^{-\frac{1}{2}})$. Parameters are $q := e^{\hbar/\sqrt{\beta}} = e^{g_s R}$, $t := q^\beta = e^{\hbar\sqrt{\beta}} = e^{g_s \beta R}$, $p := q/t = e^{-\hbar(\sqrt{\beta}-1/\sqrt{\beta})}$, $u := t^\gamma$ and $v := (q/t)^{\frac{1}{2}}$. We will use the same letter p also for the set of power sums $p := (p_1, p_2, \dots)$, but this appears only at $P_\lambda(x[p])$ or $Z_2(p)$. The integral $\oint \frac{dz}{2\pi iz} f(z)$ denotes the constant term in f .

§ 2. Quantum deformation of $\mathcal{W}_N$ algebra

We start with recapitulating the results of the q - $\mathcal{W}_N$ algebra [21, 22] and define primary fields.

§ 2.1. q - $\mathcal{W}_N$ algebra

We use three kinds of basis for bosons. First we define fundamental bosons h_n^i and Q_h^i for $i = 1, 2, \dots, N$ and $n \in \mathbb{Z}$ such that¹

$$(2.1) \quad [h_n^i, h_m^j] = \frac{1}{n} (q^{\frac{n}{2}} - q^{-\frac{n}{2}}) (t^{\frac{n}{2}} - t^{-\frac{n}{2}}) \frac{[\delta_{ij} N - 1]_{p^n}}{[N]_{p^n}} p^{\frac{n}{2} N \text{sgn}(j-i)} \delta_{n+m,0},$$

$$(2.2) \quad [h_n^i, Q_h^j] = \left(\delta_{ij} - \frac{1}{N} \right) \delta_{n,0}, \quad [Q_h^i, Q_h^j] = 0, \quad \sum_{i=1}^N p^{in} h_n^i = 0, \quad \sum_{i=1}^N Q_h^i = 0$$

with $q, t := q^\beta \in \mathbb{C}$, $p := q/t$, $[n]_p := (p^{\frac{n}{2}} - p^{-\frac{n}{2}})/(p^{\frac{1}{2}} - p^{-\frac{1}{2}})$ and $\text{sgn}(i) := 1, 0$ or -1 for $i > 0, i = 0$ or $i < 0$, respectively. Here $[A, B] := AB - BA$. This bosons correspond to the weights $\vec{h}_i$ of the vector representation whose inner product is $(\vec{h}_i, \vec{h}_j) = \delta_{ij} - 1/N$. This algebra is invariant under the following involutions: $\omega_\pm^2 = 1$,

$$(2.3) \quad \omega_+ : \quad \sqrt{\beta} \mapsto 1/\sqrt{\beta}, \quad (q, t) \mapsto (t, q), \quad h_n^i \mapsto h_n^{N-i+1}, \quad Q_h^i \mapsto Q_h^{N-i+1},$$

$$(2.4) \quad \omega_- : \quad \sqrt{\beta} \mapsto -\sqrt{\beta}, \quad (q, t) \mapsto (q^{-1}, t^{-1}), \quad h_n^i \mapsto h_n^{N-i+1}, \quad Q_h^i \mapsto Q_h^{N-i+1}.$$

We also use root type bosons $\alpha_n^a := h_n^a - h_n^{a+1}$ and $Q_\alpha^a := Q_h^a - Q_h^{a+1}$ and weight type bosons $\Lambda_n^a := \sum_{b=1}^a h_n^b p^{(b-a-\frac{1}{2})n}$ and $Q_\Lambda^a := \sum_{b=1}^a Q_h^b$ for $a = 1, 2, \dots, N-1$.

Let us define fundamental vertices $\Lambda_i(z)$ and q - $\mathcal{W}_N$ generators $W^i(z)$ for $i = 1, 2, \dots, N$ as follows:

$$(2.5) \quad \Lambda_i(z) := \bullet \exp \left\{ \sum_{n \neq 0} h_n^i z^{-n} \right\} \bullet q^{\sqrt{\beta} h_0^i p^{\frac{N+1}{2} - i}},$$

$$(2.6) \quad W^i(z p^{\frac{1-i}{2}}) := \sum_{1 \leq j_1 < \dots < j_i \leq N} \bullet \Lambda_{j_1}(z) \Lambda_{j_2}(z p^{-1}) \cdots \Lambda_{j_i}(z p^{1-i}) \bullet$$

¹To obtain the $q = 1$ limit, we need to change the normalization of bosons by

$h_n^{\text{old}} = h_n^{\text{new}} \sqrt{(q^{\frac{n}{2}} - q^{-\frac{n}{2}})(t^{\frac{n}{2}} - t^{-\frac{n}{2}})/n^2} = h_n^{\text{new}} \sqrt{[\beta]_{q^n} (q^{\frac{n}{2}} - q^{-\frac{n}{2}})/n}$ ($n \neq 0$) with $h_0^{\text{old}} = h_0^{\text{new}}$ and $Q_h^{\text{old}} = Q_h^{\text{new}}$ unchanged. Letting $q \rightarrow 1$ yields the four-dimensional case.[3]

and $W^0(z) := 1$. Here $\bullet * \bullet$ stands for the usual bosonic normal ordering such that the bosons h_n^i with non-negative mode $n \geq 0$ are in the right. These generators are obtained from the following quantum Miura transformation:

$$(2.7) \quad \sum_{i=0}^N (-1)^i W^i(z p^{\frac{1-i}{2}}) p^{(N-i)D_z} \\ = \bullet (p^{D_z} - \Lambda_1(z)) (p^{D_z} - \Lambda_2(z p^{-1})) \cdots (p^{D_z} - \Lambda_N(z p^{1-N})) \bullet$$

with $D_z := z \frac{\partial}{\partial z}$. Remark that p^{D_z} is the p -shift operator such that $p^{D_z} f(z) = f(pz)$. The mode n generator W_n^i is defined by $\sum_{n \in \mathbb{Z}} W_n^i z^{-n} := W^i(z)$.

By using root type bosons we define screening currents $S_{\pm}^a(z)$ as follows:

$$(2.8) \quad S_{\pm}^a(z) := \bullet \exp \left\{ \mp \sum_{n \neq 0} \frac{\alpha_n^a}{\xi_{\pm}^{\frac{n}{2}} - \xi_{\pm}^{-\frac{n}{2}}} z^{-n} \right\} \bullet e^{\pm \sqrt{\beta} \pm 1 Q_{\alpha}^a z^{\pm \sqrt{\beta} \pm 1} \alpha_0^a}, \quad \xi_+ = q, \quad \xi_- = t,$$

with $\alpha_n^a := h_n^a - h_n^{a+1}$ and $Q_{\alpha}^a := Q_h^a - Q_h^{a+1}$. Note that the Langlands duality $\omega_- \omega_+ S_+^a(z) = S_-^a(z)$. We denote the negative mode part of $S_{\pm}^a(z)$ by $(S_{\pm}^a(z))_- := \exp \left\{ \mp \sum_{n < 0} \frac{\alpha_n^a}{\xi_{\pm}^{\frac{n}{2}} - \xi_{\pm}^{-\frac{n}{2}}} z^{-n} \right\}$. Then the screening charges defined by $\oint dz S_{\pm}^a(z)$ commute with any q - $\mathcal{W}_N$ generators

$$(2.9) \quad [\oint dz S_{\pm}^a(z), W^b(w)] = 0, \quad a, b = 1, 2, \dots, N-1.$$

For parameters u and γ with $u := t^{\gamma}$, let us define the following vertex operators

$$(2.10) \quad V_u^a(z) := \bullet \exp \left\{ \sum_{n \neq 0} \frac{(u^{\frac{n}{2}} - u^{-\frac{n}{2}}) \Lambda_n^a}{(q^{\frac{n}{2}} - q^{-\frac{n}{2}})(t^{\frac{n}{2}} - t^{-\frac{n}{2}})} z^{-n} \right\} \bullet e^{-\gamma \sqrt{\beta} Q_{\Lambda}^a z^{-\gamma \sqrt{\beta} \Lambda_0^a},$$

with $\Lambda_n^a := \sum_{b=1}^a h_n^b p^{(b-a-\frac{1}{2})n}$ and $Q_{\Lambda}^a := \sum_{b=1}^a Q_h^b$. They satisfy

$$(2.11) \quad g_{u,p}^{a,L} \left(\frac{w}{z} \right) \Lambda_i(z) V_u^a(w) - V_u^a(w) \Lambda_i(z) g_{u,p}^{a,R} \left(\frac{z}{w} \right) \\ = (u^{-1} - 1) \sum_{b=1}^a \delta_{i,b} \delta \left(\frac{w}{z u^{\frac{1}{2}}} \right) \bullet \Lambda_i(z) V_u^a(w) \bullet,$$

where $g_{u,p}^{a,L}(x)$ and $g_{u,p}^{a,R}(x)$ are inverse of the OPE factors,

$$(2.12) \quad g_{u,p}^{a,L}(x) := \frac{\bullet \Lambda_j(z) V_u^a(w) \bullet}{\Lambda_j(z) V_u^a(w)} = \exp \left\{ \sum_{n > 0} \frac{u^{-\frac{n}{2}} - u^{\frac{n}{2}}}{n} \frac{[a]_{p^n}}{[N]_{p^n}} p^{\frac{n}{2}(a-N)} x^n \right\} u^{-\frac{a}{N}},$$

$$(2.13) \quad g_{u,p}^{a,R}(x) := \frac{\bullet V_u^a(w) \Lambda_j(z) \bullet}{V_u^a(w) \Lambda_j(z)} = \exp \left\{ \sum_{n > 0} \frac{u^{\frac{n}{2}} - u^{-\frac{n}{2}}}{n} \frac{[a]_{p^n}}{[N]_{p^n}} p^{\frac{n}{2}(N-a)} x^n \right\}$$

for any $j > a$.

§ 2.2. Highest weight module of q - $\mathcal{W}_N$ algebra

Next we refer to the representation of the q - $\mathcal{W}_N$ algebra. Let $\mathcal{F}_\alpha$ be the boson Fock space generated by the highest weight state $|\alpha\rangle$ such that $\alpha_n^a|0\rangle = 0$ for $n \geq 0$ and $|\alpha\rangle := \exp\{\sum_{a=1}^{N-1} \alpha^a Q_\Lambda^a\}|0\rangle$. Note that $\alpha_0^a|\alpha\rangle = \alpha^a|\alpha\rangle$. The dual module $\mathcal{F}_\alpha^*$ is generated by $\langle\alpha|$ such that $\langle\alpha| \alpha_{-n}^a = 0$ for $n \geq 0$ and $\langle\alpha| := \langle 0| \exp\{-\sum_{a=1}^{N-1} \alpha^a Q_\Lambda^a\}$. The bilinear form $\mathcal{F}_\alpha^* \otimes \mathcal{F}_\alpha \rightarrow \mathbb{C}$ is uniquely defined by $\langle 0|0\rangle = 1$.

Let $|\lambda\rangle$ be the highest weight vector of the q - $\mathcal{W}_N$ algebra which satisfies $W_n^a|\lambda\rangle = 0$ for $n > 0$ and $a = 1, 2, \dots, N-1$ and $W_0^a|\lambda\rangle = \lambda^a|\lambda\rangle$ with $\lambda^a \in \mathbb{C}$. Let M_λ be the Verma module over the q - $\mathcal{W}_N$ algebra generated by $|\lambda\rangle$. The dual module M_λ^* is generated by $\langle\lambda|$ such that $\langle\lambda| W_n^a = 0$ for $n < 0$ and $\langle\lambda| W_0^a = \lambda^a \langle\lambda|$. The bilinear form $M_\lambda^* \otimes M_\lambda \rightarrow \mathbb{C}$ is uniquely defined by $\langle\lambda|\lambda\rangle = 1$. A singular vector $|\chi\rangle \in M_\lambda$ is defined by $W_n^a|\chi\rangle = 0$ for $n > 0$ and $W_0^a|\chi\rangle = (\lambda^a + N^a)|\chi\rangle$ with $N^a \in \mathbb{C}$.

The highest weight vector $|\alpha\rangle \in \mathcal{F}_\alpha$ of the boson algebra is also that of the q - $\mathcal{W}_N$ algebra, i.e., $W_n^a|\alpha\rangle = 0$ for $n > 0$ and $a = 1, 2, \dots, N-1$. Note that $W_0^a|0\rangle = [N]_p^a|0\rangle$ with $[N]_p := (p^{\frac{N}{2}} - p^{-\frac{N}{2}})/(p^{\frac{1}{2}} - p^{-\frac{1}{2}})$.

For a set of non-negative integers s_a and $r_a \geq r_{a+1} \geq 0$ with $a = 1, \dots, N-1$, let

$$(2.14) \quad \pm \alpha_{r,s}^{\pm,a} := (1 + r_a - r_{a-1})\sqrt{\beta}^{\pm 1} - (1 + s_a)\sqrt{\beta}^{\mp 1}, \quad r_0 := 0,$$

$$(2.15) \quad \pm \tilde{\alpha}_{r,s}^{\pm,a} := (1 - r_a + r_{a+1})\sqrt{\beta}^{\pm 1} - (1 + s_a)\sqrt{\beta}^{\mp 1}, \quad r_N := 0.$$

The singular vectors $|\chi_{r,s}^\pm\rangle \in \mathcal{F}_{\alpha_{r,s}^\pm}$ are realized by the screening currents as follows:

$$(2.16) \quad |\chi_{r,s}^\pm\rangle = \oint \prod_{a=1}^{N-1} \prod_{j=1}^{r_a} \frac{dz_j^a}{2\pi i} \cdot S_\pm^1(z_1^1) \cdots S_\pm^1(z_{r_1}^1) \cdots S_\pm^{N-1}(z_1^{N-1}) \cdots S_\pm^{N-1}(z_{r_{N-1}}^{N-1}) |\tilde{\alpha}_{r,s}^\pm\rangle \\ = \oint \prod_{a=1}^{N-1} \prod_{j=1}^{r_a} \frac{dz_j^a}{2\pi i z_j^a} (z_j^a)^{-s_a} (S_\pm^a(z_j^a))_- \cdot \Delta(z^a; \xi_\pm, \xi_\mp) \Pi(\bar{z}^a, pz^{a+1}; \xi_\pm, \xi_\mp) |\alpha_{r,s}^\pm\rangle$$

with $z^N := 0$, $\bar{z} := 1/z$, $\xi_+ := q$ and $\xi_- := t$. Note that $\omega_- \omega_+ |\chi_{r,s}^+\rangle = |\chi_{r,s}^-\rangle$. Here

$$(2.17) \quad \Pi(z, w) := \Pi(z, w; q, t) := \prod_{i,j} \exp \left\{ \sum_{n>0} \frac{[\beta]_{q^n}}{n} p^{-\frac{n}{2}} z_i^n w_j^n \right\} = \prod_{i,j} \prod_{\ell \geq 0} \frac{1 - q^\ell t z_i w_j}{1 - q^\ell z_i w_j},$$

$$(2.18) \quad \Delta(z) := \Delta(z; q, t) := \prod_{i<j} \exp \left\{ - \sum_{n>0} [2]_{p^n} \frac{[\beta]_{q^n}}{n} \frac{z_j^n}{z_i^n} \right\} \cdot \prod_{i=1}^r z_i^{(r+1-2i)\beta} \\ = \prod_{i<j} (1 - z_j/z_i) \prod_{\ell \geq 0} \frac{1 - q^\ell p z_j/z_i}{1 - q^\ell t z_j/z_i} \cdot \prod_{i=1}^r z_i^{(r+1-2i)\beta}, \quad |q| < 1$$

with $\beta := \log t / \log q$. Note that $\Delta(cz) = \Delta(z)$.

§ 3. Quantum deformation of β -ensemble

Note that the singular vector in (2.16) is naturally mapped to the Macdonald polynomial [24] defined in the appendix A. [22] As a generalization of this map one can define, under the strategy of Ref. [3], a quantum deformation of the generalized matrix model, i.e., q -deformed β -ensemble.

§ 3.1. q -deformed β -ensemble

With a new parameters $p^{(a)} := (p_1^{(a)}, p_2^{(a)}, \dots)$ let us define the following vertex operator

$$(3.1) \quad V_N := \prod_{a=1}^{N-1} \exp \left\{ \sum_{n>0} \frac{\Lambda_n^a}{q^{\frac{n}{2}} - q^{-\frac{n}{2}}} p_n^{(a)} \right\},$$

with $\Lambda_n^a := \sum_{b=1}^a h_n^b p^{(b-a-\frac{1}{2})n}$ and $Q_\Lambda^a := \sum_{b=1}^a Q_h^b$. Note that $[\Lambda_n^a, \Lambda_m^b] = 0$ for $n, m > 0$. Then $\langle \alpha | V_N$ defines the isomorphism between the boson algebras $\langle h_n^a \rangle_{n \in \mathbb{Z}}^{1 \leq a \leq N}$ and $\langle p_n^{(a)}, \alpha^a, \frac{\partial}{\partial p_n^{(a)}} \rangle_{n \in \mathbb{N}}^{1 \leq a \leq N}$ by

$$(3.2) \quad \langle \alpha | V_N h_{-n}^i = \frac{t^{\frac{n}{2}} - t^{-\frac{n}{2}}}{n} \sum_{b=1}^{N-1} A^{i,b}(p^{-n}) p_n^{(b)} \langle \alpha | V_N,$$

$$(3.3) \quad \langle \alpha | V_N h_n^i = (q^{\frac{n}{2}} - q^{-\frac{n}{2}}) \sum_{b=1}^{N-1} B^{i,b}(p^n) \frac{\partial}{\partial p_n^{(b)}} \langle \alpha | V_N$$

for $n > 0$ and $\langle \alpha | V_N h_0^i = h^i \langle \alpha | V_N$ with $h^i = \left[\sum_{b=i}^{N-1} - \sum_{b=1}^{N-1} b/N \right] \alpha^b$. Here

$$(3.4) \quad A^{i,b}(p) := \frac{[N\theta(i \leq b) - i]_p}{[N]_p} p^{\frac{1}{2}(b - N\theta(i > b))},$$

$$(3.5) \quad B^{i,b}(p) := p^{\frac{1}{2}} \delta_{i,b} - p^{-\frac{1}{2}} \delta_{i-1,b}$$

with $\theta(P) := 1$ or 0 if the proposition P is true or false, respectively.

The vector $|S_{r,s}^+\rangle := \prod_{a=1}^{N-1} \prod_{k=1}^{r_a} (S_+^a(z_k^a))_- \cdot |\alpha_{r,s}^+\rangle$ in (2.16) also defines another linear map from $\langle h_n^a \rangle_{n \in \mathbb{Z}}^{1 \leq a \leq N}$ to $\langle \sum_{k=1}^{r_a} (z_k^a)^n \rangle_{n \in \mathbb{N}}^{1 \leq a \leq N}$ by

$$(3.6) \quad h_n^i |S_{r,s}^+\rangle = |S_{r,s}^+\rangle \frac{t^{\frac{n}{2}} - t^{-\frac{n}{2}}}{n} \sum_{b=1}^{N-1} B^{i,b}(p^n) \sum_{k=1}^{r_b} (z_k^b)^n, \quad n > 0.$$

Let us define the following partition function

Definition 3.1. Let $Z_N := Z_N(\{p^{(a)}\}_{a=1}^{N-1}) := \langle \alpha_{r,s}^+ | V_N | \chi_{r,s}^+ \rangle$.

Then by (2.16), (2.8) and (3.3), we have

$$\begin{aligned} Z_N &= \oint \prod_{a=1}^{N-1} \prod_{j=1}^{r_a} \frac{dz_j^a}{2\pi i} \langle \alpha_{r,s}^+ | V_N S_+^1(z_1^1) \cdots S_+^1(z_{r_1}^1) \cdots S_+^{N-1}(z_1^{N-1}) \cdots S_+^{N-1}(z_{r_{N-1}}^{N-1}) | \tilde{\alpha}_{r,s}^+ \rangle \\ &= \oint \prod_{a=1}^{N-1} \prod_{j=1}^{r_a} \frac{dz_j^a}{2\pi i z_j^a} (z_j^a)^{-s_a} \exp \left\{ \sum_{n>0} \frac{[\beta]_{q^n}}{n} (z_j^a)^n p_n^{(a)} \right\} \cdot \Delta(z^a) \Pi(\bar{z}^a, p z^{a+1}) \\ &= \oint \prod_{a=1}^{N-1} \prod_{j=1}^{r_a} \frac{dz_j^a}{2\pi i} \cdot \prod_{a=1}^{N-1} \Delta(z^a) e^{W(z^a, z^{a+1})} \end{aligned}$$

with

(3.7)

$$W(z^a, z^{a+1}) := \sum_{i=1}^{r_a} \left\{ \sum_{n>0} \frac{[\beta]_{q^n}}{n} \left((z_i^a)^n p_n^{(a)} + \sum_{j=1}^{r_{a+1}} \left(\frac{p^{\frac{1}{2}} z_j^{a+1}}{z_i^a} \right)^n \right) - (s_a + 1) \log z_i^a \right\}.$$

Here $z^N := 0$. This Z_N is regarded as a q -deformation of the partition function of the generalized matrix model,[3] i.e., β -ensemble. One can define other type of partition functions by acting involutions (2.3), (2.4) and (A.8).

We can calculate this integral by using the Macdonald polynomials $P_\lambda(x)$ with the Young diagram λ , their fusion coefficient $f_{\lambda,\mu}^\nu$ and the inner products $\langle *, * \rangle$ and $\langle *, * \rangle_r''$ defined in the appendix A.

Proposition 3.2.

$$(3.8) \quad Z_N = \prod_{a=1}^{N-1} \sum_{\lambda_a, \mu_a} f_{\mu_{a-1}, \lambda_a}^{\mu_a + (s_a^{r_a})} P_{\mu_a + (s_a^{r_a})}(z^a) \frac{P_{\lambda_a}(x[p^a])}{\langle \lambda_a \rangle} p^{|\mu_a|} \frac{r_a! \langle \mu_a + (s_a^{r_a}) \rangle_{r_a}''}{\langle \mu_a \rangle}$$

with $\langle 0 \rangle := 1$. Here λ_a , μ_a and ν_a are Young diagrams such that $\lambda_{a,i} \geq \lambda_{a,i+1}$, and so on. $P_\lambda(x[p])$ denotes the Macdonald function in power sums $p := (p_1, p_2, \dots)$.

One can show that (3.8) is summed over $(N-2) + (N-3)$ Young diagrams for $N \geq 3$.

For any function $\mathcal{O}$ in z_j^a 's, the correlation function with respect to $\mathcal{O}$ is defined by

$$(3.9) \quad \langle\langle \mathcal{O} \rangle\rangle := \frac{1}{Z_N} \oint \prod_{a=1}^{N-1} \prod_{j=1}^{r_a} \frac{dz_j^a}{2\pi i} \cdot \mathcal{O} \prod_{a=1}^{N-1} \Delta(z^a) e^{W(z^a, z^{a+1})}.$$

The effective action S_{eff} defined by $Z_N =: \oint \prod_{a=1}^{N-1} \prod_{j=1}^{r_a} \frac{dz_j^a}{2\pi i} \cdot e^{S_{\text{eff}}}$ is now

(3.10)

$$S_{\text{eff}} = \sum_{a=1}^{N-1} W(z^a, z^{a+1}) - \sum_{n>0} [2]_{p^n} \frac{[\beta]_{q^n}}{n} \sum_{a=1}^{N-1} \sum_{i<j} \left(\frac{z_j^a}{z_i^a} \right)^n + \beta \sum_{a=1}^{N-1} \sum_{i=1}^{r_a} (r_a + 1 - 2i) \log z_i^a.$$

§ 3.2. q - $\mathcal{W}_N$ constraint, Loop equation and quantum spectral curve

Next let us define $\hat{\Lambda}_i(z)$ and $\mathcal{W}^i(z)$ by the isomorphism (3.3) as follows:

$$(3.11) \quad \hat{\Lambda}_i(z) \langle \alpha_{r,s} | V_N := \langle \alpha_{r,s} | V_N \Lambda_i(z),$$

$$(3.12) \quad \mathcal{W}^i(z) \langle \alpha_{r,s} | V_N := \langle \alpha_{r,s} | V_N W^i(z)$$

and $\sum_{n \in \mathbb{Z}} \mathcal{W}_n^i z^{-n} := \mathcal{W}^i(z)$, which are the power sum realization of fundamental vertices $\Lambda_i(z)$ and q - $\mathcal{W}_N$ generators $W^i(z)$, respectively. Then the highest weight condition for the singular vector $W_n^a |\chi\rangle = 0$ for $n > 0$ is equivalent to the following q - $\mathcal{W}_N$ constraint:

Theorem 3.3.

$$(3.13) \quad \mathcal{W}_n^a Z_N = 0, \quad n > 0.$$

Let us define $\tilde{\Lambda}_i(z)$ and $\tilde{\mathcal{W}}^i(z)$ by linear maps (3.3) and (3.6) as follows:

$$(3.14) \quad \langle \alpha_{r,s}^+ | V_N | S_{r,s}^+ \rangle \tilde{\Lambda}_i(z) := \langle \alpha_{r,s}^+ | V_N \Lambda_i(z) | S_{r,s}^+ \rangle,$$

$$(3.15) \quad \langle \alpha_{r,s}^+ | V_N | S_{r,s}^+ \rangle \tilde{\mathcal{W}}^i(z) := \langle \alpha_{r,s}^+ | V_N W^i(z) | S_{r,s}^+ \rangle$$

and $\sum_{n \in \mathbb{Z}} \tilde{\mathcal{W}}_n^i z^{-n} := \tilde{\mathcal{W}}^i(z)$. Hence

$$(3.16) \quad \left\langle \left\langle \tilde{\mathcal{W}}^i(z) \right\rangle \right\rangle = \frac{1}{Z_N} \langle \alpha_{r,s}^+ | V_N W^i(z) | \chi_{r,s}^+ \rangle.$$

Therefore the highest weight condition for the singular vector $W_n^a |\chi\rangle = 0$ for $n > 0$ is equivalent to the following loop equation:

Theorem 3.4.

$$(3.17) \quad \left\langle \left\langle \tilde{\mathcal{W}}_n^a \right\rangle \right\rangle = 0, \quad n > 0.$$

The quantum spectral curve should be

$$(3.18) \quad \left\langle \left\langle \left(p^{D_z} - \tilde{\Lambda}_1(z) \right) \left(p^{D_z} - \tilde{\Lambda}_2(zp^{-1}) \right) \cdots \left(p^{D_z} - \tilde{\Lambda}_N(zp^{1-N}) \right) \right\rangle \right\rangle = 0$$

which regularity in z is guaranteed by the loop equation (3.17).

Let $(q, t) =: (e^{R\epsilon_2}, e^{-R\epsilon_1}) =: (e^{g_s R}, e^{g_s \beta R})$ with the radius $R \in \mathbb{R}$ of the 5th dimensional circle S^1 . Let us rescale the variables as $\tilde{p}_n^{(a)} := g_s p_n^{(a)}$, $\tilde{r}_a := g_s r_a$ and $\tilde{s}_a := g_s s_a$. Under the limit $g_s \rightarrow 0$ and $r_a s_a \rightarrow \infty$ with fixed $\tilde{r}_a$ and $\tilde{s}_a$, the sift operator p^{D_z} tends to a commutative variable and the quantum spectral curve reduces to the usual one.

§ 4. $N = 2$ case

Here we give an example when $N = 2$, i.e., the q -deformed Virasoro case. The partition function Z_2 is now

$$(4.1) \quad Z_2(p) = \oint \prod_{j=1}^r \frac{dz_j}{2\pi i z_j} z_j^{-s} \exp \left\{ \sum_{n>0} \frac{[\beta]_q^n}{n} z_j^n p_n \right\} \cdot \Delta(z) = p^{\frac{rs}{2}} \frac{r! \langle s^r \rangle_r''}{\langle s^r \rangle} P_{(sr)}(x[p]).$$

Then we have

Proposition 4.1. *The partition function $Z_2(p)$ substituting $p_n = \sum_i x_i^n + \frac{1-u^n}{1-t^n} y^n$ and $\frac{1-t^n}{1-q^n} p_n = (-1)^{n-1} (\sum_i x_i^n + \frac{1-u^n}{1-t^n} y^n)$ are*

$$(4.2) \quad \frac{Z_2 \left(\sum_i x_i + \frac{1-u}{1-t} y \right)}{Z_2 \left(\frac{1-u}{1-t} y \right)} = {}_2\varphi_1^{(q,t)} \left[\begin{matrix} q^{-s}, t^r \\ q^{1-s} t^{r-1}/u \end{matrix}; \frac{qx}{uy} \right],$$

$$(4.3) \quad \omega_{q,t} \frac{Z_2 \left(\sum_i x_i + \frac{1-u}{1-t} y \right)}{Z_2 \left(\frac{1-u}{1-t} y \right)} = {}_2\varphi_1^{(t,q)} \left[\begin{matrix} t^{-r}, q^s \\ t^{1-r} q^{s-1}/u \end{matrix}; \frac{tx}{uy} \right].$$

with $\omega_{q,t}$ in (A.8). Here ${}_2\varphi_1^{(q,t)} \left[\begin{matrix} a,b \\ c \end{matrix}; x \right]$ is the multivariate q -hypergeometric function [28]

$$(4.4) \quad {}_2\varphi_1^{(q,t)} \left[\begin{matrix} a,b \\ c \end{matrix}; x \right] := \sum_{\substack{\lambda \\ \ell(\lambda) \leq M}} P_\lambda(x) \prod_{(i,j) \in \lambda} \frac{(t^{i-1} - aq^{j-1})(t^{i-1} - bq^{j-1})}{(t^{i-1} - cq^{j-1})(1 - q^{\lambda_i - j + 1} t^{\lambda'_j - i})}.$$

Since $P_\lambda(x; q, t) = P_\lambda(x; q^{-1}, t^{-1})$, ${}_2\varphi_1^{(q,t)} \left[\begin{matrix} a,b \\ c \end{matrix}; x \right]$ satisfies

$$(4.5) \quad {}_2\varphi_1^{(q,t)} \left[\begin{matrix} a,b \\ c \end{matrix}; x \right] = {}_2\varphi_1^{(q^{-1}, t^{-1})} \left[\begin{matrix} a^{-1}, b^{-1} \\ c^{-1} \end{matrix}; \frac{ab}{qc} x \right].$$

When $M = \infty$,

$$(4.6) \quad \omega_{q,t} {}_2\varphi_1^{(q,t)} \left[\begin{matrix} a,b \\ c \end{matrix}; x \right] = {}_2\varphi_1^{(t,q)} \left[\begin{matrix} a,b \\ c \end{matrix}; \frac{ab}{c} x \right], \quad M = \infty.$$

When $M = 1$, ${}_2\varphi_1^{(q,t)} \left[\begin{matrix} a,b \\ c \end{matrix}; x \right]$ reduces to the usual q -hypergeometric function

$$(4.7) \quad {}_2\varphi_1^{(q,t)} \left[\begin{matrix} a,b \\ c \end{matrix}; x \right] := {}_2\varphi_1 \left[\begin{matrix} a,b \\ c \end{matrix}; q, x \right] := \sum_{n \geq 0} x^n \prod_{\ell=0}^{n-1} \frac{(1 - aq^\ell)(1 - bq^\ell)}{(1 - cq^\ell)(1 - q^{\ell+1})}, \quad M = 1.$$

In the next section we will show a relation between our $Z_2 \left(x + \frac{1-u}{1-t} y \right)$ and the 5-dimensional $SU(2)$ Nekrasov partition function.

§ 5. Five-dimensional Nekrasov partition function

Let $Q = (Q_1, \dots, Q_N)$ with $\prod_{i=1}^N Q_i = 1$ and $Q^\pm = (Q_1^\pm, \dots, Q_N^\pm)$ be sets of complex parameters. The instanton part of the five-dimensional $SU(N)$ Nekrasov partition function with $N_f = 2N$ fundamental matters² is written by a sum over N Young diagrams λ_i ($i = 1, 2, \dots, N$) as follows(double-sign corresponds):[25, 18]

$$(5.1) \quad Z^{\text{inst}}(Q) = \sum_{\{\lambda_i\}} \prod_{i,j} \frac{N_{\lambda_i \bullet} (vQ_i/Q_j^\pm) N_{\bullet \lambda_i} (vQ_j^\mp/Q_i)}{N_{\lambda_i \lambda_j} (Q_i/Q_j)} \cdot \prod_i \left(\frac{\Lambda_\alpha^\pm}{v^N} \right)^{|\lambda_i|}$$

with $v := (q/t)^{\frac{1}{2}}$, $N_{\lambda\mu}(Q) := N_{\lambda\mu}(Q; q, t)$, $\Lambda_\alpha^\pm := \Lambda^{2N} \prod_{j=1}^N \left(\frac{Q_j^\pm}{Q_j^\mp} \right)^{\frac{1}{2}}$ and

$$(5.2) \quad N_{\lambda\mu}(Q; q, t) := \prod_{(i,j) \in \lambda} \left(1 - Q q^{\lambda_i - j} t^{\mu'_j - i + 1} \right) \prod_{(i,j) \in \mu} \left(1 - Q q^{-\mu_i + j - 1} t^{-\lambda'_j + i} \right) \\ = \prod_{(i,j) \in \mu} \left(1 - Q q^{\lambda_i - j} t^{\mu'_j - i + 1} \right) \prod_{(i,j) \in \lambda} \left(1 - Q q^{-\mu_i + j - 1} t^{-\lambda'_j + i} \right).$$

Here $\lambda = (\lambda_1, \lambda_2, \dots)$ is a Young diagram such that $\lambda_i \geq \lambda_{i+1}$. λ' is its conjugate Young diagram and $|\lambda| = \sum_i \lambda_i$. $Z^{\text{inst}}(Q; Q^+; Q^-)$ is symmetric in masses $Q_j^\pm$'s. Note that $N_{\lambda\mu}(Q; q, t)$ satisfies

$$(5.3) \quad N_{\lambda\mu}(vQ; q, t) = N_{\mu\lambda}(Q/v; q^{-1}, t^{-1}) = N_{\mu'\lambda'}(Q/v; t, q),$$

$$(5.4) \quad N_{\lambda \bullet} (vQ) N_{\bullet \lambda} (vQ') = N_{\bullet \lambda} (v/Q) N_{\lambda \bullet} (v/Q') (QQ')^{|\lambda|}.$$

There exists Q such that $N_{\lambda \bullet}(Q)$ vanishes except for $\lambda = (0)$, (n) or (1^n) . Hence one can adjust N out of $N_f = 2N$ parameters $Q_i^\pm$'s so that (5.1) reduces to all $\lambda_i = (0)$ but a $\lambda_j = (n)$ or (1^n) with $n \in \mathbb{Z}_{\geq 0}$ same as Ref. [26]. For example, if $(Q_1, \dots, Q_{N-1}, Q_N) = (Q_1^\pm, \dots, Q_{N-1}^\pm, tQ_N^\pm)/v$ with $\prod_{i=1}^N Q_i^\pm = v^N/t$ then the right hand side of (5.1) is summed over only $(\lambda_1, \dots, \lambda_{N-1}, \lambda_N) = ((0), \dots, (0), (n))$ with $n \in \mathbb{Z}_{\geq 0}$. On the other hand,

if $(Q_1, \dots, Q_{N-1}, Q_N) = (Q_1^\pm, \dots, Q_{N-1}^\pm, Q_N^\pm/q)/v$ with $\prod_{i=1}^N Q_i^\pm = qv^N$ then only $(\lambda_1, \dots, \lambda_{N-1}, \lambda_N) = ((0), \dots, (0), (1^n))$ contributes. Therefore we obtain

²The parameters (q, t) are related with those (ϵ_1, ϵ_2) of the Ω background through $(q, t) = (e^{R\epsilon_2}, e^{-R\epsilon_1})$ where R is the radius of the 5th dimensional circle. The parameter Q is related with the vacuum expectation value a of the scalar fields in the vector multiplets and the mass m of the fundamental matter as $Q_i = q^{a_i}$, $Q_i^+ = q^{-m_i}$ and $Q_i^- = q^{-m_{N+i}}$.

Proposition 5.1.

$$\begin{aligned}
(5.5) \quad Z^{\text{inst}}(Q_1^\pm/v, \dots, Q_{N-1}^\pm/v, tQ_N^\pm/v) &= {}_N\varphi_{N-1} \left[\begin{matrix} \frac{Q_1^\mp}{vQ_N}, \dots, \frac{Q_N^\mp}{vQ_N} \\ \frac{tQ_1}{qQ_N}, \dots, \frac{tQ_{N-1}}{qQ_N} \end{matrix}; q^{-1}, \frac{\Lambda_N^\pm}{v^N} \right] \\
&= {}_N\varphi_{N-1} \left[\begin{matrix} v\frac{Q_N}{Q_1^\mp}, \dots, v\frac{Q_N}{Q_N^\mp} \\ \frac{qQ_N}{tQ_1}, \dots, \frac{qQ_N}{tQ_{N-1}} \end{matrix}; q, v^N \Lambda_N^\mp \right], \\
(5.6) \quad Z^{\text{inst}}(Q_1^\pm/v, \dots, Q_{N-1}^\pm/v, Q_N^\pm/qv) &= {}_N\varphi_{N-1} \left[\begin{matrix} \frac{Q_1^\mp}{vQ_N}, \dots, \frac{Q_N^\mp}{vQ_N} \\ \frac{tQ_1}{qQ_N}, \dots, \frac{tQ_{N-1}}{qQ_N} \end{matrix}; t, \frac{\Lambda_N^\pm}{v^N} \right]
\end{aligned}$$

with $\prod_{i=1}^N Q_i^\pm = v^N/t$ for (5.5) and $\prod_{i=1}^N Q_i^\pm = qv^N$ for (5.6) and

$$(5.7) \quad {}_r\varphi_s \left[\begin{matrix} a_1, \dots, a_r \\ b_1, \dots, b_s \end{matrix}; q, x \right] := \sum_{n \geq 0} x^n \prod_{\ell=0}^{n-1} \frac{(-q^\ell)^{s+1-r} \prod_{i=1}^r (1 - q^\ell a_i)}{(1 - q^{\ell+1}) \prod_{i=1}^s (1 - q^\ell b_i)}.$$

Note that

$$(5.8) \quad {}_r\varphi_{r-1} \left[\begin{matrix} a_1, \dots, a_r \\ b_1, \dots, b_{r-1} \end{matrix}; q, x \right] = {}_r\varphi_{r-1} \left[\begin{matrix} a_1^{-1}, \dots, a_r^{-1} \\ b_1^{-1}, \dots, b_{r-1}^{-1} \end{matrix}; q^{-1}, \tilde{x} \right], \quad \tilde{x} := \frac{x \prod_{i=1}^r a_i}{q \prod_{i=1}^{r-1} b_i}.$$

When $N = 2$, Z^{inst} coincides with the $M = 1$ case of the partition function Z_2 of the q -deformed β -ensemble (4.2) similar to Ref. [6]

$$(5.9) \quad Z^{\text{inst}}(Q_1^\pm/v, tQ_2^\pm/v) = {}_2\varphi_1 \left[\begin{matrix} v\frac{Q_2}{Q_1^\mp}, v\frac{Q_2}{Q_2^\mp} \\ \frac{qQ_2}{tQ_1} \end{matrix}; q, v^2 \Lambda_2^\mp \right] = \frac{Z_2 \left(x + \frac{1-u}{1-t} y \right)}{Z_2 \left(\frac{1-u}{1-t} y \right)},$$

$$(5.10) \quad Z^{\text{inst}}(Q_1^\pm/v, Q_2^\pm/qv) = {}_2\varphi_1 \left[\begin{matrix} \frac{Q_1^\mp}{vQ_2}, \frac{Q_2^\mp}{vQ_2} \\ \frac{tQ_1}{qQ_2} \end{matrix}; t, \frac{\Lambda_2^\pm}{v^2} \right] = \omega_{q,t} \frac{Z_2 \left(\sum_i x_i + \frac{1-u}{1-t} y \right)}{Z_2 \left(\frac{1-u}{1-t} y \right)}$$

with

$$(5.11) \quad Q_1^\pm Q_2^\pm = \frac{q}{t^2}, \quad q^s = \frac{Q_1 Q_1^\mp}{v}, \quad t^{-r} = \frac{Q_1 Q_2^\mp}{v}, \quad u^{-1} = \frac{tQ_1^\mp Q_2^\mp}{q}, \quad \frac{qx}{y} = Q_1^\mp Q_2^\mp \Lambda_2^\mp$$

for (5.9) and

$$(5.12) \quad Q_1^\pm Q_2^\pm = \frac{q^2}{t}, \quad q^s = \frac{Q_1 Q_1^\mp}{v}, \quad t^{-r} = \frac{Q_1 Q_2^\mp}{v}, \quad u = \frac{tQ_1^\mp Q_2^\mp}{q}, \quad \frac{y}{tx} = \frac{Q_1^\mp Q_2^\mp}{\Lambda_2^\pm}$$

for (5.10). In the $SU(N)$ case, the Nekrasov partition function (5.5) may coincide with our partition function Z_N by using the formulas (A.11) and the Cor. 1.6 in Ref. [27].

§ A. Macdonald polynomial

Here we recapitulate basic properties of the Macdonald polynomial.[24] Let $\lambda := (\lambda_1, \lambda_2, \dots, \lambda_r)$ with $\lambda_i \geq \lambda_{i+1} \geq 0$ be a Young diagram. λ' is its conjugate. For any λ with $\lambda_1 \leq s$, $|\lambda| := \sum_i \lambda_i$. Let $x := (x_1, \dots, x_r)$ and $p := (p_1, p_2, \dots)$ with the power sum $p_n := p_n(x) := \sum_{i=1}^r x_i^n$. For any symmetric function f in x with $r = \infty$, $f(x[p])$ stands for the function f expressed in the power sums p .

The Macdonald polynomials $P_\lambda(x) := P_\lambda(x; q, t)$ are degree $|\lambda|$ homogeneous symmetric polynomials in x defined as eigenfunctions of the Macdonald operator H as follows:

$$(A.1) \quad HP_\lambda(x) = \varepsilon_\lambda P_\lambda(x),$$

$$(A.2) \quad H := \sum_{i=1}^r \prod_{j \neq i} \frac{tx_i - x_j}{x_i - x_j} \cdot q^{D_{x_i}}, \quad \varepsilon_\lambda := \sum_{i=1}^r q^{\lambda_i} t^{r-i}$$

with a normalization condition $P_\lambda(x) = x_1^{\lambda_1} x_2^{\lambda_2} \cdots x_r^{\lambda_r} + \dots$. Where q^{D_x} with $D_x := x \frac{\partial}{\partial x}$ is the q -shift operator such that $q^{D_x} f(x) = f(qx)$. Note that $P_\bullet(x) := P_{(0)}(x) = 1$.

Two kinds of inner products are known in which the Macdonald polynomials are orthogonal each other. For any symmetric functions f and g in x , let us define inner product $\langle *, * \rangle$ and another one $\langle *, * \rangle_r''$ as follows: ³

$$(A.3) \quad \langle f, g \rangle := \oint \prod_{n>0} \frac{dp_n}{2\pi i p_n} \cdot f(x[p^*]) g(x[p]), \quad p_n^* := n \frac{1 - q^n}{1 - t^n} \frac{\partial}{\partial p_n},$$

$$(A.4) \quad \langle f, g \rangle_r'' := \frac{1}{r!} \oint \prod_{j=1}^r \frac{dx_j}{2\pi i x_j} \cdot \Delta(x) f(\bar{x}) g(x), \quad \bar{x}_j := \frac{1}{x_j}$$

with $\Delta(x)$ in (2.18). Here we must treat the power sums p_n as formally independent variables, *i.e.*, $\frac{\partial}{\partial p_n} p_m = \delta_{n,m}$ for all $n, m > 0$. The inner products of Macdonald polynomials are given by

$$(A.5) \quad \langle P_\lambda, P_\mu \rangle = \delta_{\lambda,\mu} \langle \lambda \rangle, \quad \langle \lambda \rangle := \prod_{(i,j) \in \lambda} \frac{1 - q^{\lambda_i - j + 1} t^{\lambda'_j - i}}{1 - q^{\lambda_i - j} t^{\lambda'_j - i + 1}},$$

$$(A.6) \quad \langle P_\lambda, P_\mu \rangle_r'' = \delta_{\lambda,\mu} \langle \lambda \rangle_r''.$$

Let us denote by $f\left(x\left[\frac{1-u}{1-t}\right]\right)$ the function $f(x[p])$ in the specialization $p_n := (1 - u^n)/(1 - t^n)$ with $u \in \mathbb{C}$, then [24]

$$(A.7) \quad P_\lambda\left(x\left[\frac{1-u}{1-t}\right]\right) = \prod_{(i,j) \in \lambda} \frac{t^{i-1} - uq^{j-1}}{1 - q^{\lambda_i - j} t^{\lambda'_j - i + 1}}.$$

³The usual another inner product $\langle *, * \rangle_r'$ is defined with a different kernel

$\Delta'(x) := \prod_{i \neq j}^r \exp\left\{-\sum_{n>0} (1-t^n)/(1-q^n) (x_j^n/x_i^n)/n\right\} = \prod_{i \neq j}^r \prod_{\ell \geq 0} (1 - q^\ell x_j/x_i)/(1 - tq^\ell x_j/x_i)$
($|q| < 1$). Note that $C(x) := \Delta(x)/\Delta'(x)$ is a pseudo-constant, *i.e.*, $q^{D_{x_i}} C(x) = C(x)$.

With the involution $\omega_{q,t}$,

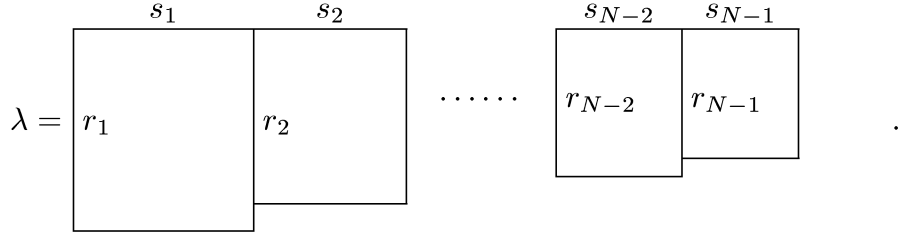
$$(A.8) \quad \frac{1}{\langle \lambda \rangle} \omega_{q,t} P_\lambda(x; q, t) = P_{\lambda'}(x; t, q), \quad \omega_{q,t}(p_n) := (-1)^{n-1} \frac{1 - q^n}{1 - t^n} p_n.$$

Let us denote a function f in the set of variables $(x_1, x_2, \dots, y_1, y_2, \dots)$ by $f(x, y)$. Let $f_{\lambda, \mu}^\nu$ be the fusion coefficient $f_{\lambda, \mu}^\nu := \langle P_\lambda P_\mu, P_\nu \rangle / \langle P_\nu, P_\nu \rangle$, then we have

$$(A.9) \quad P_\lambda(x) P_\mu(x) = \sum_\nu f_{\lambda, \mu}^\nu P_\nu(x),$$

$$(A.10) \quad \frac{P_\nu(x, y)}{\langle \nu \rangle} = \sum_{\substack{\lambda, \mu \\ \lambda, \mu \subset \nu}} \frac{P_\lambda(x)}{\langle \lambda \rangle} f_{\lambda, \mu}^\nu \frac{P_\mu(y)}{\langle \mu \rangle}.$$

Let us denote the Young diagram decomposing into rectangles as $\lambda = \sum_{i=1}^{N-1} (s_i^{r_i})$, $r_i \geq r_{i+1}$, i.e., $\lambda' = (r_1^{s_1} r_2^{s_2} \dots r_{N-1}^{s_{N-1}})$,



Then we have the following integral representation of the Macdonald polynomial [22]

$$(A.11) \quad P_\lambda(x) = C_\lambda^+ \oint \prod_{a=1}^{N-1} \prod_{j=1}^{r_a} \frac{dz_j^a}{2\pi i z_j^a} (z_j^a)^{-s_a} \cdot \Pi(x, pz^1) \prod_{a=1}^{N-1} \Pi(\overline{z^a}, pz^{a+1}) \Delta(z^a) \\ = C_\lambda^+ \langle \alpha_{r,s}^+ | \exp \left\{ - \sum_{n>0} \frac{h_n^1}{1 - q^n} \sum_{i=1}^M x_i^n \right\} | \chi_{r,s}^+ \rangle, \quad C_\lambda^+ := \prod_{a=1}^{N-1} \frac{p^{-a r_a s_a} \langle \lambda^{(a)} \rangle}{r_a! \langle \lambda^{(a)} \rangle_{r_a}''}$$

with a singular vector $|\chi_{r,s}^+\rangle$ in (2.16). Here $z_i^N := 0$ and $\lambda^{(1)} := \lambda$, $\lambda^{(a)} := \sum_{i=a}^{N-1} (s_i^{r_i})$, i.e., $\lambda^{(a)'} = (r_a^{s_a} r_{a+1}^{s_{a+1}} \dots r_{N-1}^{s_{N-1}})$. Acting $\omega_- \omega_+ \omega_{q,t}$ on (A.11) gives

$$(A.12) \quad P_{\lambda'}(x) = C_\lambda^- \langle \alpha_{r,s}^- | \exp \left\{ - \sum_{n>0} \frac{h_n^1}{1 - q^n} \sum_{i=1}^M (-q x_i)^n \right\} | \chi_{r,s}^- \rangle, \quad C_\lambda^- := \omega_- \omega_+ \frac{C_\lambda^+}{\langle \lambda \rangle}.$$

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