

# On regular non-additive measures

(正則非加法的測度について)

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## 1 Introduction

Non-additive set functions on measurable space is used in economics, decision theory and artificial intelligence, called by various name, such as cooperative game , capacity or fuzzy measure. In this paper, we distinguish the term "fuzzy measure" from "non-additive measure". Sugeno's original axioms [15] for a fuzzy measure has some continuity. On the other hand, some authors define a fuzzy measure, that is monotone set function vanishing at  $\emptyset$ , and is not assumed any continuity. In order to avoid confusion, in this paper, we say that monotone set function vanishing at  $\emptyset$  is non-additive measure.

Generally, considering an infinite set, if nothing is assumed, it is too general and is sometimes inconvenient. Then we assume the universal set  $X$  to be a locally compact Hausdorff space. Considering the topology, various regularities are proposed [12, 17, 16].

In this paper, we arrange the various regularities and clarify their correlation. We consider the relation among the regularities and the Choquet integral with respect to a regular non-additive measure and introduce some new results about the approximation

of Choquet integral of a integrable function.

The structure of this paper is as follows: In section 2 we present some basic definition and properties without topological assumption of non-additive measure and Choquet integral with respect to a non-additive measure as a preliminaries. In Section 3, we assume that the universal space  $X$  is locally compact space. We introduce some regularities of non-additive measure and show some properties. In Section 4, we show some properties of the Choquet integral with respect to a regular non-additive measure. We show some representation theorem of functionals on the class of continuous function with compact support and approximation theorem of Choquet integral of integrable functions. In Section 5, we finish with some concluding remark.

## 2 Preliminaries

In this section, we present some basic definition and properties of non-additive measure theory.  $X$  denotes the universal set and  $\mathcal{B}$  its  $\sigma$ -algebra. No topological assumption is needed in this section.

**Definition 2.1.** [2] A *non-additive measure*  $\mu$  is an extended real valued set function,  $\mu : \mathcal{B} \longrightarrow \overline{\mathbb{R}^+}$  with the following properties: (1)  $\mu(\emptyset) = 0$ , (2)  $\mu(A) \leq \mu(B)$  whenever  $A \subset B$ ,  $A, B \in \mathcal{B}$ , where  $\overline{\mathbb{R}^+} = [0, \infty]$  is the set of extended nonnegative real numbers. In this paper we assume that  $\mu$  is finite, that is,  $\mu(X) < \infty$ .

Next we will present the continuous properties of non-additive measures.

**Definition 2.2.** Let  $\mu$  be a non-additive measure on  $(X, \mathcal{B})$ .

- (1) [18]  $\mu$  is called null-additive if  $\mu(A \cup B) = \mu(A)$  whenever  $A, B \in \mathcal{B}$ ,  $A \cap B = \emptyset$  and  $\mu(B) = 0$ .

- (2) [19]  $\mu$  is called weakly null-additive if  $\mu(A \cup B) = \mu(A)$  whenever  $A, B \in \mathcal{B}$ ,  $A \cap B = \emptyset$ ,  $\mu(A) = 0$  and  $\mu(B) = 0$ .
- (3) [15]  $\mu$  is said to be continuous from below (resp. above) if for every increasing (decreasing) sequence  $\{A_n\}$  of measurable sets,  $\mu(\lim_{n \rightarrow \infty} A_n) = \lim_{n \rightarrow \infty} \mu(A_n)$  holds. We say that a non-additive measure which is continuous from both above and below is a fuzzy measure.
- (4) [4] We say that  $\mu$  has a pseudo metric generating property (for short "p.g.p."), if both  $\lim_{n \rightarrow \infty} \mu(A_n) = 0$  and  $\lim_{n \rightarrow \infty} \mu(B_n) = 0$  implies  $\lim_{n \rightarrow \infty} \mu(A_n \cup B_n) = 0$  for  $\{A_n\}, \{B_n\} \subset \mathcal{B}$ .
- (5) [5]  $\mu$  is said to be exhaustive if  $\lim_{n \rightarrow \infty} \mu(A_n) = 0$  for any infinite disjoint sequence  $\{A_n\} \subset \mathcal{B}$ .
- (6) [18]  $\mu$  is said to be autocontinuous from above if for  $A \in \mathcal{B}$   $\lim_{n \rightarrow \infty} \mu(A \cup B_n) = \mu(A)$  whenever  $\lim_{n \rightarrow \infty} \mu(B_n) = 0$ .

The class of non-negative measurable functions is denoted by  $\mathcal{M}^+$ .

**Definition 2.3.** [1, 2] Let  $\mu$  be a non-additive measure on  $(X, \mathcal{B})$ .

- (1) The Choquet integral of  $f \in \mathcal{M}^+$  with respect to  $\mu$  is defined by

$$C_\mu(f) = \int_0^\infty \mu_f(r) dr,$$

where  $\mu_f(r) = \mu(\{x | f(x) \geq r\})$ .

$L_1^+(\mu)$  denotes the class of nonnegative Choquet integrable functions. That is,

$$L_1^+(\mu) := \{f | f \in \mathcal{M}^+, C_\mu(f) < \infty\}.$$

**Definition 2.4.** [3] Let  $f, g \in \mathcal{M}^+$ . We say that  $f$  and  $g$  are comonotonic if  $f(x) < f(x')$  implies  $g(x) \leq g(x')$  for  $x, x' \in X$ .  $f \sim g$  denotes that  $f$  and  $g$  are comonotonic.

The Choquet integral of  $f \in M$  with respect to a non-additive measure has the next basic properties.

**Theorem 2.5.** [2] Let  $f, g \in \mathcal{M}^+$ .

- (1) If  $f \leq g$ , then  $C_\mu(f) \leq C_\mu(g)$
- (2) If  $f \sim g$ , then  $C_\mu(f + g) = C_\mu(f) + C_\mu(g)$ .

### 3 Regular non additive measures

In the following we assume that  $X$  is a locally compact Hausdorff space,  $\mathfrak{O}$  the class of open subsets of  $X$  and  $\mathfrak{C}$  the class of compact subsets of  $X$ . We suppose that  $\mathcal{B}$  is the class of all Borel subsets of  $X$ : that is the smallest  $\sigma$ -algebra containing all open subsets, although we may suppose that  $\mathcal{B}$  is the class of Baire subset, that is, the smallest  $\sigma$ -algebra containing all compact subset, since a non-additive measure has not  $\sigma$ -additivity. If  $X$  is metric space, both the class of Baire subset coincides with the class of Borel subset.

**Definition 3.1.** Let  $\mu$  be a fuzzy measure on the measurable space  $(X, \mathcal{B})$ .

- (1)  $\mu$  is said to be  $\sigma$ -continuous from below  $O_n \uparrow O \Rightarrow \mu(O_n) \uparrow \mu(O)$  where  $n = 1, 2, 3, \dots$  and both  $O_n$  and  $O$  are open sets .
- (2)  $\mu$  is said to be  $c$ -continuous from above if  $C_n \downarrow C \Rightarrow \mu(C_n) \downarrow \mu(C)$  where  $n = 1, 2, 3, \dots$  and both  $C_n$  and  $C$  are compact sets.
- (3)  $\mu$  is said to be  $\mathfrak{C}$ -exhaustive if  $\lim_{n \rightarrow \infty} \mu(A_n) = 0$  for any infinite disjoint sequence  $\{A_n\} \subset \mathfrak{C}$ .

First, we define the regular non-additive measures.

**Definition 3.2.** Let  $\mu$  be a non-additive measure on measurable space  $(X, \mathcal{B})$ .  $\mu$  is said to be *inner regular* if  $\mu(B) = \sup\{\mu(C) | C \in \mathfrak{C}, C \subset B\}$  for all  $B \in \mathcal{B}$ . Inner regular non-additive measure is called *i-regular* if  $\mu(C) = \inf\{\mu(O) | O \in \mathfrak{D}, C \subset O\}$  for all  $C \in \mathfrak{C}$ .  $\mu$  is said to be *outer regular* if  $\mu(B) = \inf\{\mu(O) | O \in \mathfrak{D}, O \supset B\}$  for all  $B \in \mathcal{B}$ . Outer regular non-additive measure is called *o-regular* if  $\mu(O) = \sup\{\mu(C) | C \in \mathfrak{D}, C \subset O\}$  for all  $O \in \mathfrak{D}$ .

**Proposition 3.3.** Let  $\mu_i$  be an *i-regular non-additive measure* and  $\mu_o$  be an *o-regular non-additive measure*.  $\mu_i(O) = \mu_o(O)$  for  $O \in \mathfrak{D}$  if and only if  $\mu_i(C) = \mu_o(C)$  for  $C \in \mathfrak{C}$ .

**Proposition 3.4.** Let  $\mu_i$  be an *i-regular non-additive measure* and  $\mu_o$  be an *o-regular non-additive measure*. Then we have  $\mu_i(A) \leq \mu_o(A)$  for all  $A \in \mathcal{B}$ .

The next two results follow from the definition immediately.

**Proposition 3.5.** Let  $\mu$  be a *i- (resp. o-) regular non-additive measure*. Then  $\mu$  is both *o-continuous from below*. and *c-continuous from above*.

Since *i-(o-) regular non-additive measure* is  $\mathfrak{C}$ -continuous from above, we have the next proposition.

**Proposition 3.6.** The *i-(o-) regular non-additive measure* is *c-exhaustive*.

Next we define another regularity. It is a generalization of completion regularity in classical measure theory [6]. So we will call it completion regular in non-additive measure theory. The completion regular fuzzy measure has been studied by [7, 13, 17].

**Definition 3.7.** Let  $\mu$  be a non-additive measure on  $(X, \mathcal{B})$ .

- (1)  $\mu$  is said to be outer completion regular if for every  $\varepsilon > 0$  there exist an open set  $O_\varepsilon$  such that  $\mu(O_\varepsilon \setminus A) < \varepsilon$ .
- (2)  $\mu$  is said to be inner completion regular if for every  $\varepsilon > 0$  there exist a compact set  $C_\varepsilon$  such that  $\mu(A \setminus C_\varepsilon) < \varepsilon$ .
- (3)  $\mu$  is said to be completion regular if for every  $\varepsilon > 0$  there exist an open set  $O_\varepsilon$  a compact set  $C_\varepsilon$  such that  $\mu(O_\varepsilon \setminus C_\varepsilon) < \varepsilon$ .

The next proposition follows from the definition of p.g.p. immediately.

**Proposition 3.8.** *Let  $\mu$  be a non-additive measure that has a p.g.p. If  $\mu$  is both outer and inner completion regular, then  $\mu$  is completion regular.*

Suppose that  $\mu$  is a fuzzy measure. If  $\mu$  is autocontinuous from below,  $\mu$  has a p.g.p.. Therefore if a fuzzy measure  $\mu$  is both outer and inner completion regular and continuous from below, then  $\mu$  is completion regular [7].

Next we will consider the relation among (i-) regularity and completion regularity.

The next proposition follows from Proposition 3.6. The proof is essentially similar to [12].

**Proposition 3.9.** *Let  $\mu$  be a non-additive measure on  $(X, \mathcal{B})$  that has p.g.p. If  $\mu$  is inner regular, then  $\mu$  is completion regular.*

**Proposition 3.10.** *Let  $\mu$  be a non-additive measure on  $(X, \mathcal{B})$  that is null-additive and continuous from above. If  $\mu$  is completion regular, then  $\mu$  is regular.*

The next proposition is a generalization of the result proved in [8, 14].

**Proposition 3.11.** *Let  $\mu$  be a fuzzy measure on  $(X, \mathcal{B}_f)$ .*

- (1) *If  $\mu$  is weak null-additive, then  $\mu$  is completion regular.*

(2) If  $\mu$  is null-additive,  $\mu$  is regular.

Since null-additivity implies weak null-additivity, a null additive fuzzy measure on locally compact space is both regular and completion regular.

## 4 Choquet integral

The class of non-negative continuous functions with compact support is denoted by  $C_0^+(X)$ .

First we will present the basic properties of Choquet integral with respect to  $i$ - ( $o$ -) regular non-additive measure.

The next proposition follows from monotone convergence theorem of classical measure theory and definition of Choquet integral.

**Proposition 4.1.** *Let  $\mu$  be a  $i$ - ( $o$ -) regular non-additive measure on  $(X, \mathcal{B})$ . If  $f_n \downarrow f$  for  $\{f_n\} \subset C_0(X)$ , then  $C_\mu(f_n) \downarrow C_\mu(f)$ .*

The next proposition is proved by Narukawa et al [9] in the  $o$ -regular case. The  $i$ -regular case is proved similarly.

**Proposition 4.2.** *Let  $\mu_1$  and  $\mu_2$  be  $i$ - ( $o$ -) regular non-additive measures. If for all  $f \in C_0(X)$   $C_{\mu_1}(f) = C_{\mu_2}(f)$  then  $\mu_1 = \mu_2$ .*

Next we discuss the representation of functional on  $C_0^+(X)$ .

**Definition 4.3.** Let  $I$  be a real valued functional on  $C_0^+(X)$ . We say that  $I$  is *comonotonically additive* iff  $f \sim g \Rightarrow I(f + g) = I(f) + I(g)$  for  $f, g \in C_0^+(X)$ , and that  $I$  is *comonotonically monotone* iff  $f \leq g \Rightarrow I(f) \leq I(g)$  for comonotonic  $f, g \in C_0^+(X)$ .

If a functional  $I$  is comonotonically additive and comonotonically monotone, we say that  $I$  is a *c.a.c.m. functional*.

The next theorem is proved in the similar way to Sugeno et al [16].

**Theorem 4.4.** *Let  $I$  be a c.a.c.m. functional on  $C_0^+(X)$ . We put  $\mu_I(O) = \sup\{I(f) | f \in C_0^+(X), \text{supp}(f) \subset O, 0 \leq f \leq 1\}$ ,  $\mu_I(C) = \inf\{\mu_I(O) | O \in \mathfrak{D}, O \supset C\}$  and  $\mu_I(B) = \sup\{\mu_I(C) | C \in \mathfrak{C}, B \supset C\}$  for  $O \in \mathfrak{D}, C \in \mathfrak{C}$  and  $B \in \mathfrak{B}$ , where  $\text{supp}(f)$  is a support of  $f \in C_0^+(X)$ .*

*Then  $\mu_I$  is an  $i$ -regular fuzzy measure and  $I(f) = C_{\mu_I}(f)$ .*

Since Choquet integral with respect to a non-additive measure is a c.a.c.m functional, applying the theorem above, we have the next proposition.

**Proposition 4.5.** *Let  $\mu$  be a non-additive measure on  $(X, \mathfrak{B})$ . There exists an  $i$ -regular non-additive measure  $\mu_r$  on  $(X, \mathfrak{B})$  such that  $C_\mu(f) = C_{\mu_r}(f)$  for all  $f \in C_0^+(X)$ .*

**Definition 4.6.** Let  $I$  be a c.a.c.m. functional on  $C_0^+(X)$ . We say that  $\mu_I$  defined in Theorem 4.4 is an  $i$ -regular fuzzy measure induced by  $I$ .

In the case of  $i$ -( $o$ )-regular non-additive measure, the Choquet integral of any measurable function can be approximated by the Choquet integral of continuous function with compact support. In the following, we state this fact.

The next lemma follows from the definition of regular non-additive measure. The proof is similar to the case of  $o$ -regular [11].

**Lemma 4.7.** *Let  $\mu$  be a  $i$ -regular non-additive measure. For every measurable set  $A \in \mathfrak{B}$  and an arbitrary  $\epsilon > 0$ , there exists a continuous function with compact support  $f \in C_o(X)$  and a compact set  $C \in \mathfrak{C}$  such that  $C \subset A$ ,  $1_C \leq f$  and  $C_\mu(f) - \epsilon \leq \mu(C)$  and  $|\mu(A) - C_\mu(f)| < \epsilon$ .*

Conversely an approximation property in lemma above implies  $i$ -regularity.



**Lemma 4.8.** *Let  $\mu$  be a non-additive measure on  $(X, \mathcal{B})$ . Suppose that  $\mu$  has an approximation property, that is, For every measurable set  $A \in \mathcal{B}$  and an arbitrary  $\epsilon > 0$ , there exists a continuous function with compact support  $f \in C_0^+(X)$  and a compact set  $C \in \mathfrak{C}$  such that  $C \subset A$ ,  $1_C \leq f$  and  $C_\mu(f) - \epsilon \leq \mu(C)$  and  $|\mu(A) - C_\mu(f)| < \epsilon$ . Then  $\mu$  is  $i$ -regular.*

Applying the lemmas above, we have the next theorem.

**Theorem 4.9.** *Let  $\mu$  be a non-additive measure on  $(X, \mathcal{B})$ ,  $\mu$  is  $i$ -regular if and only if for every  $\epsilon > 0$  and  $f \in L_1^+(\mu)$ , there exists a continuous function with compact support  $g \in C_0^+(X)$  and compact set  $C \in \mathfrak{C}$  such that*

$$C \subset \text{supp}(f), 1_C \leq 1_{\text{supp}(g)}, |C_\mu(g) - \mu(C)| < \epsilon \text{ and } |C_\mu(f) - C_\mu(g)| < \epsilon.$$

Next we will consider additional condition for a c.a.c.m. functional.

**Definition 4.10.** *Let  $I$  be a c.a.c.m. functional on  $C_0^+(X)$ . We say that  $I$  is additively continuous at 0 if and only if  $I(f_n) \rightarrow 0$  and  $I(g_n) \rightarrow 0$  as  $n \rightarrow \infty$  for  $\{f_n\}, \{g_n\} \subset C_0^+(X)$  imply  $I(f_n + g_n) \rightarrow 0$ .*

We have the next lemma from Definition 4.10 and Theorem 4.4.

**Lemma 4.11.** *Let  $\mu_I$  be a  $i$ -regular non-additive measure induced by a c.a.c.m. functional with additively continuity at 0. Then  $\mu_I$  has a p.g.p..*

**Lemma 4.12.** *Let  $\mu$  be a completion regular non-additive measure on  $(X, \mathcal{B})$ .*

- (1) *For every  $A \in \mathcal{B}$  and an arbitrary  $\epsilon > 0$  there exist a continuous function  $f \in C_0^+(X)$  such that  $C_\mu(|1_A - f|) < \epsilon$ .*
- (2) *If  $\mu$  has a p.g.p., for every simple function  $s$  and an arbitrary  $\epsilon > 0$  there exist a continuous function  $f \in C_0^+(X)$  such that  $C_\mu(|s - f|) < \epsilon$ .*

Let  $f$  be a bounded non-negative measurable function. There exists a sequence  $\{s_n\}$  of simple function such that  $s_n \uparrow f$  as  $n \rightarrow \infty$ . Applying monotone convergence theorem in classical measure theory to non decreasing function  $\mu(|s_n - f| > \alpha)$ , we have the next proposition.

**Proposition 4.13.** *Let  $\mu$  be a completion regular non-additive measure on  $(X, \mathcal{B})$ . If  $\mu$  has a p.g.p., for every measurable function  $f \in L_1^+(\mu)$  and an arbitrary  $\epsilon > 0$  there exist a continuous function  $g \in C_0^+(X)$  such that  $C_\mu(|f - g|) < \epsilon$ .*

Let  $I$  be a c.a.c.m. functional on  $C_0^+(X)$  with additively continuity at 0. Since the induced regular non-additive measure  $\mu_I$  has a p.g.p. and  $\mathfrak{C}$ -exhaustivity, it follows from Proposition 3.9 that  $\mu_I$  is completion regular. Therefore we have the next theorem.

**Theorem 4.14.** *Let  $I$  be a c.a.c.m. functional on  $C_0^+(X)$  with additively continuity at 0 and  $\mu_I$  be a induced  $i$ -regular non-additive measure. Then for every measurable function  $f \in L_1^+(\mu)$  and an arbitrary  $\epsilon > 0$ ,*

- (1) *there exists a continuous function  $g$  with compact support such that*

$$|C_{\mu_I}(f) - C_{\mu_I}(g)| < \epsilon,$$

- (2) *there exists a continuous function  $g$  with compact support such that*

$$C_{\mu_I}(|f - g|) < \epsilon.$$

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