

On Type II and Type III Principal Graphs

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§0. Introduction

Let us consider a pair of type III_1 factors with finite index. Then it is known that there exists the simultaneous continuous crossed product decomposition of the pair. Therefore we have two principal graphs, so-called the type II principal graph and the type III principal graph [KL]. In this note we shall characterize the condition that above two graphs are different, in terms of Longo's sectors.

In the case of type $\text{III}_\lambda (0 < \lambda < 1)$ factors, we can show the same type of characterization when the pair has simultaneous discrete crossed product decomposition.

§1. The type III_1 case

Let $M \supset N$ be a pair of type III_1 factors with finite index. Then we can construct the simultaneous continuous crossed product decomposition as follows. Let ω be a faithful normal state on N and $E : M \rightarrow N$ the minimal conditional expectation of Hiai [H]. We define a pair of type II_∞ factors $\widetilde{M} \supset \widetilde{N}$ as follows.

$$\widetilde{M} = M \rtimes_{\sigma^{\omega \cdot E}} \mathbf{R} \supset \widetilde{N} = N \rtimes_{\sigma^\omega} \mathbf{R}.$$

Let τ be the trace on $\widetilde{M}$ and θ the dual action of $\sigma^{\omega \cdot E}$, which coincides with the dual action of σ^ω restricted to $\widetilde{N}$. Thanks to Takesaki duality we can identify $M \supset N$ with

$\widetilde{M} \rtimes_{\theta} \mathbf{R} \supset \widetilde{N} \rtimes_{\theta} \mathbf{R}$. Let

$$M_n \supset M_{n-1} \supset \cdots \supset M \supset N$$

$$\widetilde{M}_n \supset \widetilde{M}_{n-1} \supset \cdots \supset \widetilde{M} \supset \widetilde{N}$$

be the towers associated with $M \supset N$ and $\widetilde{M} \supset \widetilde{N}$. In the same way as above M_n can be identified with $\widetilde{M}_n \rtimes_{\theta} \mathbf{R}$. (Note that θ is canonically extended to $\widetilde{M}_n$.)

Definition 1.1. We call the principal graph of $\widetilde{M} \supset \widetilde{N}$ the type II principal graph and that of $M \supset N$ the type III principal graph.

It is known that the following relation holds [KL].

$$(\widetilde{M}_n \cap \widetilde{N}')_{\theta} = M_n \cap N'.$$

So in general two principal graphs do not coincide. Actually there exist examples having different principal graphs [S]. We show the necessary and sufficient condition that this phenomenon happens in terms of Longo's sectors. (For the definition of the sectors, see [L2, I].)

Theorem 1.2. Let $M \supset N$ be a pair of type III_1 factors with finite index and $\gamma : M \longrightarrow N$ the canonical endomorphism [L3]. The type II and the type III principal graphs of $M_1 \supset M$ do not coincide if and only if there appears the modular automorphism $_M[\sigma_t^{\hat{\tau}}]_M$ ($t \neq 0$) in $_M[\gamma^n]_M$ for some $n \in \mathbf{N}$.

As a corollary we have the following.

Corollary 1.3. Moreover if the depth of $M \supset N$ is finite the type II and type III principal graphs coincide.

It is easy to show the sufficiency. So we sketch the proof of the necessity. Assume that the two graphs are different. Since a one-parameter action can not move any central

projections in the higher relative commutant algebras, we can see that there exist two different ($M - M$ or $M - N$) sectors $[\rho_1], [\rho_2]$ which coincide restricted to $\widetilde{M}$. This means that there exists $t \neq 0$ and $[\rho_2] = [\sigma_t^{\hat{\tau}} \cdot \rho_1]$. Therefore

$$[\rho_2][\overline{\rho_1}] = [\sigma_t^{\hat{\tau}}][\rho_1 \overline{\rho_1}]$$

contains $[\sigma_t^{\hat{\tau}}]$. Corollary 2.3 follows from the fact that no aperiodic automorphism can appear in the decsendent $M - M$ sectors when the depth is finite.

§2. The type III_λ ($0 < \lambda < 1$) case

In the case of type III_λ ($0 < \lambda < 1$) factors, we have to assume that there exist a pair of type II_∞ factors $\widetilde{M} \supset \widetilde{N}$ and $\theta \in \text{Aut}(M)$ scaling the trace and globally preserving N such that

$$M = \widetilde{M} \rtimes_{\theta} \mathbf{Z} \supset N = \widetilde{N} \rtimes_{\theta} \mathbf{Z}.$$

Note that the assumption is satisfied if M is isomorphic to N and the center of $M \cap N'$ is trivial [Li].

We define the two principal graphs in the same way as before using the discrete crossed product decomposition. Then the same type of theorem holds.

Theorem 1.2. *Let $M \supset N$ be a pair of type III_λ ($0 < \lambda < 1$) factors with finite index and $\gamma : M \rightarrow N$ the canonical endomorphism. We assume that the pair has the simultaneous discrete crossed product decomposition. Then The type II and the type III principal graphs of $M_1 \supset M$ do not coincide if and only if there appears the modular automorphism $M[\sigma_t^{\hat{\tau}}]_M$ ($t \notin T(M)$) in $M[\gamma^n]_M$ for some $n \in \mathbf{N}$.*

In contrast to the type III_1 case the discrete action can move central projections in the higher relative commutant algebras. So we have to treat an additional case. Assume

that θ moves some central projections. Then we can show that there exists a decendant $(M - M$ or $M - N)$ sector $[\rho]$ satisfying $[\sigma_t^{\hat{\tau}} \rho] = [\rho]$ ($t \notin T(M)$). Hence,

$$[\rho \bar{\rho}] = [\sigma_t^{\hat{\tau}} \rho \bar{\rho}]$$

contains $[\sigma_t^{\hat{\tau}}]$.

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