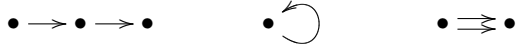


The Clebsch-Gordan problem for quiver representations

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Quivers. A **quiver** Q is a directed graph consisting of a set of vertices Q_0 and a set of arrows Q_1 . Below are some examples of quivers:



Representations. A **representation** V of a quiver Q consists of a collection of vector spaces V_x , where $x \in Q_0$ and linear maps $V(\alpha) : V_x \rightarrow V_y$, where $x \xrightarrow{\alpha} y \in Q_1$. Given representations V and W we define their direct sum $V \oplus W$ and tensor product $V \otimes W$ by

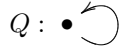
$$(V \oplus W)_x = V_x \oplus W_x, \quad (V \oplus W)(\alpha) = V(\alpha) \oplus W(\alpha), \\ (V \otimes W)_x = V_x \otimes W_x, \quad (V \otimes W)(\alpha) = V(\alpha) \otimes W(\alpha).$$

Clebsch-Gordan problem. Find the decomposition

$$V \otimes W = \bigoplus_i U_i$$

into indecomposable representations U_i for each pair of indecomposable representations V, W .

Known solutions. For the loop quiver



the Clebsch-Gordan problem was solved by Aitken (1935) over an algebraically closed ground field of characteristic zero. For algebraically closed ground fields of positive characteristic the solution was found by Iima-Iwamatsu (2006). Finally, over any perfect ground field the solution was found by Darpö-H (2008).

For Dynkin quivers the solution is known for type A , D , and E_6 . The Clebsch-Gordan problem has also been solved for a fairly large class of tame algebras called string algebras.

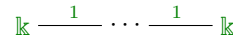
String algebras. Let I be an ideal in the path algebra $\mathbb{k}Q$ generated by a set of paths such that the quotient algebra $\Lambda = \mathbb{k}Q/I$ is finite dimensional. Then Λ is called a **string algebra** if

1. For each vertex $x \in Q_0$ there are at most two arrows starting (respectively ending) at x .
2. For each arrow $\alpha \in Q_1$ there is at most one $\beta \in Q_1$ such that $\alpha\beta \notin I$ and at most one $\gamma \in Q_1$ such that $\gamma\alpha \notin I$.

Strings and bands. For a quiver morphism $F : P \rightarrow Q$ we call (F, P) a shape over Q if for distinct arrows $x \xrightarrow{\alpha} y$ and $x' \xrightarrow{\alpha'} y'$ in P , $F\alpha = F\alpha'$ implies $x \neq x'$ and $y \neq y'$. With each shape we associate two functors

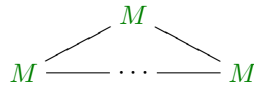
$$\text{rep}_{\mathbb{k}} P \xrightleftharpoons[F^*]{F_*} \text{rep}_{\mathbb{k}} Q.$$

A shape $\mathbf{F} = (F, L)$ is called linear if the underlying graph of L is Dynkin of type A . Let V be the representation



and set $S_{\mathbf{F}} = F_* V$. The representations $S_{\mathbf{F}}$ are called **strings**.

A shape $\mathbf{G} = (G, Z)$ is called cyclic if it has trivial automorphism group and the underlying graph of Z is extended Dynkin of type $\tilde{A}$. Let M be a $\mathbb{k}[T, T^{-1}]$ -module and $\gamma \in Z_1$. Furthermore, let W be the representation



where all arrows act as identity except γ which acts by multiplication with T . Set $B_{\mathbf{G}}(M, \gamma) = G_* W$. The representations $B_{\mathbf{G}}(M, \gamma)$ are called **bands**.

It is a classical result due to several authors including Gelfand-Ponomarev and Ringel-Butler that for string algebras, the indecomposable representations are classified by strings and bands. Let $\mathcal{L}(\mathbf{F}, \mathbf{F}')$ be the set of linear connected components of the fibre product $\mathbf{F} \times_Q \mathbf{F}'$. An example of the fibre product of two linear shapes can be found below.

Theorem. For linear shapes $\mathbf{F}, \mathbf{F}'$, non-isomorphic cyclic shapes $\mathbf{G}, \mathbf{G}'$, and $\mathbb{k}[T, T^{-1}]$ -modules M, M' the following formulae hold:

$$S_{\mathbf{F}} \otimes S_{\mathbf{F}'} \xrightarrow{\sim} \bigoplus_{\mathbf{H} \in \mathcal{L}(\mathbf{F}, \mathbf{F}')} S_{\mathbf{H}} \\ S_{\mathbf{F}} \otimes B_{\mathbf{G}}(M, \gamma) \xrightarrow{\sim} \bigoplus_{\mathbf{H} \in \mathcal{L}(\mathbf{F}, \mathbf{G})} \dim M S_{\mathbf{H}} \\ B_{\mathbf{G}}(M, \gamma) \otimes B_{\mathbf{G}'}(M', \gamma') \xrightarrow{\sim} \bigoplus_{\mathbf{H} \in \mathcal{L}(\mathbf{G}, \mathbf{G}')} \dim M \dim M' S_{\mathbf{H}} \\ B_{\mathbf{G}}(M, \gamma) \otimes B_{\mathbf{G}}(M', \gamma) \xrightarrow{\sim} B_{\mathbf{G}}(M \otimes_{\mathbb{k}} M', \gamma) \oplus \bigoplus_{\mathbf{H} \in \mathcal{L}(\mathbf{G}, \mathbf{G})} \dim M \dim M' S_{\mathbf{H}}$$

Example. $Q : \bullet \xrightleftharpoons[\beta]{\alpha} \bullet \xrightarrow{\gamma} \bullet$ $I = \langle \alpha\beta, \gamma^2, (\beta\gamma\alpha)^n \rangle$

